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# Asymptotic Regularity and Uniform Attractor for Non-autonomous Viscoelastic Equations with Memory

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***Abstract*—In this paper, long-time behavior of a class of non-autonomous viscoelastic equations with fading memory is investigated. We establish the existence of a compact uniform attractor together with its structure in $H_0^1(\Omega)\times H_0^1(\Omega)\times L_\mu^2(R^+;H_0^1(\Omega))$. The compact uniform attractor is bounded in $D(A)\times D(A)\times L_\mu^2(R^+;D(A))$ and attracts every bounded set of $H_0^1(\Omega)\times H_0^1(\Omega)\times L_\mu^2(R^+;H_0^1(\Omega))$.**

***Keywords-non-autonomous wave equations; asymptotic regularity; uniform attractor; memory;viscoelasticity***

## I. INTRODUCTION

In this paper, we consider the dynamical behavior of the solutions for the following non-autonomous evolutionary equations with a fading memory

$$u_{tt}-\Delta u_t-\Delta u-\Delta u_{tt}(t)-\int_0^\infty \mu(s)\Delta\eta^t(s)ds+f(u)=g, \qquad (1)$$

and

$$\eta_t^t=-\eta_s^t+u_t.$$

The problem is supplemented with the boundary condition

$$u(x,t)\big|_{\partial\Omega}=0 \qquad \text{for all} \quad t\ge\tau,\tau\in R$$

and initial condition

$$u(x,t)=u_\tau(x,t),u_t(x,t)=\frac{\partial}{\partial t}u_\tau(x,t) \quad t\le\tau,\tau\in R.$$

Where $\Omega$ is a bounded smooth domain in $R^3$, $g=g(t)$ is a given external time-dependent forcing, $f$ is the critical nonlinearity.

Problem(1)is related to the following equations like

$$u_{tt}-u_{xxt}-u_{xx}-u_{xxtt}=0,$$

Which appear as a class of nonlinear evolution equations, and that is used to represent the propagation problems of lengthways-wave in nonlinear elastic rods and Ion-sonic of space transformation by weak nonlinear effect (see for instance[1,3]). Since (1) contain terms $\Delta u_{tt}$, it is essentially different from D'Alembert wave equation.

Let us recall some results concerning the problem (1). In [10, 11] etc, authors studied this equations with Dirichlet boundary conditions as $\mu=0$. Recently, Araújo et al.[5] and M. Conti [4], H. Yassine and A. Abbas [9] studied the well posedness for this equations. In particular, Qin[8] obtain the existence of uniform attractors as $f=0$.

Maybe, we could establish the existence of uniform attractors of (1) using the method in [16, 17], but the regularity and structure cannot obtain directly. In this paper, we will apply the techniques introduced in Sun [14] to overcome the difficulty due to the critical nonlinearity, and establish the asymptotic regularity of the solutions. Based on this regularity result, we obtain the asymptotic compactness of the non-autonomous system and prove the existence of a uniform attractor together with its structure in $H_0^1(\Omega)\times H_0^1(\Omega)\times L_\mu^2(R^+;H_0^1(\Omega))$. It is noteworthy that the compact uniform attractor is bounded in $D(A)\times D(A)\times L_\mu^2(R^+;D(A))$.

For conveniences, hereafter let $|u|$ be the modular (or absolute value) of $u$ and $|\cdot|_p$ be the norm of $L^P(\Omega)(P>1)$. Denote $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and $\|\cdot\|_{H^{-1}}$ be the norm of $H^{-1}(\Omega)$ .Let $(\mathcal{V},\|\cdot\|_{\mathcal{V}})$ be a Banach space, we denote respectively the inner product and norm of the weighted space $L_\mu^2(R^+;\mathcal{V})$ by

$$\langle\varphi,\psi\rangle_{\mu,\mathcal{V}}=\int_0^\infty\mu(s)\langle\varphi(s),\psi(s)\rangle_{\mathcal{V}}\,ds$$

and

$$\|\varphi\|_{\mu,\mathcal{V}}^2=\int_0^\infty\mu(s)\|\varphi(s)\|_{\mathcal{V}}^2\,ds.$$

Denote $A=-\Delta$ with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, and for $r\in R$, let $\mathcal{E}_r=D(A^{\frac{r+1}{2}})$ and $\|\cdot\|_r$ be the norm of $\mathcal{E}_r$. We also define the system state space for $(u,u_t,\eta)$ as $H_r$, together with a dense subspace $M$:

$$H_r=\mathcal{E}_r\times\mathcal{E}_r\times L_\mu^2(R^+;\mathcal{E}_r),$$
$$M=D(A)\times D(A)\times(L_\mu^2(R^+;D(A))\cap H_\mu^2(R^+;H_0^1(\Omega))).$$

We also define the norm of the product space $H_r$ as follows

$$\|z\|_{H_r}^2=\|(u,v,\eta^t)\|_{H_r}^2=\frac{1}{2}(\|u\|_r^2+\|v\|_r^2+\|\eta^t\|_{u,\mathcal{E}_r}^2),$$

for any $z=(u,v,\eta^t)\in H_r$.

Let $C$ be an arbitrarily positive constant, which may be differential from line to line, even in the same line.

For the memory kernel $\mu(s)$, we assume the following hypotheses: for all $s\in R^+$ and some $\delta>0$

$$\mu\in C^1(R^+)\bigcap L^1(R^+),\ \ \mu(s)\ge 0,\ \ \mu'(s)\le 0\ , \tag{2}$$

$$\mu'(s)+\delta\mu(s)\le 0 \tag{3}$$

We introduce a new variable of the system,

$$\eta=\eta^t(x,s):=u(x,t)-u(x,t-s),\ \ s\in R^+, \tag{4}$$

which will be ruled by a supplementary equation. Denoting

$$\eta_t^t=\frac{\partial}{\partial t}\eta^t\quad,\qquad \eta_s^t=\frac{\partial}{\partial s}\eta^t.$$

Then the following estimate holds(See[17])

$$\left\langle \eta^t,\eta_s^t\right\rangle_{\mu,\nu}\ge\frac{\delta}{2}\|\eta^t\|_{\mu,\nu}^2\ . \tag{5}$$

The past history $u_\tau(\tau-s)$ of the variable $u$ satisfies the condition as follows: there exist two positive constants $\Re$ and $\kappa\le\delta$ such that

$$\int_0^\infty e^{-\kappa s}\|u_\tau(\tau-s)\|_0^2\,ds\le\Re. \tag{6}$$

The nonlinearity $f\in C^1(R,R)$, fulfills $f(0)=0$ satisfies the following decomposition

$$|f'(s)|\le c(1+|s|^4)\quad\text{for all}\quad s\in R \tag{7}$$

and

$$\liminf_{|s|\to\infty}\frac{f(s)}{s}>-\lambda_1\,, \tag{8}$$

for any $s\in R$, where $c,\lambda_1$ are positive constants and $\lambda_1$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with the Dirichlet boundary condition.

Calling $F(s)=\int_0^s f(y)dy$ . Notice that by (8), the following inequalities hold for some $0<\lambda<\lambda_1$ and $c_0\ge 0$

$$2\int_\Omega f(u)u\ge 2\int_\Omega F(u)-\lambda|u|_2^2-c_0 \tag{9}$$

For the time-dependent forcing $g$, we assume the following hypotheses: $g\in L_b^2(R;L^2(\Omega))$ (translation bounded in $L_{w,loc}^2(R;L^2(\Omega))$), and with the norm

$$\|g\|_{L_b^2}^2=\sup_{t\in R}\int_t^{t+1}|g(s)|_2^2ds<\infty\,.$$

## II. Preliminaries

We will complete our task exploiting the transitivity property of exponential attraction[15], that we recall below for the readers convenience.

**Lemma 2.1.***[15] Let $(H;d)$ be an abstract metric space, $U(t;\tau)$ be a Lipschitz continuous dynamical process in $H$, i.e*

$$\|U(t+\tau,\tau)z_1-U(t+\tau,\tau)z_2\|_H\le L_0e^{v_0t}\|z_1-z_2\|_H\,,$$

*for appropriate constants $v_0\ge 0$ and $L_0\ge 0$ which are independent of $z_i,\tau$ and $t$. We further assume that there exist three subsets $\mathcal{K}_1,\mathcal{K}_2,\mathcal{K}_3\subset H$ such that*

$$dist_H(U(t+\tau,\tau)\mathcal{K}_1,\mathcal{K}_2)\le L_1e^{-v_1t}\,,$$
$$dist_H(U(t+\tau,\tau)\mathcal{K}_2,\mathcal{K}_3)\le L_2e^{-v_2t}\,,$$

*for some $v_1,v_2\ge 0$ and $L_1,L_2\ge 0$. Then it follows that*

$$dist_H(U(t+\tau,\tau)\mathcal{K}_1,\mathcal{K}_3)\le Le^{-vt}\,,$$

*where $v=\dfrac{v_1v_2}{v_0+v_1+v_2}$ and $L=L_0L_1+L_2$*

**Lemma 2.2.** *[12] Let $X\subset\subset H\subset Y$ be Banach spaces, with $X$ reflexive. Suppose that $u_n$ is a sequence that is uniformly bounded in $L^2(0,T;X)$ and $\dfrac{du_n}{dt}$ is uniformly bounded in $L^p(0,T;Y)$, for some $p>1$. Then there is a subsequence of $u_n$ that converges strongly in $L^2(0,T;H)$.*

## III. Uniform Attractor in $H_0$

Throughout the paper, we assume $g_0 \in L_b^2(R; L^2(\Omega))$ and $\sum$ is the hull of $g_0$ in $L_{w,loc}^2(R; L^2(\Omega))$ and $g \in \sum$. Assume further that (2)-(3) and (6)-(8).

### *A. The Well-Posedness*

By the standard Faedo-Galerkin methods, it easy to obtain the following result.

**Lemma 3.1.** *for any $T > 0$ and $z_\tau = (u_\tau, v_\tau, \eta^\tau) \in H_0$. problem (1.1) admits a unique week solution*

$$z = (u(x,t), u_t(x,t), \eta^t) \in C([\tau, T], H_0),$$

*satisfying*

$$u \in L^\infty(R_\tau; H_0^1(\Omega)), u_t \in L^\infty(R_\tau; H_0^1(\Omega)),$$
$$u_{tt} \in L^2([\tau, T]; H_0^1(\Omega)), \eta \in L^\infty(R_\tau; L_\mu^2(R^+; H_0^1(\Omega)))$$

The proof of *Lemma3.1* is similar to that of *Theorem 2.1* of Araújo et al.[5] and hence is omitted.

Form *Lemma 3.1* above, for each $g \in L_b^2(R; L^2(\Omega))$ we define a process

$$U_g(t,\tau): H_0 \to H_0,$$
$$z_\tau = (u_\tau, v_\tau, \eta^\tau) \to (u(t), v(t), \eta^t) = U_g(t,\tau) z_\tau.$$

### *B. Dissipativity*

First of all, we can obtain the following theorem from [4]

**Theorem 3.2.***There exists a positive constant $M_0$ with following property: given any $Y \ge 0$ there exist $T_0 = T_0(Y, \tau) \ge \tau$ such that, whenever $\|z_\tau\|_{H_0} \le Y$ it follows that*

$$\|U_g(t,\tau) z_\tau\|_{H_0}^2 \le M_0, \qquad \forall t \ge T_0.$$

*Consequently, the set*

$$B_0 = \left\{ z_\tau \in H_0 : \|z_\tau\|_{H_0}^2 \le M_0 \right\}$$

*is a bounded uniformly (w.r.t $\sigma \in \sum$) absorbing set for $U_g(t,\tau)$ on $H_0$, that is, for any bounded (in $H_0$) subsets $B$ ,there is a $T_0 = T_0(\|B\|_{H_0}, \tau) \ge \tau$ such that*

$$\underset{g\in\sum}{U} U_g(t,\tau) B \subset B_0$$

*for every $t \ge T_0$.*

Combining *Lemma 3.1*, we know that for any $\tau \in R$, $U_g$ maps the bounded set of $H_0$ into a bounded set of $H_0$ for all $t \ge \tau$, that is

**Corollary 3.3.***Given any $R > 0$, there is $M_R = M_R(R, \|g\|_{L_b^2})$ such that for all $\|z_\tau\|_{H_0} \le R$,*

$$\|U_g(t,\tau) z_\tau\|_{H_0}^2 \le M_R, \forall t > \tau.$$

**Lemma 3.4.** *Given any $R > 0$, let $z_{1\tau}, z_{2\tau} \in H_0$ $g_1, g_2 \in L_b^2(R; L^2(\Omega))$ ,be two initial data, and $\|z_{i\tau}\|_{H_0} \le R (i = 1,2)$ .Then the following estimate holds,*

$$\|U_{g_1}(t,\tau) z_{1\tau} - U_{g_2}(t,\tau) z_{2\tau}\|_{H_0}^2 \le Q(R) e^{k(t-\tau)} (\|z_{1\tau} - z_{2\tau}\|_{H_0}^2 + \|g_1 - g_2\|_{L_b^2}^2) \tag{10}$$

*for any $t \ge \tau$ and some $k = k(R)$.*

### *C. Asymptotic Regularity*

For the nonlinear function $f(u)$ from[2], we know that $f$ has the following decomposition

$$f = f_0 + f_1$$

where $f_0, f_1 \in C(R)$ and satisfy

$$f_0(s)s \ge 0 \quad \text{for all} \quad s \in R, \tag{11}$$

$$|f_0(s)| \le c(1 + |s|^5) \quad \text{for all} \quad s \in R, \tag{12}$$

$$|f_1(s)| \le c(1 + |s|^\gamma) \quad \text{for all } s \in R \text{ with some } \gamma < 5, \tag{13}$$

$$\liminf_{|s|\to\infty} \frac{f_1(s)}{s} > -\lambda_1, \tag{14}$$

where $c, \lambda_1$ are positive constants and $\lambda_1$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with the Dirichlet boundary condition. Denote

$$\sigma = \min\{\frac{1}{4}, \frac{5-\gamma}{2}\}. \tag{15}$$

In order to obtain the regularity estimates later, we decompose the solution $U_g(t,\tau) z_\tau = (u(t), u_t(t), \eta^t)$ into the sum:

$$U_g(t,\tau) z_\tau = S(t,\tau) z_\tau + K_g(t,\tau) z_\tau.$$

$S(t,\tau) z_\tau = (v(t), v_t(t), \xi^t)$, $K_g(t,\tau) z_\tau = (w(t), w_t(t), \zeta^t)$ are the solutions the following equations respectively

$$\begin{cases} v_{tt}-\Delta v_{tt}-\Delta v_t-\Delta v-\int_0^\infty \mu(s)\Delta\xi^t(s)ds+f_0(v)=0, \\ \xi_t^t=-\xi_s^t+v_t(t), \\ (v(\tau),v_t(\tau),\xi^\tau)=z_\tau,\quad v\big|_{\partial\Omega}=0,\xi\big|_{\partial\Omega\times R^+}=0, \end{cases} \tag{16}$$

and

$$\begin{cases} w_{tt}-\Delta w_{tt}-\Delta w_t-\Delta w-\int_0^\infty \mu(s)\Delta\zeta^t(s)ds+f(u)-f_0(v)=g(x,t), \\ \zeta_t^t=-\zeta_s^t+w_t(t), \\ (w(\tau),w_t(\tau),\zeta^\tau)=0,\quad w\big|_{\partial\Omega}=0,\zeta\big|_{\partial\Omega\times R^+}=0. \end{cases} \tag{17}$$

We will establish a priori estimates about the solutions of (16) and(17), which are the basis of our works.

**Lemma 3.5.** *For any initial data $z_\tau\in H_0$, the solutions of (16) satisfy the following estimates: There exists constant $k_0$ such that for every $t\ge\tau$,*

$$\|S(t,\tau)z_\tau\|_{H_0}^2=\|v(t)\|_0^2+\|v_t(t)\|_0^2+\|\xi^t\|_{\mu,\varepsilon_0}^2\le Q_1(\|z_\tau\|_{H_0})e^{-k_0(t-\tau)}$$

*where $Q_1(\cdot)$ is an increasing function on $[0,\infty)$, $Q_1$ and $k_0$ only depend on the $H_0$ - bound of $z_\tau$, but both are independent of $\tau$.*

**Proof** Repeating word by word the proof of *Theorem 3.2*, that applies to the present case with $S(t,\tau)z_\tau$ in place of $U_g(t,\tau)z_\tau$ (with the further simplification that $C=0$, for now $f_1\equiv 0$ and $g\equiv 0$), It follows that

$$\|S(t,\tau)z_\tau\|_{H_0}^2=\|v(t)\|_0^2+\|v_t(t)\|_0^2+\|\xi^t\|_{\mu,\varepsilon_0}^2\le Q_1(\|z_\tau\|_{H_0})e^{-k_0(t-\tau)}$$

For the solution of (17), we have

**Lemma 3.6.** *For any $\tau\in R$, the solutions of (17) satisfy the following estimates: There exists constant $k_1$ such that for every $t\ge\tau$,*

$$\begin{aligned}\|K_g(t,\tau)z_\tau\|_{H_\sigma}^2&=\|w(t)\|_\sigma^2+\|w_t(t)\|_\sigma^2+\|\xi^t\|_{\mu,\varepsilon_\sigma}^2, \\ &\le Q_2(\|z_\tau\|_{H_0})e^{k_1(t-\tau)}(1+\|g\|_{L_b^2}^2)\end{aligned}$$

*where $Q_2(\cdot)$ is an increasing function on $[0,\infty)$, and $\sigma$ is given in (15).*

**Proof.** Multiplying(17) by $A^\sigma w_t(t)$, and integrating in $dx$ over $\Omega$, we get that

$$\frac{1}{2}\frac{d}{dt}\left(\left|A^{\frac{\sigma}{2}}w_t(t)\right|_2^2+\|w(t)\|_\sigma^2+\|w_t(t)\|_\sigma^2\right)+\|w_t(t)\|_\sigma^2-\left\langle\int_0^\infty\mu(s)\Delta\zeta^t(s)ds,A^\sigma w_t(t)\right\rangle$$
$$=-\left\langle f(u)-f_0(v),A^\sigma w_t(t)\right\rangle+\left\langle A^\sigma w_t(t),g(x,t)\right\rangle. \tag{18}$$

Similar to that in *Theorem 3.2* above, we get

$$-\int_\Omega\int_0^\infty\mu(s)\Delta\zeta^t(s)A^\sigma\zeta_t(s)dsdx=\frac{1}{2}\frac{d}{dt}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2\ , \tag{19}$$

and

$$-\int_\Omega\int_0^\infty\mu(s)\Delta\zeta^t(s)A^\sigma\zeta_s^t(s)dsdx\ge\frac{\delta}{2}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2. \tag{20}$$

Next we deal with the nonlinearity, we have

$$\left|\left\langle f(u)-f_0(v),A^\sigma w_t(t)\right\rangle\right|\le\left|\left\langle f(u)-f(v),A^\sigma w_t(t)\right\rangle\right|+\left|\left\langle f_1(v),A^\sigma w_t(t)\right\rangle\right|$$

and by *Corollary (3.3)* and *Lemma (3.5)*, we have

$$\|u(t)\|_0^2+\|v(t)\|_0^2\le M_1\quad\text{for all}\quad t\ge\tau, \tag{21}$$

where the constant $M_1$ depends on $\|z_\tau\|_{H_0}$ but independent of $\tau$.

From (7), (21) and Hölder's inequality, then we have

$$\left|\left\langle f(u)-f(v),A^\sigma w_t(t)\right\rangle\right|\le C_{M_1}\|w(t)\|_\sigma^2+\frac{1}{4}\|w_t(t)\|_\sigma^2. \tag{22}$$

Note that $\sigma\le\frac{5-\gamma}{2}$, so we can get the following estimates

$$\left|\left\langle f_1(v),A^\sigma w_t(t)\right\rangle\right|\le C+\frac{1}{4}\|w_t(t)\|_\sigma^2. \tag{23}$$

Moreover,

$$\left|\left\langle A^\sigma w_t(t),g(t)\right\rangle\right|\le C|g(t)|_2^2+\frac{1}{2}\|w_t(t)\|_\sigma^2. \tag{24}$$

Combined with (19)-(20) and(22)-(23), by (18), we have that

$$\frac{d}{dt}\left(\left|A^{\frac{\sigma}{2}}w_t(t)\right|_2^2+\|w_t(t)\|_\sigma^2+\|w(t)\|_\sigma^2+\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2\right)\le C(1+|g(t)|_2^2)+C_{M_1}\|w(t)\|_\sigma^2.$$

Applying the Gronwall's inequality, we deduce that

$$\left|A^{\frac{\sigma}{2}}w_t(t)\right|_2^2+\|w_t(t)\|_\sigma^2+\|w(t)\|_\sigma^2+\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2$$
$$\le Q_2(\|z_\tau\|_{H_0})e^{k_1(t-\tau)}(1+\|g\|_{L_b^2}^2)$$

here $k_1 = C_{M_1}$ and $C_{M_1}$ depend on $\|z_\tau\|_{H_0}$ .

**Lemma 3.7.** *For any $\varepsilon > 0$ $u_t(t)$ is decomposed as*

$$u(t) = v_1(t) + w_1(t) ,$$

*$v_1(t)$ satisfies: there is a positive constant $M_1 = M_1(\|z_\tau\|_{\varepsilon_0})$ such that the following estimates are true*

$$\|v_1(t)\|_0^2 \le M_1 ,$$

*and*

$$\int_s^t \|v_1(\nu)\|_0^2 d\nu \le \varepsilon(t-s) + C_\varepsilon \quad \text{for all} \quad t \ge s \ge \tau . \qquad (25)$$

*As well as $w_1(t)$ satisfies the following estimate*

$$\|w_1(t)\|_\sigma^2 \le K_\varepsilon \quad \text{for all} \quad t \ge \tau , \qquad (26)$$

*with the constants $C_\varepsilon$ and $K_\varepsilon$ depending on $\varepsilon, \|z_\tau\|_{H_0}$ and $\|g\|_{L_b^2}$, but both being independent of $\tau$ .*

The proof of this *lemma* is similar to that in Sun [14].

In what follows we begin to establish the asymptotic regularity of the solutions of (1).

**Lemma 3.8.** *There exists constant $\Upsilon_0$ which depends only on the $H_0$-bounds of $B(\subset H_0)$ ,such that for any $\tau \in R$*

$$\|K_g(t,\tau)z_\tau\|_{H_\sigma}^2 \le \Upsilon_0 \quad \text{for all} \quad t \ge \tau \quad \text{and} \quad z_\tau \in B,$$

*where $\sigma$ is given in (15).*

**Proof.** Taking inner product of the first equation of (17) and $A^\sigma(w_t + \varepsilon w)$ ($\varepsilon$ is an positive undetermined constant), we get that

$$\left\langle w_{tt} - \Delta w - \Delta w_t - \Delta w_{tt} - \int_0^t \mu(s)\Delta\zeta^t(s)ds, A^\sigma(w_t+\varepsilon w) \right\rangle$$
$$= -\left\langle f(u) - f_0(v), A^\sigma(w_t+\varepsilon w)\right\rangle + \left\langle g(x,t), A^\sigma(w_t+\varepsilon w)\right\rangle, \qquad (27)$$

In the following, we will deal with the left side of (27) one by one. Similar to that (19) and (20), we get that

$$-\left\langle \int_0^\infty \mu(s)\Delta\zeta^t(s)ds, A^\sigma(w_t+\varepsilon w) \right\rangle \ge \frac{1}{2}\frac{d}{dt}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2 + \frac{\delta}{2}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2 - \varepsilon\|\zeta^t\|_{\mu,\varepsilon_\sigma}\|w\|_\sigma .$$

Now we rewrite (27) as

$$\frac{d}{dt}E_2(t) + I_2(t) = -\left\langle f(u) - f_0(v), A^\sigma(w_t+\varepsilon w) \right\rangle + \left\langle g(x,t), A^\sigma(w_t+\varepsilon w) \right\rangle \qquad (28)$$

here

$$E_2(t) = \frac{1}{2}\left|A^{\frac{\sigma}{2}} w_t\right|_2^2 + \varepsilon\left\langle w_t, A^\sigma w\right\rangle + \frac{1+\varepsilon}{2}\|w\|_\sigma^2 + \frac{1}{2}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2 + \frac{1}{2}\|w_t\|_\sigma^2 + \varepsilon\left\langle Aw_t, A^\sigma w\right\rangle \qquad (29)$$

and

$$I_2(t) = -\varepsilon\left|A^{\frac{\sigma}{2}} w_t\right|_2^2 + \varepsilon\|w\|_\sigma^2 + \|w_t\|_\sigma^2 - \varepsilon\|w_t\|_\sigma^2 + \frac{\delta}{2}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2 - \varepsilon\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2\|w\|_\sigma \qquad (30)$$

Applying the Hölder's inequality in (29), we get that

$$E_2(t) \le \alpha_1(\|w_t\|_\sigma^2 + \|w\|_\sigma^2 + \|\zeta^t\|_{\mu,\varepsilon_\sigma}^2), \qquad (31)$$

where $\alpha_1 = \max\{\frac{1+\varepsilon}{2}(1+\frac{1}{\lambda_1}), \frac{\varepsilon}{2\lambda_1} + \frac{1+2\varepsilon}{2}\}$.

On the other hand, we have

$$E_2(t) \ge \frac{1}{2}(1-\frac{\varepsilon^2}{\lambda_1})\|w\|_\sigma^2 + \frac{1}{2}(1-\varepsilon)\|w_t\|_\sigma^2 + \frac{1}{2}\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2 . \qquad (32)$$

choose

$$\varepsilon \le \frac{1}{2}\min\{1, \sqrt{\lambda_1}\}. \qquad (33)$$

Let $\beta_1 = \min\{\frac{1}{2}(1-\varepsilon), \frac{1}{2}(1-\frac{\varepsilon^2}{\lambda_1})\} > 0$ ,then

$$E_2(t) \ge \beta_1(\|w_t\|_\sigma^2 + \|w\|_\sigma^2 + \|\zeta^t\|_{\mu,\varepsilon_\sigma}^2). \qquad (34)$$

Toward $I_2(t)$, we have

$$I_2(t) \ge \frac{\varepsilon}{2}\|w\|_\sigma^2 + (1-(1+\frac{1}{\lambda_1})\varepsilon)\|w_t\|_\sigma^2 + \frac{1}{2}(\delta-\varepsilon)\|\zeta^t\|_{\mu,\varepsilon_\sigma}^2 \qquad (35)$$

Combined with (31), choose

$$\varepsilon = \frac{1}{2}\min\{\frac{\lambda_1}{1+\lambda_1}, \delta, \sqrt{\lambda_1}\}.$$

Let $\alpha_2 = \frac{1}{2}\min\{\varepsilon, 2(1-(1+\frac{1}{\lambda_1})\varepsilon), \delta-\varepsilon\}$ .

$$I_2 \ge \alpha_2(\|w_t\|_\sigma^2 + \|w\|_\sigma^2 + \|\zeta^t\|_{\mu,\varepsilon_\sigma}^2). \qquad (36)$$

From *Corollary 3.3* and *Lemma 3.5*, there is a positive constant $M_2 = M_2(\|z_\tau\|_{\varepsilon_0})$ such that

$$\|\mathcal{K}_g(t,\tau)z_\tau\|_{\varepsilon_0}^2 \le M_2$$

holds for any $\tau \in R$.

Since $\frac{1+\delta}{2} < 1$, employing the interpolation inequality, we can get that

$$\left|\left\langle g(t), A^\sigma(w_t+\varepsilon w)\right\rangle\right| \le C_{\alpha_2}|g(t)|_2^2 + \frac{\alpha_2}{8}(\|w_t\|_\sigma^2 + \|w\|_\sigma^2), \quad (37)$$

and employing *Lemma 3.7* to deal with the nonlinear term:

$$\begin{aligned}&\left|\left\langle f(u)-f_0(v), A^\sigma(w_t+\varepsilon w)\right\rangle\right| \\ &\le \left|\left\langle f(u)-f(v), A^\sigma(w_t+\varepsilon w)\right\rangle\right| + \left|\left\langle f_1(v), A^\sigma(w_t+\varepsilon w)\right\rangle\right|.\end{aligned} \quad (38)$$

From (7) and *Lemma 3.5*, we have

$$\begin{aligned}&\left|\left\langle f(u)-f(v), A^\sigma(w_t+\varepsilon w)\right\rangle\right| \\ &\le CM_2 + C\int_\Omega(|u(t)|^4+|v(t)|^4)|w(t)|\left|A^\sigma(w_t+\varepsilon w)\right|.\end{aligned} \quad (39)$$

Using *Lemma 3.7*, we have

$$\int_\Omega |u(t)|^4|w(t)|\left|A^\sigma(w_t+\varepsilon w)\right| \le \int_\Omega(|v_1(t)|^4+|w_1(t)|^4)|w(t)|\left|A^\sigma(w_t+\varepsilon w)\right| \quad (40)$$

and

$$\int_\Omega |v_1|^4|w|\left|A^\sigma(w_t+\varepsilon w)\right| \le \mathrm{M}_1\|v_1\|_0^2(\|w_t\|_\sigma^2+\|w\|_\sigma^2). \quad (41)$$

Therefore, note that $\sigma = \min\{\frac{1}{4}, \frac{5-\gamma}{2}\}$ ,we have $\frac{12}{5} < \frac{6}{1+6\sigma} < 6$, and then

$$\int_\Omega |w_1|^4|w|\left|A^\sigma(w_t+\varepsilon w)\right| \le \frac{2K_\varepsilon^8 M_2^2}{\alpha_2} + \frac{\alpha_2}{4}(\|w_t\|_\sigma^2+\|w\|_\sigma^2). \quad (42)$$

where $K_\varepsilon$ is given in (26).

$$\int_\Omega |v|^4|w|\left|A^\sigma(w_t+\varepsilon w)\right| \le 2\mathrm{Q}_1(\|z_\tau\|_{H_0})\|v\|_0^2(\|w_t\|_\sigma^2+\|w\|_\sigma^2). \quad (43)$$

where $\mathrm{Q}_1(\|z_\tau\|_{H_0})$ from *Lemma 3.5.*

Substitute (40)-(43) into (39), we get that

$$\begin{aligned}\left|\left\langle f(u)-f(v), A^\sigma(w_t+\varepsilon w)\right\rangle\right| &\le C(\|v\|_0^2+\|v_1\|_0^2)(\|w_t\|_\sigma^2+\|w\|_\sigma^2) \\ &+\frac{\alpha_2}{4}(\|w_t\|_\sigma^2+\|w\|_\sigma^2)+K_1\end{aligned} \quad (44)$$

where $K_1 = CM_2 + \frac{2CK_\varepsilon^8 M_2^2}{\alpha_2}$.

Similarly,

$$\left|\left\langle f_1(v), A^\sigma(w_t+\varepsilon w)\right\rangle\right| \le K_2 + \frac{\alpha_2}{8}(\|w_t\|_\sigma^2+\|w\|_\sigma^2), \quad (45)$$

Moreover, it follows *Lemma 3.7*,

$$\int_\tau^\infty \|v(s)\|_0^2 ds \le \frac{\mathrm{Q}_1(\|z_\tau\|_{H_0})}{k_0} \quad ,$$

then for any $\varepsilon > 0$

$$\int_s^t(\|v(s)\|_0^2+\|v_1(s)\|_0^2)ds \le \varepsilon(t-s) + \frac{\mathrm{Q}_1(\|z_\tau\|_{H_0})}{k_0} + C_\varepsilon.$$

Hence, combining the above estimates into (28), we see that for all $t \ge \tau$,

$$\frac{d}{dt}E_2(t) + \frac{\alpha_2}{2\alpha_1}E_2(t) \le \frac{C}{\beta_1}(\|v(t)\|_0^2+\|v_1(t)\|_0^2)E_2(t)+K_1+K_2. \quad (46)$$

Then Gronwall's inequality yields, for any $t \ge T > \tau$

$$E_2(t) \le \beta E_2(T)e^{-\gamma(t-T)} + \rho,$$

here $\beta > 0$ is a constant which depended on initial data and $\gamma, \rho$ are positive constants which depended on initial data.

At the last, by *Lemma 3.6*, (31), (34) and noting that $T > \tau$ is fixed, then the proof is completed.

**Lemma 3.9.** *Assume $B_\sigma$ is bounded in $H_\sigma$. Then there exists a constant $M_\sigma(>0)$ which only depends on the $H_\sigma$-bounds of $B_\sigma$ such that for any $\tau \in R$*

$$\|U_g(t,\tau)z_\tau\|_{H_\sigma} \le M_\sigma \quad \textit{for all } t \ge \tau \textit{ and } z_\tau \in B_\sigma.$$

**Proof** Multiply (1) by $A^\sigma(u_t+\varepsilon u)$ ($\varepsilon$ is a positive undetermined constant), we get that

$$\frac{d}{dt}E_3(t)+I_3(t) = -\left\langle f(u), A^\sigma(u_t+\varepsilon u)\right\rangle + \left\langle g(x,t), A^\sigma(u_t+\varepsilon u)\right\rangle, \quad (47)$$

here

$$E_3(t)=\frac{1}{2}\left|A^{\frac{\sigma}{2}}u_t\right|_2^2+\varepsilon\left\langle u_t,A^\sigma u\right\rangle+\frac{1+\varepsilon}{2}\|u\|_\sigma^2$$
$$+\frac{1}{2}\|\xi^t\|_{\mu,\varepsilon_\sigma}^2+\frac{1}{2}\|u_t\|_\sigma^2+\varepsilon\left\langle Au_t,A^\sigma u\right\rangle, \quad (48)$$

and

$$I_3(t)=-\varepsilon\left|A^{\frac{\sigma}{2}}u_t\right|_2^2+\varepsilon\|u\|_\sigma^2+\|u_t\|_\sigma^2-\varepsilon\|u_t\|_\sigma^2$$
$$+\frac{\delta}{2}\|\xi^t\|_{\mu,\varepsilon_\sigma}^2-\varepsilon\|\xi^t\|_{\mu,\varepsilon_\sigma}^2\|u\|_\sigma. \quad (49)$$

Applying the Hölder's inequality in (48), we get that

$$E_3(t)\le\alpha_1(\|u_t\|_\sigma^2+\|u\|_\sigma^2+\|\xi^t\|_{\mu,\varepsilon_\sigma}^2), \quad (50)$$

where $\alpha_1$ from (31).

On the other hand, we have

$$E_3(t)\ge\frac{1}{2}(1-\frac{\varepsilon^2}{\lambda_1})\|u\|_\sigma^2+\frac{1}{2}(1-\varepsilon)\|u_t\|_\sigma^2+\frac{1}{2}\|\xi^t\|_{\mu,\varepsilon_\sigma}^2. \quad (51)$$

choose

$$\varepsilon\le\frac{1}{2}\min\{1,\sqrt{\lambda_1}\}. \quad (52)$$

then

$$E_3(t)\ge\beta_1(\|u_t\|_\sigma^2+\|u\|_\sigma^2+\|\xi^t\|_{\mu,\varepsilon_\sigma}^2). \quad (53)$$

where $\beta_1=\min\{\frac{1}{2}(1-\varepsilon),\frac{1}{2}(1-\frac{\varepsilon^2}{\lambda_1})\}>0$ (from (34)).

Toward $I_3(t)$, we have

$$I_3(t)\ge\frac{\varepsilon}{2}\|u\|_\sigma^2+(1-(1+\frac{1}{\lambda_1})\varepsilon)\|u_t\|_\sigma^2+\frac{1}{2}(\delta-\varepsilon)\|\xi^t\|_{\mu,\varepsilon_\sigma}^2 \quad (54)$$

Combined with (52), choose

$$\varepsilon=\frac{1}{2}\min\{\frac{\lambda_1}{1+\lambda_1},\delta,\sqrt{\lambda_1}\}.$$

Similar to (36), let $\alpha_2=\frac{1}{2}\min\{\varepsilon,2(1-(1+\frac{1}{\lambda_1})\varepsilon),\delta-\varepsilon\}$.

$$I_3\ge\alpha_2(\|u_t\|_\sigma^2+\|u\|_\sigma^2+\|\xi^t\|_{\mu,\varepsilon_\sigma}^2). \quad (55)$$

From *Corollary 3.3*, there is a positive constant $Y=Y(\|z_\tau\|_{H_0})$ such that

$$\|U_g(t,\tau)z_\tau\|_{H_0}^2\le Y$$

holds for any $\tau\in R$.

Since $\frac{1+\delta}{2}<1$, employing the interpolation inequality, we can get that

$$\left|\left\langle g(t),A^\sigma(u_t+\varepsilon u)\right\rangle\right|\le C_1|g(t)|_2^2+\frac{\alpha_2}{6}(\|u_t\|_\sigma^2+\|u\|_\sigma^2). \quad (56)$$

where $C_1$ is a constant which depends on $\alpha_2$ and the measure of $\Omega$ .Next, we deal with the nonlinear term,

$$\left|\left\langle f(u),A^\sigma v\right\rangle\right|\le c\int_\Omega(1+|u|^5)|A^\sigma v|\le c\int_\Omega|A^\sigma v|+c\int_\Omega|u|^5|A^\sigma v|.$$

Using the Hölder inequality and the Sobolev embedding theorem, it follows

$$c\int_\Omega|A^\sigma v|\le C_2+\frac{\alpha_2}{6}(\|u_t\|_\sigma^2+\|u\|_\sigma^2). \quad (57)$$

where $C_1$ is a constant which depends on $\alpha_2$, $c$ (from (7)) and the measure of $\Omega$

On the other hand, by *Lemma 3.7*, we get

$$c\int_\Omega|u|^5|A^\sigma v|\le C_M\|v_1\|_0^2(\|u_t\|_\sigma^2+\|u\|_\sigma^2)+\frac{\alpha_2}{6}(\|u_t\|_\sigma^2+\|u\|_\sigma^2)+C_3. \quad (58)$$

where $C_M$ is a constant which depends on $c$ (from (7)) and the $H_\sigma$-bounds of initial data (see *Corollary 3.3*).

So we have

$$\frac{d}{dt}E_3(t)+\frac{\alpha_2}{2\alpha_1}E_3(t)\le\frac{C_M\alpha_2}{\beta_1}\|v_1(t)\|_0^2E_3(t)+C_1|g(t)|_2^2+C_2+C_3, \quad (59)$$

where the positive constants $C_i, i=1,2,3$ depend on $\delta,K_\varepsilon$ and $\|z_\tau\|_{H_0}$.

Using *Lemma 3.1* and integrating over $[\tau,T]$, we get that

$$E_3(t)\le\alpha_1\|z_\tau\|_{H_\sigma}^2e^{-w(t-\tau)+m_1}+\frac{m_2e^{w+m_1}}{1-e^{-w}}, \quad (60)$$

where

$$w = \frac{\alpha_2}{4\alpha_1}, \quad m_1 = C_w \text{ (from (25))}, \quad m_2 = C_2 + C_3 + \|g\|_{L_b^2}^2 .$$

We then complete the proof.

**Lemma 3.10.** *For each $\theta \in [\sigma, 1]$, let $B$ be any bounded subset of $H_\theta$. Then there exists a constant $M_\theta$ which only depends on the $H_\theta$-bounds of $B$, such that for any $\tau \in R$,*

$$\|U_g(t,\tau)z_\tau\|_{H_\theta} \le M_\theta \quad \text{for all } t \ge \tau \text{ and } z_\tau \in B.$$

**Lemma 3.11.** *For each $\theta \in [\sigma, 1-\sigma]$, if the initial data set $B$ be any bounded subset of $H_\theta$, then the decomposed ingredient $(w(t), w_t(t))$ (the solutions of (17)) satisfies, for any $\tau \in R$,*

$$\|K_g(t,\tau)z_\tau\|_{H_{\theta+\sigma}} \le \Upsilon_\theta \quad \text{for all } t \ge \tau \text{ and } z_\tau \in B,$$

*where the constant $\Upsilon_\theta$ only depends on the $H_\theta$-bounds of $B$.*

**Theorem 3.12.** *There exist a bounded (in $H_1$) set $B_1 \subset H_1$, a positive constants $\nu$ and a monotonically increasing function $Q(\cdot)$ such that: For any bounded (in $H_0$) set $B \subset H_0$, any $g \in \Sigma$, $\tau \in R$ and $t \ge \tau$, the following estimate holds:*

$$dist_{H_0}(U_g(t,\tau)B, B_1) \le Q(\|B\|_{H_0})e^{-\nu(t-\tau)}. \tag{61}$$

*where $dist_{H_0}(\cdot,\cdot)$ denotes the usual Hausdorff semi-distance in $H_0$.*

**Proof** Let $B_0$ be the bounded uniformly (w.r.t $\sigma \in \Sigma$) absorbing set in $H_0$ (see *Theorem 3.2*).

By *Lemma 3.5* and *Lemma 3.8*, set $A_\sigma = \{z \in H_\sigma : \|z\|_{H_\sigma} \le \Upsilon_0\}$ then

$$dist_{H_0}(U_g(t,\tau)B_0, A_\sigma) \le dist_{H_0}(S(t,\tau)B_0, A_\sigma) \le Q_1(\|B_0\|_{H_0})e^{-k_0(t-\tau)},$$

where $\Upsilon_0$ is a constant from *Lemma 3.8* corresponding to $B_0$.

Using $A_\sigma$ to replace $B_0$ in *Lemma 3.11* and *Lemma 3.5*, then there is $A_{2\sigma} \subset H_0$ which is bounded in $H_{2\sigma}$ such that

$$dist_{H_0}(U_g(t,\tau)A_\sigma, A_{2\sigma}) \le dist_{H_0}(S(t,\tau)A_\sigma, A_{2\sigma}) \le Q_1(\|A_\sigma\|_{H_0})e^{-k_0'(t-\tau)},$$

for two appropriate constants $C$ and $k_0''$.

Since $\sigma = \min\{\frac{1}{4}, \frac{5-\gamma}{2}\}$ is fixed, by finite steps, we can infer that there is a bounded (not only in $H_0$, but also $H_1$*) set* $B_1 \subset H_1$ such that

$$dist_{H_0}(U_g(t,\tau)B_0, B_1) \le Q(\|B_0\|_{H_0})e^{-\nu(t-\tau)}. \tag{62}$$

Note further that all the constants in (62) only depend on $\|B_0\|_{H_0}$ and $\|g\|_{L_b^2}$.

Now, for any bounded (in $H_0$*)* $B$, from *Theorem 3.2* There is $T_0 \ge \tau$ such that

$$\bigcup_{\sigma\in\Sigma} U_g(t,\tau)B \subset B_0 \quad \text{for all} \quad t \ge T_0$$

Combined with *Lemma 3.4*, it follows that

$$dist_{H_0}(U_g(t,\tau)B, B_0) \le Qe^{\nu T_0}e^{-\nu(t-\tau)}, \tag{63}$$

where $Q = \sup\{\|U_g(t,\tau)B\|_{H_0} : g \in \Sigma, \tau \le t \le T_0\} < \infty.$

Finally, we apply *Lemma 2.1*, again to (62) and (63), and the proof of *Theorem* is completed.

## IV. Uniformly Attractors

Now collecting *Theorem 3.2*, *Lemma 3.5*, and *Theorem 3.12*, we establish that $\{U_g(t,\tau)\}, g \in \Sigma$ corresponding to (1) is asymptotically compactness. Therefore, by means of well-known results of the theory of dynamical systems we get that the family of processes $\{U_g(t,\tau)\}, g \in \Sigma$ corresponding to (1), posses a compact (in $H_0$)uniform (w.r.t $g \in \Sigma$) attractor $\mathcal{A}$, and $\mathcal{A} \subset H_0$. We remark that the above existence does not require any continuity of the family of processes. However, in order to obtain the explicit form of $\mathcal{A}$, we need some continuity. Moreover, since the symbol space $\Sigma$ now has only weak compactness, we need to verify the corresponding of weak continuity. First, by the results of Chepyzhov and Vishik[6], we see that $\Sigma$ with the local weak convergence topology of $L^2_{loc}(R; L^2(\Omega))$ forms a sequentially compact and metrizable complete space. We denote the equivalent metric by $d(,)$. Thus $(\Sigma, d)$ is a compact metric space. Moreover, through *Lemma 4.1*, Chapter V[6], we also have the following conclusion.

**Lemma 4.1.** *[13] The translation semigroup $\{T(t)\}_{t\ge0}$ acts on $\Sigma$ (i.e.), $T(t)g(x,s) = g(x,t+s)$ for any $g \in \Sigma$ and any $t \ge 0$ is invariant and continuous in $\Sigma$ with respect to the local weak convergence topology of $L^2_{loc}(R; L^2(\Omega))$, equivalently, with respect to the metric $d$.*

In the following, we also recall an useful lemma, whose proof is simple and we omit it.

**Lemma 4.2.** *[13] Let $X$ be a reflexive Banach and $x_n \xrightarrow{弱} 0$ in $X$. then for each compact (in $X^*$) subset* $B \subset X^*$, the uniform convergence hold: For any $\varepsilon > 0$ there is a $N_\varepsilon$, depending only on $\varepsilon$, such that

$$\left| < f, x_n >_{X^*} \right| \le \varepsilon \quad \text{for all} \quad n \ge N_\varepsilon \ \text{ and all } \ f \in B \quad (64)$$

**Theorem 4.3.** *The family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, corresponding to (1.1), has a compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\Im$ in $H_0$. Moreover, this attractor is bounded in $H_1$ and can be decomposed as follows*

$$\Im = \bigcup_{\sigma \in \Sigma} \kappa_\sigma(0) \quad (65)$$

*where $\kappa_\sigma$ is the kernel of the process $\bigcup_\sigma$, and $\kappa_\sigma(0)$ is the kernel section at time $0$.*

**Proof** We only need to verify continuity claim on the attractor $\Im$ in $H_0$, i.e., for the attractor $\Im$ *in* $H_0$, $\Im \subset H_0$ any fixed $\tau \in R$ and $t \ge \tau$, *if* $z_{n\tau} \to z_\tau$ in $\Im$ and $g_n \to g$ with respect to the local weak convergence topology of $L^2_{loc}(R; L^2(\Omega))$, then $U_{g_n}(t,\tau) z_{n\tau}$ converges to $U_g(t,\tau) z_\tau$ in $\Im$.

Denoted

$z_\tau = z_{1\tau} - z_{2\tau}$, $(u_i(t), u_{it}(t), \eta^t) = U_{g_i}(t,\tau) z_{i\tau} (i = 1,2)$ and $(w(t), w_t(t), \zeta^t) = U_{g1}(t,\tau) z_{1\tau} - U_{g2}(t,\tau) z_{2\tau}, z_{i\tau} \in \Im, i = 1,2$. Then $(w(t), w_t(t), \zeta^t)$ satisfies the following equation

$$w_{tt} - \Delta w - \Delta w_t - \Delta w_{tt} - \int_0^\infty \mu(s) \Delta \zeta^t(s) ds + f(u_1) - f(u_2) = g_1(x,t) - g_2(x,t), \quad (66)$$

and

$$\zeta^t(s) = w(t) - w(t-s), \quad (w(\tau), w_t(\tau), \zeta^\tau) = z_\tau, \quad w|_{\partial\Omega} = 0.$$

Since $\Im$ is bound in $H_1$, following *Theorem 3.12*, then there is a positive constant $R_0$ such that

$$\sup_{g \in \Sigma} \sup_{\tau \in R} \sup_{t \ge \tau} \left\| \bigcup_g (t,\tau) \Im \right\|_{H_1}^2 \le R_0 < \infty \quad (67)$$

Multiplying (66) by $w_t(t)$ and using (67), it follows that

$$\frac{d}{dt}(|w_t(t)|_2^2 + \|w(t)\|_0^2 + \|w_t(t)\|_0^2 + \|\zeta^t\|_{\mu,\varepsilon_0}^2) + 2\|w_t(t)\|_0^2 + \delta\|\zeta^t\|_{\mu,\varepsilon_0}^2$$
$$\le C(|w_t(t)|_2^2 + \|w(t)\|_0^2 + \|w_t(t)\|_0^2 + \|\zeta^t\|_{\mu,\varepsilon_0}^2) + 2 < g_1(t) - g_2(t), w_t(t) >, \quad (68)$$

and by integrating over $[\tau, t]$, then we get, for each $\tau \le t \le T$,

$$\begin{aligned} &|w_t(t)|_2^2 + \|w(t)\|_0^2 + \|w_t(t)\|_0^2 + \|\zeta^t\|_{\mu,\varepsilon_0}^2 \\ &\le e^{C(T-\tau)}(\|z_\tau\|_{H_0}^2 + \left| \int_\tau^T < g_1(s) - g_2(s), w_t(s) > ds \right|). \end{aligned} \quad (69)$$

By *Theorem 3.2*, then we have

$\bigcup_{g \in \Sigma} \{\prod_2 \bigcup_g (t,\tau) z_\tau : t \in [\tau, T], z_\tau \in \Im\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$

and

$\bigcup_{g \in \Sigma} \{\partial_t \prod_2 \bigcup_g (t,\tau) z_\tau : t \in [\tau, T], z_\tau \in \Im\}$ is bounded in $L^2(\tau, T; H_0^1(\Omega))$

then

$\bigcup_{g \in \Sigma} \{\partial_t \prod_2 \bigcup_g (t,\tau) z_\tau : t \in [\tau, T], z_\tau \in \Im\}$ is bounded in $L^2(\tau, T; H^{-1}(\Omega))$

where $\prod_2$ is the projector from $X \times Y$ to $Y$. Then by *Lemma 2.2*, we get

$\bigcup_{g \in \Sigma} \{\prod_2 \bigcup_g (t,\tau) z_\tau : t \in [\tau, T], z_\tau \in \Im\}$ is compact in $L^2(\tau, T; L^2(\Omega))$

By *Lemma 4.2*, it does show that if $g_n \to g$ in $L^2_{w,loc}(R; L^2(\Omega))$, then

$$\left| \int_\tau^t < g_1(s) - g_2(s), w_t(s) > ds \right| \to 0 \quad (70)$$

uniformly on a compact subset of $L^2(\tau, t; L^2(\Omega))$.

Based on the continuity claim above, and by constructing a skew-product flow on $\Im \times \Sigma$ and applying *Theorem 5.1*,IV[6], then the structure equality (65)is proved. So the proof is completed.

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