Heterogeneous Complex Function

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***Abstract*—In this paper, we discuss heterogeneous complex analysis function and heterogeneous harmonic function, and establish the relationship between heterogeneous complex analysis function and heterogeneous harmonic function, and then obtain heterogeneous Liouville theorem.**

Keywords-heterogeneous complex analytic function; heterogeneous C.R. equation; heterogeneous cauchy integral theorem; heterogeneous cauchy integral formula; liouville theorem

I. INTRODUCTION

Heterogeneous complex number field and heterogeneous complex function, introduced in [1], are means that generalized complex number field and generalized complex function, ,and then authors establish the relationship between heterogeneous complex analytic function and partial differential equation. By using the property on heterogeneous complex function[2], authors introduce the definitions of integration of heterogeneous complex function, establish heterogeneous Cauchy integral theorem and Cauchy integral formula. Considering the importance of Application for partial differential equation and heterogeneous analysis in [3][4], we will discuss the boundness of heterogeneous complex analytic function and its relation with heterogeneous harmonic function.

II. HETEROGENEOUS COMPLEX FUNCTION

A. Definition of Heterogeneous Complex Numbers Field

We first introduce some of basic notations and terminologies, the details can be found in [1].

Definition 2.1 The heterogeneous complex numbers field C_k consists, by definition $C_k=\{z|z=a+jb\}$ where $a,b\in R, j^2=-k, k>0$, and the vector operations are defined by

$$z=a+jb, m\in R, \text{τηεν}, ma=ma+jmb,$$

$z_1=a_1+jb_1$ and $z_2=a_2+jb_2$,then

$$z_1+z_2=a_1+a_2+j(b_1+b_2)$$

It is obvious that C_k is a vector space.

If we give a product operation by

$z_1=a_1+jb_1$ and $z_2=a_2+jb_2$, then $z_1z_2=a_1a_2-kb_1b_2+j(a_1b_2+a_2b_1)$,

And the length of $z=a+jb$ be definited by $|z|_k=\sqrt{a^2+kb^2}$, and then inverse element of $z=a+jb$ be $z^{-1}=\dfrac{a-jb}{|z|_k^2}$, hence C_k is a field. It is called heterogeneous complex numbers field.

Remark 2.1 When $k=1$, heterogeneous complex numbers field is the same as complex number field, hence denote C_1 is usually complex number field, denote $|z|_1=\sqrt{a^2+b^2}$ is the length in usually complex number field .

B. Heterogeneous Complex Analytic Function

Definition 2.2. Let f be function from C_k to C_k , z_0 be a fix point in C_k, f be called differentiable function in z_0 ,if the following limit exists:

$$A=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z)-f(z_0)}{z-z_0}$$, and then A be called derivative of $f(z)$ in z_0 by being denoted by $f'(z_0)$.

Suppose f be a function in set D, f be called a differentiable function in D, if every $z\in D$, $f(z)$ is differentiable.

Definition 2.3 Let f be function from domain D to C_k, f be called an analytic function in domain D, if f be a differentiable function in D, f be called analysis in point z , if f be a differentiable function in some domain D, where $z\in D$.

Theorem 2.1 Suppose $f(z)=p(x,y)+jq(x,y)$ be a function from domain D and $z=x+jy$. If f be a differentiable function in D, then

(1) Partial derivative p_x, p_y, q_x, q_y is exist,

(2) Heterogeneous C.-R. equation is exist for $p(x,y)$, $q(x,y)$ at point (x,y)

Where heterogeneous C.-R. equation satisfy the following equation system:

$$\frac{\partial p}{\partial x}=\frac{\partial q}{\partial y},\quad \frac{1}{k}\frac{\partial p}{\partial y}=-\frac{\partial q}{\partial x}$$

Theorem 2.2 Suppose $f(z)=p(x,y)+jq(x,y)$ be a function from domain D and $z=x+jy$, then f be a differentiable function in D ,if and only if

(1) $p(x,y)$, $q(x,y)$ differentiable function for every (x,y),

(2) Heterogeneous C.-R.equation is exist for $p(x,y)$, $q(x,y)$ at point (x,y).

III. Integration of Heterogeneous Complex Function

We first introduce the definition of integration of heterogeneous complex function and its some properties, the details can be found in [2].

A. Definition of Integration of Heterogeneous Complex Function

Definition 3.1 Let C_j be a directing curve in heterogeneous complex number field C_k by

$z=z(t)\quad (\alpha\le t\le\beta$

where $a=z(\alpha)$ is starting point and $b=z(\beta)$ is end point. Let $f(z)$ is a function along C_j , some points: $a=z_0,z_1,...,z_{n-1},z_n=b$ will be identified along directing curve C_j from a to b . A point ξ_t be took from z_{t-1} to z_t in the curve C_j . $f(z)$ be called integrability along directing curve C_j ,if the following limit exists:

$$\lim_{\delta\to 0} S_n=\lim_{\delta\to 0}\sum_{l=1}^{n} f(\xi_l)\Delta z_l$$

Where $\Delta z_l=z_l-z_{l-1}$ and $\delta=\max\limits_{1\le l\le n}|\Delta z_l|_k$, then this limit be called integration of $f(z)$ along directing curve C_j from a to b, by being denoted by $\int_{C_j} f(z)\mathrm{d}z$.

Remark 3.1 Denote $\int_{C_1} f(z)\mathrm{d}z$ is usually the integration on complex number field.

Theorem **3.1** Suppose $f(z)=p(x,y)+jq(x,y)$ be a continuous function along directing curve C_j and $z=x+jy$, then f be integrability along directing curve C_j with

$$\int_{C_j} f(z)\mathrm{d}z=\int_{C_j} p\mathrm{d}x-kq\mathrm{d}y+j\int_{C_j} q\mathrm{d}x+p\mathrm{d}y$$

Theorem **3.2** Let C_j be a directing curve in heterogeneous complex number field C_k by

$$C_j:\ z=z(t)=x(t)+jy(t)\quad (\alpha\le t\le\beta)=\zeta(\tau),$$

where $a=z(\alpha)$ is starting point, $b=z(\beta)$ is end point, $z'(t)=x'(t)+jy'(t)$ is a continuous function on $[\alpha,\beta]$ and $z'(t)\neq 0$. Let $f[z(t)]=p[x(t),y(t)]+jq[x(t),y(t)]$ is a continuous function along C_j , then

$$\int_C f(z)dz=\int_\alpha^\beta f[z(t)]z'(t)dt.$$

Theorem 3.3 Let C_j be a directing curve in heterogeneous complex number field C_k , the length of C_j be L , $f(z)$ be a continuous function along directing curve C_j and $\left|f(z)\right|_k\le M$,then

$$\left|\int_{C_j} f(z)\mathrm{d}z\right|\le(1+k)LM$$

Proof:
$\left|\int_{C_j} f(z)\mathrm{d}z\right|_k\le\int_{C_j}\left|f(z)\right|_k|dz|_k\le(1+k)\int_{C_1}\left|f(z)\right|_1|dz|_1$, it is very obvious to us by using the Theorem 3.2 in [5]pp:100.

B. Heterogeneous Cauchy Integral Theorem and Cauchy Integral Formula

Theorem 3.4 Suppose $f(z)$ be an analytic function from simply connected domain D in C_k and C_j be a closed curve in D, then $\int_{C_j} f(z)dz = 0$.

Theorem 3.5 Suppose $f(z)$ be an analytic function from simply connected domain D in C_k then

$F(z) = \int_{z_0}^{z} f(\xi)\mathrm{d}\xi,$ be a analytic function on z, where z_0 is a fix point in D and $z \in D$.

Theorem 3.6 Suppose the boundary of domain D in C_k be closed curve C_j, $f(z)$ be an analytic function in domain D and continuous function on $\bar{D} = D + C_j$, then

$$f(z) = \frac{1}{2\pi i}\int_{C_j} \frac{f(\xi)}{\xi - z}\mathrm{d}\xi, \ (z \in D)$$

Theorem 3.7 Suppose the boundary of domain D in C_k be closed curve C_j, $f(z)$ be an analytic function in domain D and continuous function on $\bar{D} = D + C_j$, then there exists each order derivative for $f(z)$ and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_j} \frac{f(\xi)}{(\xi - z)^{n+1}}\mathrm{d}\xi, \ (z \in D)$$

IV. Heterogeneous Harmonic Function and Heterogeneous Liouville Theorem

A. Heterogeneous Complex Analysis Function and Heterogeneous Harmonic Function

In this subsection, we will establish the relationship between heterogeneous complex analysis function and heterogeneous harmonic function. $k\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0,$

Theorem 4.1 Suppose $f(z) = p(x, y) + jq(x, y)$ be a heterogeneous analysis function from domain D and $z = x + jy$. Then

$$k\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, \ k\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0.$$

Definition 4.1 Denote

$$H_k = \{p(x, y) \mid k\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0, z = x + jy \in D\},$$

If $p(x, y) \in H_k$, p is called heterogeneous harmonic function with coefficient k. Suppose $f(z) = p(x, y) + jq(x, y)$ be a heterogeneous analysis function in domain D, q is called heterogeneous conjugate harmonic function of p.

Remark 4.1 It is obviously that $f(z) = p(x, y) + jq(x, y)$ cannot be a heterogeneous analysis function for some $p, q \in H_k$

Theorem 4.2 Let $p(x, y) \in H_k$, then there exists q such that q is a heterogeneous conjugate harmonic function of p, that is, $f(z) = p(x, y) + jq(x, y)$ be a heterogeneous analysis function in domain D.

Proof: $\forall z = x + jy \in D$, $p(x, y) \in H_k$ then we have $k\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0,$ and then $\frac{\partial}{\partial x}(\frac{\partial p}{\partial x}) = \frac{\partial}{\partial x}(-\frac{1}{k}\frac{\partial p}{\partial y}),$ further, there exists a total differential $q(x, y)$ such that $dq(x, y) = \frac{\partial p}{\partial x}dy - \frac{1}{k}\frac{\partial p}{\partial y}dx$ due to mathematical analysis, hence satisfied the following heterogeneous C.-R. equation

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \frac{\partial q}{\partial x} = -\frac{1}{k}\frac{\partial p}{\partial y},$$

By appropriate treatment, we can find such a $q(x, y)$, which $\frac{\partial q}{\partial y}, \frac{\partial q}{\partial x}$ are two continuous functions and above heterogeneous C.-R. equation is right . Hence $f(z) = p(x, y) + jq(x, y)$ be a heterogeneous analysis function due to theorem 2.2.

B. Heterogeneous Cauchy Inequality and Liouville Theorem

According to theorem3.3, theorem3.7, we can obtain the following heterogeneous Cauchy inequality,

Theorem 4.3 Suppose $f(z) = p(x, y) + jq(x, y)$ be a heterogeneous analysis function from domain D and $z = x + jy$. $\forall a \in D$, denote and $\Gamma = \left\{\xi \middle| \left|\xi - a\right|_k = R\right\}$

Such that $\overline{\Gamma} = \{\xi \| \xi - a|_k \leq R\} \subset D$, Then

$$\left| f^{(n)}(a) \right|_k \leq \frac{n!(1+k)}{R^n} M(R)$$

Where $M(R) = \max_{z \in \Gamma} \left| f(z) \right|_k$

Proof: According to theorem3.7, we have,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} \mathrm{d}\xi \text{，（} a \in D \text{）}$$

$$\left| f^{(n)}(a) \right|_k = \left| \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - a)^n} d\xi \right|_k$$

Consider theorem 3.3, then

$$\frac{n!}{2\pi} \int_{C_k} \left| \frac{f(\xi)}{(\xi - a)^{n+1}} \right|_k |d\xi|_k \leq \frac{n!}{2\pi} \frac{M(R)}{R^{n+1}} (1+k) R.$$

Theorem 4.4(Liouville theorem) If $f(z)$ be a bound analytic function in heterogeneous complex number field C_k, then $f(z)$ is a hetergeneous complex constant.

Proof: $\forall z \in C_k$, suppose $\exists M > 0,$ such that $\left| f(z) \right|_k \leq M$, by using theorem 4.3, we have

$\left| f'(z) \right|_k \leq \dfrac{(1+k)}{R} M,$ let $R \to \infty$, then $\left| f'(z) \right|_k \to 0,$ hence $f'(z) = 0,$ we use the same method as complex function theory, we can obtain $f(z) = C.$

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