Advances in Intelligent Systems Research (AISR), volume 141
International Conference on Applied Mathematics, Modelling and Statistics Application (AMMSA 2017)

Domination Numbers of Trees

Min-Jen Jou* and Jenq-Jong Lin
Ling Tung University, Taichung 40852, Taiwan
*Corresponding author

***Abstract*—A set S of vertices is a dominating set of G if $N_G[S]=V(G)$. The domination number $\gamma(G)$ of a graph G is the minimum cardinality among all dominating sets of G. The decision problem of determining the domination number for arbitrary graphs is NP-complete. Here we focus on trees. If x and x' are duplicated leaves adjacent to the same support vertex in a tree T, then $\gamma(T - x')= \gamma(T)$. If T' can be obtained from T by adding some duplicated leaves, we can see that $\gamma(T')= \gamma(T)$. So the maximum order of a tree T, which is $\gamma(T)=k$, is infinity. In this paper, we focus on trees which are without duplicated leaves. For $k\geqq 1$, we determine the minimum and maximum orders of the trees T which are without duplicated leaves and $\gamma(T)=k$. Moreover, we characterize the trees of minimum and maximum orders.**

Keywords- domination number; tree; order; duplicated leaves

I. INTRODUCTION

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The cardinality of $V(G)$ is called the *order* of G, denoted by $|G|$. The (open) *neighborhood* $N_G(v)$ of a vertex v is the set of vertices adjacent to v in G, and the *close neighborhood* $N_G[v]$ is $N[v]=N(v)\bigcup\{v\}$. For any subset $A\subseteq V(G)$, denote $N(A)=\bigcup_{v\in A}N(v)$ and $N[A]=\bigcup_{x\in A}N[v]$ .The *degree* of v is the cardinality of $N_G(v)$, denoted by $\deg_G(v)$. Two distinct vertices u and v are called *duplicated* in G if $N_G(u)= N_G(v)$. A vertex x is said to be *leaf* if $\deg_G(v)=1$. A vertex of G is a *suppor vertex* if it is adjacent to a leaf in G. We denote by $L(G)$ and $U(G)$ the collections of all leaves and support vertices of G, respectively. For a subset $A\subseteq V(G)$, the *deletion* of A from G is the graph $G-A$ obtained by removing all vertices in A and all edges incident to these vertices. The *union* of two disjoint graphs G_1 and G_2 is the graph $G_1\cup G_2$ with vertex set $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and edge set $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$. A *forest* is a graph with no cycles, and a *tree* is a connected forest. If u and v are duplicated vertices in a tree, then they are both leaves. The *n-path* P_n is a path of order n. For other undefined notions, the reader is referred to [1] for graph theory.

A set S of vertices is a *dominating set* of G if $N_G[S]=V(G)$. The *domination number* $\gamma(G)$ of G is defined to be the minimum cardinality among all dominating sets of G. A dominating set of cardinality $\gamma(G)$ in G is said to be a γ-set. A γ-set containing all support vertices of G is called a γ_U-set. One of the fastest growing areas within graph theory is the study of domination and related subset problems. A dominating set have been proposed as a virtual backbone for routing in wireless ad hoc networks (see [6]). The topology of such wireless ad hoc network can be modeled as a unit-disk graph (UDG), a geometric graph in which there is an edge between two vertices if and only if their distance is at most one. A dominating set of a wireless ad hoc network is a dominating set of the corresponding UDG. The discussion of domination in graphs are initiated by Ore [5]. Several decades later, domination and its variations in graphs are well studied, an estimated thousand papers have been written on this topic (see [2],[3],[4]).

The decision problem of determining the domination number for arbitrary graphs is NP-complete. Here we focus on trees. If x and x' are duplicated leaves adjacent to the same support vertex in a tree T, then $\gamma(T- x')= \gamma(T)$. If T' can be obtained from T by adding some duplicated leaves, we can see that $\gamma(T')=\gamma(T)$. So the maximum order of a tree T, which is $\gamma(T)=k$, is infinity. In this paper, we focus on trees which are without duplicated leaves. For $k\geq 1$, we determine the minimum and maximum orders of the trees T which are without duplicated leaves and $\gamma(T)=k$. Moreover, we characterize the trees of minimum and maximum orders.

II. PRELIMINARY

We need the following lemmas.

***Lemma* 2.1.** If uv is an edge of a connected graph G and $G-uv=G_1\bigcup G_2$, then $\gamma(G)\leqslant \gamma(G_1)+ \gamma(G_2)$.

***Proof*.** Suppose uv is an edge of a connected graph G and $G-uv=G_1\bigcup G_2$. Let S_i be a γ-set of G_i for i=1 and 2. Suppose $S=S_1\bigcup S_2$, then $N[S]=N[S_1\bigcup S_2]=N[S_1]\bigcup N[S_2]=V(G)$. So S is a dominating of G, thus $\gamma(G)\leq|S|=|S_1|+|S_2|=\gamma(G_1)+\gamma(G_2)$.

***Lemma* 2.2.** If G is a graph with at least three vertices, then there exists a γ_U-set of G.

***Proof*.** Suppose G is a graph with at least three vertices and let S be a γ-set S of G. If S is a γ_U-set of G, then we are done. So we assume that $A=U(T)-S\neq\phi$, and let $B=L(T)\bigcap N(A)$. Then $B\subseteq S$ and $|B|\geq|A|$. Let $S'=(S-B)\bigcup A$. Then $N[S']=V(G)$, so S' is a dominating set of G. Thus $|S|=\gamma(G)\leq|S'|=|S|-|B|+|A|\leq|S|$, the equalities hold and S' is a γ_U-set of G.

***Lemma* 2.3.** If x and x' are two duplicated leaves adjacent to the same support vertex in a graph *G*, then $\gamma(G-x')=\gamma(G)$.

Proof. Suppose *x* and *x*' are two duplicated leaves adjacent to the same support vertex in a graph *G*, and let *G'* =*G*- *x*. If *S* is a γ_U -set of *G*, then *S* is a γ_U -set of *G'*. So $\gamma(G-x')=\gamma(G')=|S|=\gamma(G)$.

III. MAIN THEOREM

By Lemma 2.3, we can see that the maximum order of a tree *T*, which is $\gamma(T)=k$, is infinity. Thus we focus on trees which are without duplicated leaves. First, we determine the minimum order of the trees *T* which are without duplicated leaves and $\gamma(T)=k$. Moreover, we characterize the trees of the minimum order.

Theorem 3.1. If *T* is a tree with at least two vertices and $\gamma(T)=k$, where $k\geq 1$, then $|T|\geq 2k$.

Proof. Suppose *T* is a tree with at least two vertices and $\gamma(T)=k$, where $k\geq 1$. Let *S* be a γ-set of *T*. Then $N[S]=V(G)=N[V(T)-S]$, so $S^c=V(T)-S$ is a dominating set of *T*. Hence $|S^c|\geq k$ and $|T|=|S|+|S^c|\geq k+k=2k$.

Lemma 3.2. Let *T* be a tree with at least two vertices and $\gamma(T)=k$, where $k\geq 1$. If $|T|=2k$, then *T* has no duplicated leaf.

Proof. Let *T* be a tree of order 2*k* and $\gamma(T)=k$, where $k\geq 1$. Suppose that there exist two distinct leaves *x* and *x*' adjacent to *y* in *T*, by Lemma 2.3, then $\gamma(T-x')=\gamma(T)=k$. Note that $T'=T-x'$ is a tree. By Theorem 3.1, $|T'|\geq 2k$. Thus $|T|=|T'|+1\geq 2k+1$. This is a contradiction, we complete the proof.

Theorem 3.3. Let *T* be a tree with at least two vertices and $\gamma(T)=k$, where $k\geq 1$. If |*T*|= 2*k*, then $V(T)=U(T)\bigcup L(T)$ and $|U(T)|=k$.

Proof. We prove this theorem by induction on $k\geq 1$. If $k=1$, then $T=P_2$. If $k=2$, then $T=P_4$. It's true for $k=1$ and 2. Let $k\geq 3$. Assume that it's true for all $k'<k$. Suppose that *T* is a tree of order 2*k* and $\gamma(T)=k$. Let $P_i: x_i, y_i, z, w, u, ...$ be a longest path of *T*, where $|P_i|=m\geq 5$ and $i=1,...,a$. By Lemma 3.2, then $|N(y_i)\bigcap L(T)|=1$ for every *i*. Let $A=\{y_1,...,y_a\}$. If *m* = 5, then $a=k-2$. Thus $z\in U(T)$ and $U(T)=A\bigcup\{z,w\}$. So it's true for *m* = 5. Thus we assume that $m\geq 6$.

Claim 1. $z\in U(T)$. Suppose that $z\notin U(T)$, then $N(z)=A\bigcup\{w\}$ and $H=T-N[A]$ is tree of order $|H|=|T|-(2a+1)=2(k-a)-1\geq 3$. By Theorem 3.1, $\gamma(H)\leq k-a-1$. Note that $z\in N(A)$. By Lemma 2.1, $k=\gamma(T)\leq|A|+\gamma(H)\leq a+(k-a-1)=k-1$. This is a contradiction, so $z\in U(T)$.

Let *z*' be the leaf of *z* in *T* and $T'=T-(N[A]\bigcup\{z,z'\})$ and $T^*=T-V(T')$. Then *T*' is a tree of order $|T'|=|T|-(2a+2)=2(k-a-1)\geq 3$. By Theorem 3.1, $\gamma(T')\leq k-a-1$. Note that *zw* is an edge of *T* such that $T-zw=T^*\bigcup T'$, by Lemma 2.1, $k=\gamma(T)\leq\gamma(T^*)+\gamma(T')\leq (a+1)+(k-a-1)=k$. The equalities hold, $\gamma(T')=k-a-1$. Hence *T*' is a tree of order $|T'|=2(k-a-1)$ and $\gamma(T')=k-a-1$, by induction hypothesis,

$V(T')=U(T')\bigcup L(T')$.

Claim 2. $w\in U(T')$. Suppose that $w\notin U(T')$, then $w\in L(T')$ and $T''=T'-\{w\}$ is a tree of order $|T''|=|T'|-1=2(k-a-1)-1\geq 2$. Hence, by Theorem 3.1, we have that $\gamma(T'')\leq k-a-2$. Note that $w\in N(z)$ and $T-wu=(T-V(T''))\bigcup T''$. By Lemma 2.1, $k=\gamma(T)\leq\gamma(T-V(T''))+\gamma(T'')\leq(a+1)+(k-a-2)=k-1$. This is a contradiction, so $w\in U(T')$.

By Claim 2, we can see that $U(T)=A\bigcup\{z\}\bigcup U(T')$. That is $V(T)=U(T)\bigcup L(T)$ and $|U(T)|=k$. Hence it's true for *k*, we complete the proof.

Now we determine the maximum order of the trees *T* which are without duplicated leaves and $\gamma(T)=k$. Moreover, we characterize the trees of the maximum orders. Let $\Omega(k)$ be the collection of trees *T* which hold the following properties.

(i) *T* has no duplicated leaf.

(ii) $\gamma(T)=|U(T)|=k$.

(iii)For each $v\in U(T)$, $\delta(v)=\min\{d(u,v):u\in U(T)\}=3$, where $d(u,v)$ is the distance between *u* and *v*.

Lemma 3.4. Suppose $T\in\Omega(k)$, then *T* is a tree without duplicated leaves of order $|T|=4k-2$ and $U(T)$ is a γ_U -set of *T*, where $\gamma(T)=|U(T)|=k$.

Theorem 3.5. Suppose *T* is a tree without duplicated leaves and $\gamma(T)=k$, where $k\geq 1$. Then $|T|\leq 4k-2$. The equality holds if and only if $T\in\Omega(k)$.

Proof. It's true for *k* =1, so we assume that $k\geq 2$. Suppose that *T* is a tree without duplicated leaves and $\gamma(T)=k$ such that $|T|$ is as large as possible. By Lemma 3.4, then we obtain that $|T|\geq 4k-2$. Let *S* be a γ_U -set of *T*. Since $|T|$ is as large as possible, we obtain that $S=U(T)$ and $N[u]\bigcap N[v]=\phi$ for $u\neq v$ in *S*. Thus $|V(T)-(U(T)\bigcup L(T)|\leq 2(|S|-1)=2k-2$. Hence $4k-2\leq|T|=|U(T)\bigcup L(T)|+|V(T)-(U(T)\bigcup L(T))|\leq 2k+(2k-2)=4k-2$. The equalities hold, $\gamma(T)=|U(T)|=$

$|L(T)| = k$ and $|V(T) - (U(T) \cup L(T)| = 2(|S| - 1) = 2k - 2.$ That is $T \in \Omega(k)$.

REFERENCES

[1] J.A. Bondy, USR Murty and Graph Theory with Application, New York, 1976.

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker Inc., New York, 1998.

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs:Advanced Topics, Marcel Dekker Inc., New York, 1998.

[4] M.J. Jou, "Dominating sets and independent sets in a tree," to appear in Ars Combinatoria.

[5] O. Ore, Theory of graphs, in: Amer. Math. Soc. Colleq., Vol. 38, Providence, RI, 1962.

[6] Peng-Jun Wan, Khaled M. Alzoubi and Ophir Frieder, "Distributed Construction of Connected Dominating Set in Wireless Ad Hoc Networks," Mobile Networks and Applications, Vol. 9, Issue 2, pp 141-149, April 2004.