A D-N Alternating Algorithm for Exterior Laplace Problem with Mixed Boundary Value Conditions

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Abstract—This paper develops a Dirichlet-Neumann (D-N) alternating algorithm to solve the mixed boundary value problem of Laplace equation in an infinite domain, and analyses the convergence of the algorithm. By choosing a circle surrounding the original boundary as an artificial boundary to divide the unbounded domain into two sub-domains, we can make use of the natural boundary reduction (NBR) method in the infinite sub-domain to solve a Dirichlet boundary value problem while use the finite element method in the finite sub-domain to solve a mixed boundary value problem. We prove that the algorithm is convergent geometrically for any relaxation factor between 0 and 1. The numerical experiment results also display that the sequence of iterative solutions is geometrically convergent, the convergence rate is independent of the finite element mesh size h, and the maximum nodal error on $\overline{\Omega}_1$ is roughly of $O(h^2)$.

Keywords-exterior Laplace problem; mixed boundary value conditions; D-N alternating algorithm; natural boundary reduction; Convergence analysis

I. INTRODUCTION

Many applications in science and engineering fields, such as hydrodynamics, electromagnetism, acoustics, etc., can be reduced to exterior boundary value problems of partial differential equations. It's very effective to solve boundary value problems on bounded domains by the finite element method and the finite difference method, yet those methods are difficult to be applied to problems on unbounded domains directly. Therefore, a variety of numerical methods are proposed to solve the exterior problems [1,2,3]. Among them, the D-N alternating algorithm has been studied to solve problems on unbounded domains, for examples, Laplace equation, Poisson equation and anisotropic elliptic equation with Dirichlet boundary condition [4,5,6], Helmholtz equation with Neumann boundary condition [7,8], and the anisotropic problem with mixed boundary condition in an infinite domain with a concave angle [9].

In this paper, we consider the exterior Laplace problem with mixed boundary value conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & \text{in } \Omega, \\ u = u_0, & \text{on } \Gamma_D, \\ n_x \dfrac{\partial u}{\partial x} + n_y \dfrac{\partial u}{\partial y} = g, & \text{on } \Gamma_N, \\ \lim_{\sqrt{x^2+y^2}\to\infty} u \text{ is limited,} \end{cases} \quad (1)$$

where $\Gamma_D \bigcup \Gamma_N = \Gamma_0$, Γ_0 is the planar piecewise smooth closed curve, $u_0 \in H^{\frac{1}{2}}(\Gamma_D)$ and $g \in H^{-\frac{1}{2}}(\Gamma_N)$ are given functions. Ω is the unbounded domain outside Γ_0. $n = (n_x, n_y)$ is the outward unit normal vector on Γ_0. Let meas$\Gamma_D \neq 0$ and the function u is bounded at infinity, then the problem (1) has unique solution.

Section 2 designs the D-N alternating algorithm; Section 3 analyzes the convergence of the D-N alternating algorithm; Section 4 displays some numerical results to check the accuracy and effectiveness of this algorithm.

II. THE D-N ALTERNATING ALGORITHM BASED ON NBR

For solving the problem (1), we make the artificial boundary to be a circle $\Gamma_1 = \left\{ (x, y) \mid x^2 + y^2 = R^2 \right\}$ to enclose Γ_0 and dist$(\Gamma_0, \Gamma_1) > 0$, divide Ω into two parts: inner subdomain Ω_1 and outer subdomain Ω_2, and then propose the following D-N alternating algorithm:

Step 1. For any initial value $\lambda^0 \in H^{\frac{1}{2}}(\Gamma_1)$, put $k = 0$.

Step 2. Solve a Dirichlet boundary value problem in domain Ω_2:

$$\begin{cases} \dfrac{\partial^2 u_2^{(k)}}{\partial x^2} + \dfrac{\partial^2 u_2^{(k)}}{\partial y^2} = 0, & \text{in } \Omega_2, \\ u_2^{(k)} = \lambda^{(k)}, & \text{on } \Gamma_1. \end{cases} \quad (2)$$

Step 3. Solve a mixed boundary value problem in domain Ω_1 :

$$\begin{cases} \dfrac{\partial^2 u_1^{(k)}}{\partial x^2} + \dfrac{\partial^2 u_1^{(k)}}{\partial y^2} = 0, & \text{in } \Omega_1, \\ \dfrac{\partial u_1^{(k)}}{\partial n_1} = -\dfrac{\partial u_2^{(k)}}{\partial n_2}, & \text{on } \Gamma_1, \\ \dfrac{\partial u_1^{(k)}}{\partial n} = g, & \text{on } \Gamma_N, \\ u_1^{(k)} = u_0, & \text{on } \Gamma_D. \end{cases} \quad (3)$$

Step 4. Input a relaxation factor θ_k and set

$$\lambda^{(k+1)} = \theta_k u_1^{(k)}\big|_{\Gamma_1} + (1-\theta_k)\lambda^{(k)}, \qquad 0 \le \theta_k \le 1. \quad (4)$$

Step 5. Put $k = k+1$, go to Step 2.

Noting that, only the normal derivative on boundary Γ_1 of the solution of (2) is needed for solving (3). So, instead of solving problem (2), we use the natural integral equation in [2] to obtain $\dfrac{\partial u_2^{(k)}}{\partial n}$ directly by using λ^k on boundary Γ_1 :

$$\frac{\partial u_2^{(k)}}{\partial n} = -\kappa_2(\lambda^{(k)}), \quad (5)$$

where

$$\kappa_2(\lambda^{(k)}) = \frac{1}{4\pi R} \int_0^{2\pi} \frac{\lambda^{(k)}(\varphi')}{\sin^2 \dfrac{\varphi - \varphi'}{2}} d\varphi'.$$

III. THE DISCRETE VARIATIONAL FORM AND ITS CONVERGENCE

Discrete variational form of the problem (2) and (3) is: find $u_{1,h}^{(k)} \in V_h(\Omega_1)$, such that $\forall v_h \in V_{1,h}(\Omega_1)$

$$D_1(u_{1,h}^{(k)}, v_h) = \int_{\Gamma_N} g v_h \, ds - \int_{\Gamma_1} v_h \kappa_2(\lambda^{(k)}) \, ds, \quad (6)$$

where

$$D_1(u_{1,h}^{(k)}, v) = \int_{\Omega_1} \nabla u_{1,h}^{(k)} \cdot \nabla v \, dxdy,$$

$$\lambda_h^{(k)}\big|_{\Gamma_1} = \theta_{k-1} u_{1,h}^{(k-1)} + (1-\theta_{k-1})\lambda_h^{(k-1)} \ (k=1,2,\cdots),$$

$$V_h(\Omega_1) = \{v_h \mid v_h \in H^1(\Omega_1), v_h\big|_{\Gamma_D} = u_0\},$$

$$V_{1,h}(\Omega_1) = \{v_h \mid v_h \in H^1(\Omega_1), v_h\big|_{\Gamma_D} = 0, v_h\big|_{\Gamma_1} = 0\}.$$

From the problem (6), we can get algebraic equations as follows:

$$\begin{pmatrix} K_{11} & K_{1i} & 0 \\ K_{i1} & K_{ii} & K_{i0} \\ 0 & K_{0i} & K_{00} \end{pmatrix} \begin{pmatrix} U_{1,h}^{(k)} \\ U_{i,h}^{(k)} \\ U_{0,h}^{(k)} \end{pmatrix} = \begin{pmatrix} -K_h \Lambda^{(k)} \\ 0 \\ F_0 \end{pmatrix}, \quad (7)$$

and

$$\Lambda^{(k+1)} = \theta_k U_1^{(k)} + (1-\theta_k)\Lambda^{(k)}, \quad (k=0,1,\cdots), \quad (8)$$

where the third-order block matrix on the left is a stiffness matrix obtained by finite element method in Ω_1, K_h on the right is gotten from the natural boundary element method on Γ_1, $\Lambda^{(k)}$ is a vector which is constructed by the nodal values of $\lambda^{(k)}$ on Γ_1, F_0 is the total load vector on Γ_N. $U_{1,h}^{(k)}$, $U_{i,h}^{(k)}$ and $U_{0,h}^{(k)}$ are the functional vector to be solved on Γ_1, in Ω_1 and on Γ_N respectively.

Theorem 3.1 The discrete D-N alternating algorithm (7) and (8) is equivalent to the preprocessing Richardson iteration method:

$$S_{1,h}(\Lambda^{(k+1)} - \Lambda^{(k)}) = \theta_k(\overline{F}_0 - S_h \Lambda^{(k)}), \quad (9)$$

where

$$S_{1,h} = K_{11} - K_{1i}(K_{ii} - K_{i0}K_{00}^{-1}K_{0i})^{-1}K_{i1},$$
$$S_h = S_{1,h} + K_h,$$
$$\overline{F}_0 = K_{1i}(K_{ii} - K_{i0}K_{00}^{-1}K_{0i})^{-1}K_{i0}K_{00}^{-1}F_0.$$

Theorem 3.2 The condition number of the iterative matrix $S_{1,h}^{-1}S_h$ of the discrete D-N alternating algorithm is

independent of the finite element mesh size h.

Theorem 3.3 If $0 < \min \theta_k \leq \max \theta_k < 1$, then the discrete D-N alternating algorithm (7) and (8) is convergent, and the convergence rate is independent of the finite element mesh size h.

IV. NUMERICAL RESULTS

Example. Consider to solve the problem (1), where Ω is the infinite domain outside of square $[-1,1] \times [-1,1]$, $\Gamma_N = \{(x,y) \mid x = -1, -1 \leq y \leq 1\}$ and $\Gamma_D = \Gamma_0 \setminus \Gamma_N$. The exact solution is $u = \dfrac{x}{x^2 + y^2}$, and consequently g and u_0 are determined by u.

Choosing the artificial boundary as $\Gamma_1 = \{(r,\theta) \mid r = R$, $R > \sqrt{2}\}$, and making uniform partitions on Γ_0, Γ_1 and Ω, we have four meshes: h: 8×2; $h/2$: 16×4; $h/4$: 32×8; $h/8$: 64×16, and then use the discrete D-N alternating algorithm (7) and (8).

$e_h(k)$ denotes the maximum nodal error on $\overline{\Omega}_1$:

$$e_h(k) = \sup_{P_i \in \overline{\Omega}_1} |u(P_i) - u_{1,h}^{(k)}(P_i)|;$$

$d_h(k)$ denotes the maximum nodal error of the adjacent two-steps:

$$d_h(k) = \sup_{P_i \in \overline{\Omega}_1} |u_{1,h}^{(k)}(P_i) - u_{1,h}^{(k-1)}(P_i)|;$$

$q_h(k)$ denotes the approximation of the convergence rate:

$$q_h(k) = \frac{d_h(k-1)}{d_h(k)}.$$

The numerical results are as follows (Table 1, Table 2 and Table 3):

TABLE I. THE RATIOS OF ERRORS BETWEEN $u_{1,h}^{(k)}$ AND u W.R.T. h.

Ratio	k			
	2	3	4	5
$e_h(k)/e_{h/2}(k)$	1.72693	1.70784	1.70310	1.70189
$e_{h/2}(k)/e_{h/4}(k)$	2.84887	2.72295	2.69462	2.68784
$e_{h/4}(k)/e_{h/8}(k)$	3.62506	3.55190	3.41010	3.37851

TABLE II. THE RELATIONSHIP BETWEEN MESH SIZE AND CONVERGENCE RATE (R=4, Θ=0.4)

mesh	k	0	1	2	3	4	5	6	7
h	$e_h(k)$	0.33029	0.25084	0.25705	0.25873	0.25919	0.25931	0.25935	0.25936
	$d_h(k)$		0.18062	0.05057	0.01417	0.00397	0.00112	0.00034	0.00010
	$q_h(k)$			3.57187	3.56970	3.56604	3.56097	3.29301	3.24789
$h/2$	$e_h(k)$	0.23615	0.13866	0.14885	0.15150	0.15219	0.15237	0.15242	0.15243
	$d_h(k)$		0.16664	0.04237	0.01096	0.00289	0.00076	0.00020	0.00005
	$q_h(k)$			3.93341	3.86506	3.78821	3.78758	3.78615	3.78412
$h/4$	$e_h(k)$	0.22805	0.05928	0.05225	0.05564	0.05648	0.05669	0.05674	0.05675
	$d_h(k)$		0.17702	0.04258	0.01032	0.00253	0.00064	0.00016	0.00004
	$q_h(k)$			4.15739	4.12677	4.07897	3.95567	3.95830	3.96027
$h/8$	$e_h(k)$	0.24045	0.05468	0.01441	0.01566	0.01656	0.01678	0.01683	0.01684
	$d_h(k)$		0.18578	0.04315	0.01011	0.00238	0.00058	0.00014	0.00004
	$q_h(k)$			4.30508	4.27046	4.23996	4.09522	4.06111	4.06448

TABLE III. THE RELATIONSHIP BETWEEN RELAXATION FACTOR Θ AND CONVERGENCE RATE (R=4, N=16, P=4)

θ / k		0	1	2	3	4	5	6	7
0.45	$e_h(k)$	0.23615	0.14358	0.15094	0.15218	0.15239	0.15243	0.15243	0.15243
	$d_h(k)$		0.18747	0.03044	0.00523	0.00090	0.00016	0.00003	0.00001
	$q_h(k)$			6.15826	5.81721	5.81448	5.80546	5.79278	5.77763
0.5	$e_h(k)$	0.23615	0.14850	0.15212	0.15241	0.15243	0.15243	0.15243	0.15243
	$d_h(k)$		0.20830	0.01575	0.00125	0.00010	0.00001	0.00000	0.00000
	$q_h(k)$			13.22463	12.57553	12.35673	11.82296	10.41002	8.95196
0.55	$e_h(k)$	0.23615	0.15342	0.15239	0.15244	0.15243	0.15243	0.15243	0.15243
	$d_h(k)$		0.22913	0.00583	0.00029	0.00002	0.00000	0.00000	0.00000
	$q_h(k)$			39.32322	20.27816	14.78793	12.86079	13.97590	10.39517
0.6	$e_h(k)$	0.23615	0.15834	0.15175	0.15252	0.15242	0.15243	0.15243	0.15243
	$d_h(k)$		0.24996	0.02966	0.00370	0.00049	0.00007	0.00001	0.00000
	$q_h(k)$			8.42807	8.01647	7.56591	7.11229	6.71366	6.38820
0.65	$e_h(k)$	0.23615	0.16326	0.15019	0.15290	0.15233	0.15246	0.15243	0.15243
	$d_h(k)$		0.27079	0.05737	0.01239	0.00273	0.00062	0.00014	0.00003
	$q_h(k)$			4.71983	4.63229	4.53461	4.42881	4.31954	4.21179

From the numerical results, we can see that:

(1) Table 1 shows that the sequence of iterative solutions is geometrically convergent according to the decrease rate $q_h(k)$ of $d_h(k)$; moreover, when using different mesh size, the convergence speed with the same relaxation factor is approximately same, which is consistent with Theorem 3.2.

(2) Table 2 shows the finer the mesh size is, the smaller the error between the iterative convergence solution and the accurate solution will be, and the maximum nodal error on $\overline{\Omega}_1$ is roughly of $O(h^2)$;

(3) Table 3 shows that when θ closes to 0.55, the convergence rate of the iterative algorithm will be faster.

REFERENCES

[1] A. Bayliss, M. Gunzburger, E. Turkel, "Boundary conditions for the numerical solution of elliptic equations in exterior regions", SIAMJ. Appl. Math., vol. 42, no. 2, pp. 430-451, 1982.

[2] H. D. Han, X. N. Wu, Artificial Boundary Method, Tsinghua Univ. Pr. & Springer, vol. 56, no. 1, 2012, pp. 259-298

[3] D. H. Yu, Natural Boundary Integral Method and Its Application, Kluwer Academic Publishers, Beijing/New York, 2002.

[4] D. H. Yu, "Discretization of non-overlapping domain decomposition method for unbounded domains and its convergence", Mathematica Numerica Sinica, vol. 18, no. 4, pp. 2267-2272, 1996.

[5] D. H. Yu, J. M. Wu, "A Non-overlapping Domain Decomposition Method For Exterior 3-D Problem", Comput. Math., vol. 19, no. 1, pp. 77-86, 2001.

[6] X. Q. Luo, Q. K. Du, L. B. Liu, "A D–N alternating algorithm for exterior 3-D Poisson problem with prolate spheroid boundary", Appl. Math. & Comput., vol. 269(C), pp. 252-264, 2015.

[7] D. H. Yu, Z. P. Jia, "Non-overlapping domain decomposition algorithm based on natural boundary reduction for two dimensional Helmholtz equation", Mathematica Numerica Sinica, vol. 22, no. 2, pp. 228-240, 2000.

[8] Y. J. Chen, Q. K. Du, "Infinite concave angle area of anisotropic problem of non-overlapping domain decomposition algorithm", Mathematical applicate sinica, vol. 37, no. 5, pp. 912-925, 2014.

[9] Y. J. Chen, Q. K. Du, "A non-overlapping domain decomposition method for an anisotropic problem in an infinite domain with a concave angle", Acta. Math. Appl. Sinica, vol. 37, no. 5, pp. 912-924, 2013.