

The Overlapping Domain Decomposition Method for Exterior Harmonic Problem with Mixed Boundary Conditions

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Abstract—In this paper, a Schwarz alternating algorithm is proposed by introducing two artificial boundaries for exterior mixed boundary value problems of Laplace equation. The theorem of the geometrical convergence is obtained for the algorithm. Two numerical examples confirm the theoretical results and demonstrate the advantage in accuracy and efficiency.

Keywords—laplace equation; unbounded domain; mixed boundary conditions; domain decomposition

I. INTRODUCTION

Many physical phenomena develop elliptic problems in exterior regions, for examples, the flow of an incompressible irrotational fluid about a body and the study of electrostatics exterior to given surfaces are described by the Laplace equation (see, [1] and references therein). However, the unboundedness of the domains brings about the essential difficulty for solving these problems, and therefore, various methods known as coupling of BEM and FEM, the FEM with artificial boundary conditions and the domain decomposition method (DDM) had been constructed for the numerical solution of this kind of problems [2-4]. In particular, the DDM not only overcomes shortcoming of the stiffness matrix being not sparse banded in the coupling method, but allows the establishment of different mathematical models on different sub-regions and can achieve a high degree of parallel computing.

In this paper, we propose an overlapping DDM for solving the exterior Laplace equation with mixed boundary conditions, which is more general than the usually considered Dirichlet boundary condition. In Section 2, we prove the geometric convergence for the Schwarz alternating method. The accuracy and efficiency of this method is demonstrated in Section 3. Final remarks are given in Section 4.

II. SCHWARZ ALTERNATING ALGORITHM AND ITS CONVERGENCE

Consider to solve the following exterior mixed boundary value problem of the Laplace equation:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_D, \\ n_x \frac{\partial u}{\partial x} + n_y \frac{\partial u}{\partial y} = g, & \text{on } \Gamma_N, \end{cases} \quad (1)$$

where $\Gamma_0 = \Gamma_D \cup \Gamma_N$ is a polygonal boundary surrounding the origin in R^2 , Ω is the unbounded domain outside Γ_0 , $u_0 \in H^{1/2}(\Gamma_D)$, $g \in H^{-1/2}(\Gamma_N)$, and $n = (n_x, n_y)$ is the outward unit normal vector on Γ_N . As long as u is bounded at infinity, there is a unique solution u of the problem (1),

$$u \in W_0^1(\Omega), \text{ where } W_0^1(\Omega) := \left\{ v \left| \frac{v}{\sqrt{x^2 + y^2 + 1 \ln(x^2 + y^2 + 2)}} \right., \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(\Omega) \right\}.$$

Set bilinear form

$$D(u, v) = \int_{\Omega} \nabla u \nabla v dx dy.$$

The corresponding variational form of (1) is

$$\begin{cases} \text{Find } u \in V + u_0, \text{ such that} \\ D(u, v) = F(v), \quad \forall v \in V, \end{cases} \quad (2)$$

where

$$F(v) = \int_{\Gamma_N} g v ds, \quad V = \{v \mid v \in W_0^1(\Omega), v|_{\Gamma_D} = 0\}.$$

Make a circle Γ_2 , where $\text{dist}(\Gamma_0, \Gamma_2) > 0$, such that Γ_0 is surrounded by Γ_2 . Then make a closed circle Γ_1 which surrounds Γ_2 . Let Ω_1 denotes the annular domain between Γ_0 and Γ_1 . Define Ω_2 as the exterior unbounded domain with boundary Γ_2 . Let

$$\Omega_{11} = \Omega_1 \cap (\bar{\Omega}_2)^c, \Omega_{12} = \Omega_1 \cap \Omega_2, \Omega_{22} = \Omega_2 \cap (\bar{\Omega}_1)^c,$$

where $(\bar{\Omega}_i)^c$ represents the complement of $\bar{\Omega}_i$ ($i = 1, 2$). Set $u_i^{(k)} = u^{(k)}|_{\Omega_i}$ ($i = 1, 2$). Then we construct the following Schwarz alternating algorithm:

Step 1. Put any initial data $u_2^{(0)} \in H^{\frac{1}{2}}(\Gamma_1)$, for an example, $u_2^{(0)} = 0$, and $k:=1$.

Step 2. Solve a Mixed boundary value problem in bounded domain Ω_1 :

$$\begin{cases} \frac{\partial^2 u_1^{(k)}}{\partial x^2} + \frac{\partial^2 u_1^{(k)}}{\partial y^2} = 0, & \text{in } \Omega_1, \\ u_1^{(k)} = u_0, & \text{on } \Gamma_D, \\ n_x \frac{\partial u_1^{(k)}}{\partial x} + n_y \frac{\partial u_1^{(k)}}{\partial y} = g, & \text{on } \Gamma_N, \\ u_1^{(k)} = u_2^{(k-1)}, & \text{on } \Gamma_1. \end{cases} \quad (3)$$

Step 3. Solve a Dirichlet boundary value problem in unbounded domain Ω_2 :

$$\begin{cases} \frac{\partial^2 u_2^{(k)}}{\partial x^2} + \frac{\partial^2 u_2^{(k)}}{\partial y^2} = 0, & \text{in } \Omega_2, \\ u_2^{(k)} = u_1^{(k)}, & \text{on } \Gamma_2. \end{cases} \quad (4)$$

Step 4.

$$e_h(k) = \max \left\{ \sup_{\Gamma_2} |u_1^{(k)} - u_1^{(k-1)}|, \sup_{\Gamma_1} |u_2^{(k)} - u_2^{(k-1)}| \right\}$$

Step 5. If $e_h(k)$ is small enough, stop; else turn to Step 2.

Extend $u_1^{(k)}$ and $u_2^{(k)}$ to Ω by $u_1^{(k)}|_{\Omega_{22}} = u_2^{(k-1)}$ and $u_2^{(k)}|_{\Omega_{11}} = u_1^{(k)}$. Let spaces

$$V_1 = \{v \mid v \in H^1(\Omega_1), v|_{\Gamma_D} = 0, v|_{\Gamma_1} = 0\}, \\ V_2 = \{v \mid v \in W_0^1(\Omega_2), v|_{\Gamma_2} = 0\}.$$

Then (3) is equivalent to the following variational problem:

$$\begin{cases} \text{Find } u_1^{(k)} \in V_1 + u_2^{(k-1)}, & \text{such that} \\ D_1(u_1^{(k)}, v) = F(v), & \forall v \in V_1, \end{cases} \quad (5)$$

(4) is equivalent to the following variational problem:

$$\begin{cases} \text{Find } u_2^{(k)} \in V_2 + u_1^{(k)}, & \text{such that} \\ D_2(u_2^{(k)}, v) = 0, & \forall v \in V_2, \end{cases} \quad (6)$$

where

$$D_i(u, v) = \int_{\Omega_i} \nabla u \nabla v dx dy, \quad i = 1, 2,$$

$$F(v) = \int_{\Gamma_N} g v ds.$$

Then (3) and (4) are equivalent to the following variational problems:

$$\begin{cases} D_1(u_1^{(k)} - u, v_1) = 0, \quad \forall v_1 \in V_1, \\ u_1^{(k)} - u_2^{(k-1)} \in V_1, \end{cases} \quad (7)$$

and

$$\begin{cases} D_2(u_2^{(k)} - u, v_2) = 0, \quad \forall v_2 \in V_2, \\ u_2^{(k)} - u_1^{(k)} \in V_2. \end{cases} \quad (8)$$

Let $P_{V_i} : V \rightarrow V_i$ ($i = 1, 2$) denote the projection operators under the inner product $D_i(\cdot, \cdot)$ in V . Thus (7) and (8) are equivalent to

$$D_1(u_1^{(k)} - u_2^{(k-1)}, v_1) = D_1(u - u_2^{(k-1)}, v_1), \quad \forall v_1 \in V_1,$$

$$D_2(u_2^{(k)} - u_1^{(k)}, v_2) = D_2(u - u_1^{(k)}, v_2), \quad \forall v_2 \in V_2.$$

We obtain

$$\begin{cases} u_1^{(k)} - u_2^{(k-1)} = P_{V_1}(u - u_2^{(k-1)}), \\ u_2^{(k)} - u_1^{(k)} = P_{V_2}(u - u_1^{(k)}), \end{cases} \quad k = 1, 2, \dots \quad (9)$$

As $u - u_2^{(k-1)} = u - u_1^{(k)} + u_1^{(k)} - u_2^{(k-1)}$, $u_1^{(k)} - u_2^{(k-1)} \in V_1$, $u - u_1^{(k)} \in V_1^\perp$. Similarly, we have $u - u_2^{(k)} \in V_2^\perp$. Then (9) can be written as

$$\begin{cases} u - u_1^{(k)} = P_{V_1^\perp}(u - u_2^{(k-1)}), \\ u - u_2^{(k)} = P_{V_2^\perp}(u - u_1^{(k)}), \end{cases} \quad k = 1, 2, \dots \quad (10)$$

Let $e_i^{(k)} = u - u_i^{(k)}$, $i = 1, 2$, it follows

$$\begin{cases} e_1^{(k)} = P_{V_1^\perp} e_2^{(k-1)}, \\ e_2^{(k)} = P_{V_2^\perp} e_1^{(k)}, \end{cases} \quad k = 1, 2, \dots \quad (11)$$

This yields to

$$\begin{cases} e_1^{(k+1)} = P_{V_1^\perp} P_{V_2^\perp} e_1^{(k)}, & k = 1, 2, \dots, \\ e_2^{(k+1)} = P_{V_2^\perp} P_{V_1^\perp} e_2^{(k)}, & k = 0, 1, \dots. \end{cases} \quad (12)$$

This implies that, if $\{e_i^{(k)}\}$, $i = 1, 2$, are convergent, then their limits are in $V_1^\perp \cap V_2^\perp$.

Similar to the proofs of Theorem 1 in [5], the following results are obtained.

Lemma 1. Spaces V , V_1 and V_2 satisfy: $V = V_1 + V_2$, $V_1^\perp \cap V_2^\perp = 0$.

Lemma 2. If $V = V_1 + V_2$, then for any $v \in V$ there exists a positive constant C_0 such that

$$(\|v_1\|_1^2 + \|v_2\|_1^2)^{\frac{1}{2}} \leq C_0 \|v\|, \quad v = v_1 + v_2, \quad v_1 \in V_1, \quad v_2 \in V_2,$$

$$\|v\|_1 \leq C_0 (\|P_{V_1} v\|_1^2 + \|P_{V_2} v\|_1^2)^{\frac{1}{2}}.$$

where $\|\cdot\|_{1, \Omega_i}$ denotes energy norm $\sqrt{D_i(\cdot, \cdot)}$, $i = (1, 2)$.

Theorem 1. $\lim_{k \rightarrow \infty} \|e_i^{(k)}\|_1 = 0$ ($i = 1, 2$), and there exists a constant $\alpha \in [0, 1)$, such that

$$\|P_{V_1^\perp} P_{V_2^\perp}\| \leq \alpha, \quad \|P_{V_2^\perp} P_{V_1^\perp}\| \leq \alpha,$$

and

$$\|e_1^{(k+1)}\|_1 \leq \alpha^k \|e_1^{(1)}\|_1, \quad \|e_2^{(k)}\|_1 \leq \alpha^k \|e_2^{(0)}\|_1, \quad k = 1, 2, \dots.$$

Theorem 1 shows that the above Schwarz alternating algorithm is geometrically convergent and the contraction factor is α .

III. NUMERICAL EXAMPLES

Example 1. Consider to solve problem (1), where $\Gamma_0 = \{(x, y) \mid |x| = 1, |y| \leq 1\} \cup \{(x, y) \mid |x| \leq 1, |y| = 1\}$, $\Gamma_D = \{(x, y) \mid x = 1, |y| \leq 1\} \cup \{(x, y) \mid |x| \leq 1, |y| = 1\}$, $\Gamma_N = \{(x, y) \mid x = -1, |y| \leq 1\}$, and the boundary conditions are obtained by the exact solution $u = \frac{x}{x^2 + y^2}$.

The artificial boundaries Γ_1 and Γ_2 are circles with radiuses $R_1 = 4$ and $R_2 = 2$, respectively. The initial mesh Ω_{1h} consists of uniform triangular partitions with 8 points on Γ_0 , Γ_1 and Γ_2 , respectively.

The finite element solution is obtained for problem (3) by linear FEM on Ω_{1h} . The numerical integral solution is obtained for problem (4) by composite trapezoidal formula on Γ_2 . $e(k)$ denotes the maximal nodal error between the exact solution and iterative solution, $e_h(k)$ denotes the maximal nodal error of successive iterative solutions, and $q_h(k)$ represents the iterative convergence rate, i.e.,

$$e(k) = \max \left\{ \sup_{\Gamma_{2h}} |u - u_{1h}^{(k)}|, \sup_{\Gamma_{1h}} |u - u_{2h}^{(k)}| \right\},$$

$$e_h(k) = \max \left\{ \sup_{\Gamma_{2h}} |u_{1h}^{(k-1)} - u_{1h}^{(k)}|, \sup_{\Gamma_{1h}} |u_{2h}^{(k-1)} - u_{2h}^{(k)}| \right\},$$

$$q_h(k) = e_h(k-1) / e_h(k).$$

The numerical results are given in Table 1 and Figure 1.

TABLE I. THE RELATION BETWEEN CONVERGENCE RATE AND MESH

N	meshes	k	1	2	3	4	5	6	7
8	h	$e(k)$	1.7025E-1	9.6678E-2	8.3441E-2	8.1248E-2	8.0932E-2	8.0906E-2	8.0914E-2
		$e_h(k)$	3.9545E-1	7.6610E-2	1.3957E-2	2.6150E-3	5.2500E-4	1.1900E-4	3.1000E-5
		$q_h(k)$		5.161879	5.489068	5.336852	4.980439	4.430905	3.780227
16	h/2	$e(k)$	1.4098E-1	5.8287E-2	4.1711E-2	3.8303E-2	3.7539E-2	3.7341E-2	3.7280E-2
		$e_h(k)$	4.0531E-1	8.2695E-2	1.6576E-2	3.4080E-3	7.6400E-4	1.9800E-4	6.1000E-5
		$q_h(k)$		4.901341	4.988825	4.863690	4.461336	3.861971	3.241780
32	h/4	$e(k)$	1.2912E-1	3.9048E-2	1.9014E-2	1.4164E-2	1.2795E-2	1.2333E-2	1.2153E-2
		$e_h(k)$	4.0717E-1	9.0069E-2	2.0034E-2	4.8490E-3	1.3700E-3	4.6200E-4	1.8000E-4
		$q_h(k)$		4.520673	4.495819	4.131305	3.540690	2.965827	2.568918
64	h/8	$e(k)$	1.2772E-1	3.4223E-2	1.2187E-2	6.4080E-3	4.6170E-3	3.9530E-3	3.6750E-3
		$e_h(k)$	4.0747E-1	9.3498E-2	2.2036E-2	5.7890E-3	1.8040E-3	6.6800E-4	2.8000E-4
		$q_h(k)$		4.358076	4.243033	3.806701	3.208026	2.701405	2.388426

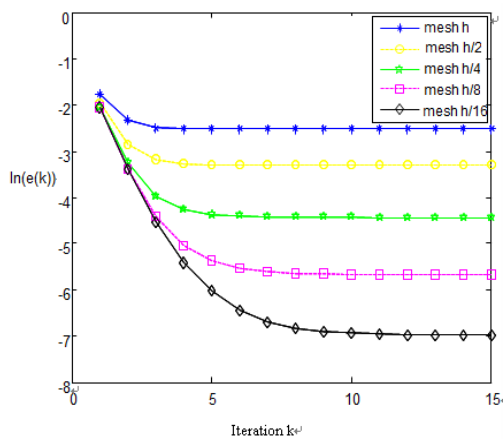


FIGURE I. THE RELATION BETWEEN RELATIVE ERRORS AND MESHES

The accurate finite element solution u_h for problem (3) on Ω_{1h} is obtained with exact boundary condition $u|_{\Gamma_1} = \frac{1}{R_1} \cos \theta$. e_0 denotes the maximal nodal error $e_0 = \sup_{\Gamma_{2h}} |u - u_h|$. Take e_0 as a reference to examine the error $e(k)$, then, from another point of view, show the accuracy of iterative solution and the validity of the algorithm, see Table 1 and 2.

TABLE II. THE COMPARISON OF $e_h(10)$, $e(10)$ AND e_0

	$e_h(10)$	$e(10)$	ratio	e_0	ratio
h	1.0000E-6	8.0925E-2		6.7550E-2	
h/2	4.0000E-6	3.7246E-2	2.1727	2.8174E-2	2.3976
h/4	1.5000E-5	1.2027E-2	3.0969	8.6720E-3	3.2488
h/8	2.7000E-5	3.4660E-3	3.4700	2.4140E-3	3.5924
h/16	3.7000E-5	9.6500E-4	3.5917	6.4000E-4	3.7719

The $q_h(k)$ in Table 1 shows that $e_h(k)$ decreases geometrically as the number of iterations k increases and gradually tends to zero. The finer the mesh is, the smaller the error will be. The comparison between $e(10)$ and e_0 in Table 2 shows that the iterative solution is almost as accurate as the accurate finite element solution is. Although the accurate finite element solution is quadratically convergent with respect to h , it is a impractical solution. Moreover, Figure 1 shows that maximal errors with different meshes dwindle quickly until approach to stable states.

Example 2. To show the relation between convergence rate and overlapping degree, we compute on mesh $h/4$ with different R_1 and R_2 in Example 1.

TABLE III. THE RELATION BETWEEN CONVERGENCE RATE AND OVERLAPPING DEGREE

R_2	R_1	$q_h(7)$	$q_h(8)$	$q_h(9)$	$e_h(9)$	$e(9)$
2	4	2.5689	2.3526	2.2494	3.3970E-5	1.2043E-2
2	8	3.4042	3.3464	3.3373	1.2586E-7	3.5529E-2
4	8	1.9467	1.7193	1.6004	8.2031E-5	1.4731E-2
2	12	3.9891	3.9906	3.9908	6.9561E-8	6.6899E-2

Table 3 indicates that the larger the overlapping of subdomains is, the faster the convergence rate is.

IV. CONCLUSIONS

In this work, we propose a Schwarz alternating algorithm for solving the Laplace equation with the mixed boundary conditions in the unbounded domain, and prove the corresponding geometric convergence. By means of this algorithm, a large scale multidimensional problem can be turned into two smaller discrete subproblems, which can reduce computing scale greatly. Moreover, its parallel computation in different subspaces can improve the efficiency. This algorithm gives a necessary complement to other methods in accordance with the feature of exterior problems. Two examples confirm the

theoretical results and demonstrate the advantage in accuracy and efficiency.

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