A 4-node plane element based quadrilateral area coordinate method QACM-II

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Abstract. Nonlinear transformation between the isoparametric and the Cartesian coordinates may make the elements more sensitive to mesh distortion. By using the second kind of quadrilateral coordinate method QACM-II, which can keep the transformation always linear between the area and the Cartesian coordinate systems, a 4-node plane element ACQλ2 was developed based on optimized generalized conforming conditions. Additional displacement field based on internal parameters was also used to make the whole displacement field quadratic complementary.

Introduction

As a local coordinate system, isoprametric coordinate method was widely used for developing finite elements as its convenience, such as the most famous elements Q6[1] and QM6[2], but the nonlinear transformation between the isoparametric and the Cartesian coordinates would lead to a problem that the serendipity family was quite sensitive to the mesh distortions.

For overcoming this inherent defect of isoparametric coordinates，Long et al. [3, 4] developed the first kind of quadrilateral area coordinate method QACM-I. For this kind of quadrilateral coordinate method, the transformation between the area and the Cartesian coordinate systems is always linear, so it is powerful for developing new elements which are much more insensitive to mesh distortion.

By using QACM-I, Chen *et al*. [5] proposed 4-node quadrilateral membrane elements AGQ6-I and AGQ6-II, which are locking free of MacNeal's thin beam. Based on this research, Cen *et al*. [6] derived out the analytical element stiffness matrix of AGQ6-I and developed a family of the quadrilateral plane membrane elements [7].

As QACM-I contains four area coordinate components (L_1, L_2, L_3, L_4), among which only two are independent, Chen *et al*. [8] proposed the second kind of quadrilateral coordinate method QACM-II which has only two independent coordinate components Z_1 and Z_2.

In this paper, based on generalized conforming theory[9], a 4-node plane element ACQλ2 was developed by using QACM-II. Its nodal displacement field was formulated with optimized conforming conditions which would introduce fewest restrains on element's sides, and additional displacement field based on internal parameters was used to make the whole displacement field quadratic complementary.

Generalized conforming conditions

According to the theory of generalized conforming，the conforming conditions for plane elements under constant stresses are as follows:

Compatibility of constant stress s_x:

$$\iint \varepsilon_x dA \equiv \iint \frac{\partial u}{\partial x} dA \equiv \oint luds = \oint l\tilde{u}ds \tag{1}$$

Compatibility of constant stress s_y:

$$\iint \varepsilon_y dA \equiv \iint \frac{\partial v}{\partial v} dA \equiv \oint mvds = \oint m\tilde{v}ds \tag{2}$$

Compatibility of constant stress t_{xy}:

$$\iint \gamma_{xy}\, dA \equiv \iint \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dA \equiv \oint \left(mu + lv \right) ds = \oint \left(m\tilde{u} + l\tilde{v} \right) ds \tag{3}$$

Compatibility of rotation：

$$\iint 2\varpi\, dA \equiv \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA \equiv \oint \left(lv - mu \right) ds = \oint \left(l\tilde{v} - m\tilde{u} \right) ds \tag{4}$$

Compatibility of motion along x :

$$\sum_{i=1}^{4} \left(u - \tilde{u} \right)_i = 0 \tag{5}$$

Compatibility of motion along y :

$$\sum_{i=1}^{4} \left(v - \tilde{v} \right)_i = 0 \tag{6}$$

Eq.3 and Eq.4 can be written in equivalent form as follows:

$$\iint \frac{\partial u}{\partial y}\, dA \equiv \oint mu\, ds = \oint m\tilde{u}\, ds$$

$$\iint \frac{\partial v}{\partial x}\, dA \equiv \oint lv\, ds = \oint l\tilde{v}\, ds \tag{7}$$

Then there are three conforming conditions for u and v respectively, and for a 4-node element, the additional condition was selected as follows:

$$\sum_{i=1}^{4} \left(u - \tilde{u} \right)_i x_i h_i = 0$$

$$\sum_{i=1}^{4} \left(v - \tilde{v} \right)_i x_i h_i = 0 \tag{8}$$

Formulation of nodal displacement

The nodal displacement field of the new element can be assumed with QACM-II as follows:

$$u_q = a_1 + a_2 Z_1 + a_3 Z_2 + a_4 Z_1 Z_2$$

$$v_q = b_1 + b_2 Z_1 + b_3 Z_2 + b_4 Z_1 Z_2 \tag{9}$$

With the conforming conditions presented in Eq.1~Eq.8, the undetermined parameters can be formulated out.

Substitute the results of $a_i (i = 1,2,3,4)$ into Eq.9，then the shape functions of nodal displacement can be written as:

$$N_{qi} = \frac{1 + 3(g_1 g_3 - g_2 g_4) g_j x_i h_i}{4(1 + g_1 g_3 + g_2 g_4)} + \frac{x_i}{4} \left[1 - \frac{h_i (g_3 - g_2) g_j}{1 + g_1 g_3 + g_2 g_4} \right] Z_1$$

$$+ \frac{h_i}{4} \left[1 - \frac{x_i (g_2 - g_1) g_j}{1 + g_1 g_3 + g_2 g_4} \right] Z_2 + \frac{3 x_i h_i g_j}{4(1 + g_1 g_3 + g_2 g_4)} Z_1 Z_2 \tag{10}$$

The Partial derivatives of shape functions are as follows:

$$\frac{\partial N_{qi}}{\partial x} = \frac{x_i \overline{b_1}}{4A}\left[1 - \frac{h_i(g_3 - g_2)g_j}{1 + g_1 g_3 + g_2 g_4}\right] + \frac{h_i \overline{b_2}}{4A}\left[1 - \frac{x_i(g_2 - g_1)g_j}{1 + g_1 g_3 + g_2 g_4}\right]$$

$$+ \frac{3x_i h_i g_j}{4(1 + g_1 g_3 + g_2 g_4)}\left(\frac{\overline{b_1}}{A}Z_2 + \frac{\overline{b_2}}{A}Z_1\right)$$

$$\frac{\partial N_{qi}}{\partial y} = \frac{x_i \overline{c_1}}{4A}\left[1 - \frac{h_i(g_3 - g_2)g_j}{1 + g_1 g_3 + g_2 g_4}\right] + \frac{h_i \overline{c_2}}{4A}\left[1 - \frac{x_i(g_2 - g_1)g_j}{1 + g_1 g_3 + g_2 g_4}\right]$$

$$+ \frac{3x_i h_i g_j}{4(1 + g_1 g_3 + g_2 g_4)}\left(\frac{\overline{c_1}}{A}Z_2 + \frac{\overline{c_2}}{A}Z_1\right)$$

$$\left(\begin{array}{l} i=1,2,3,4 \quad j=i+2 \\ \textbf{when } i=3, \ j=1 \\ \textbf{when } i=4, \ j=2 \end{array}\right) \text{(11)}$$

Then the matrix of strains $\boldsymbol{B}_q$ can be formulated out. The stiffness matrix of nodal displacement field can be written as:

$$\boldsymbol{K}_q = \iint_A \boldsymbol{B}_q^T \boldsymbol{D} \boldsymbol{B}_q t dA \tag{12}$$

Where $\boldsymbol{D}$ is the elastic matrix and t is the thickness of element.

Formulation of additional displacement

Additional displacement field is assumed as follows:

$$u_1 = I_1 N_{I1} + I_2 N_{I2}$$
$$v_1 = I_1' N_{I1} + I_2' N_{I2}$$
(13)

The shape functions of internal parameters can be selected as follows:

$$N_{I1} = Z_1^2 - \frac{2}{3}(g_2 - g_1)Z_1 - \frac{1}{3}\left[(g_3 - g_2)^2 + 1\right] + \frac{2}{9}(g_2 - g_1)^2$$

$$N_{I2} = Z_2^2 - \frac{2}{3}(g_3 - g_2)Z_2 - \frac{1}{3}\left[(g_2 - g_1)^2 + 1\right] + \frac{2}{9}(g_3 - g_2)^2$$
(14)

These two shape functions both can satisfy the conforming condition as:

$$\iint \frac{\partial N_{\lambda i}}{\partial x} dA = \oint l N_{\lambda i} ds = 0$$

$$\iint \frac{\partial N_{\lambda i}}{\partial y} dA = \oint m N_{\lambda i} ds = 0$$

$$\iint N_{\lambda i} dA = 0$$
(15)

With static condensation method, the stiffness matrix of element can be written as:

$$\boldsymbol{K}^e = \boldsymbol{K}_q - \boldsymbol{K}_{1q}^T \boldsymbol{K}_{11}^{-1} \boldsymbol{K}_{1q} \tag{16}$$

$$\boldsymbol{K}_{11} = \iint_A \boldsymbol{B}_1^T \boldsymbol{D} \boldsymbol{B}_1 t dA$$

$$\boldsymbol{K}_{1q} = \iint_A \boldsymbol{B}_1^T \boldsymbol{D} \boldsymbol{B}_q t dA$$
(19)

where $\boldsymbol{B}_1$ is the strain matrix about N_{1i}, it can be formulated with partial derivatives as follows:

$$\frac{\partial N_{I1}}{\partial x} = \frac{2\overline{b_1}}{A}Z_1 - \frac{2\overline{b_1}}{3A}(g_2 - g_1) \quad \frac{\partial N_{I2}}{\partial x} = \frac{2\overline{b_2}}{A}Z_2 - \frac{2\overline{b_2}}{3A}(g_3 - g_2)$$

$$\frac{\partial N_{I1}}{\partial y} = \frac{2\overline{c_1}}{A}Z_1 - \frac{2\overline{c_1}}{3A}(g_2 - g_1) \quad \frac{\partial N_{I2}}{\partial y} = \frac{2\overline{c_2}}{A}Z_2 - \frac{2\overline{c_2}}{3A}(g_3 - g_2)$$
(22)

This element model is denoted as ACQλ2.

Numerical examples

Example 1. Patch test

The constant strain/stress patch test is shown in Fig.1. Let Young's modulus $E=1000$, Poisson's ratio $\mu=0.25$, and thickness of the patch $t=1$. ACQλ2 can present exact solutions as listed in Tab.1.
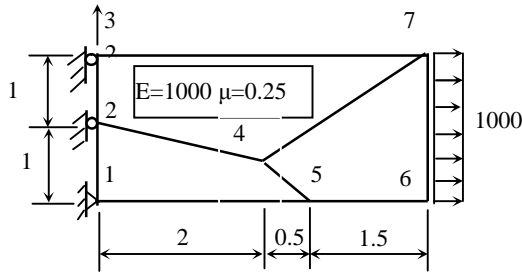

Fig.2 Patch test

Tab.1 Exact solutions of Patch test

Node	u_i	v_i
1	0.0	0.0
2	0.0	-0.25
3	0.0	-0.5
4	2.0	-0.125
5	2.5	0.0
6	4.0	0.0
7	4.0	-0.5

Example 2. Cook's skew beam

A skew cantilever beam under shear distributed load at the free edge, as shown in Fig. 2, was proposed by Cook et al. [10]. The results of vertical deflection at point C, the maximum principal stress at point A and the minimum principal stress at point B are listed in Table 2.
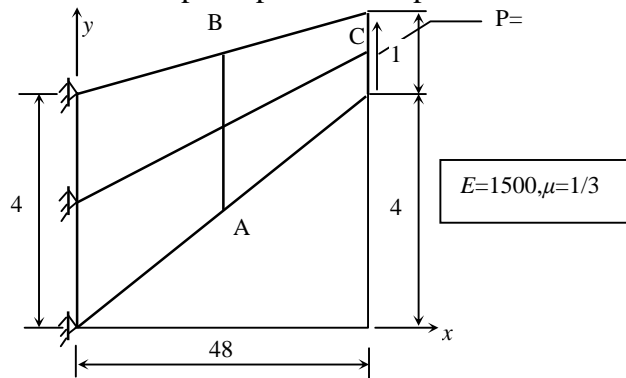

Fig.2 Cook's skew beam

Tab.2 Results of Cook's skew beam

Element	V_C			σ_{Amax}			σ_{Bmin}		
	2×2	4×4	8×8	2×2	4×4	8×8	2×2	4×4	8×8
Q4	11.80	18.29	22.08	0.1217	0.1873	0.2242	-0.0960	-0.1524	-0.1869
QM6	21.05	23.02	-	0.1928	0.2243	-	-0.1580	-0.1856	-
HL[11]	18.17	22.03	23.39	0.1582	0.1980	0.2205	-0.1335	-0.1770	-0.1931
P-S[12]	21.13	23.02	23.88	0.1854	0.2241	0.2364	-	-	-
ACQλ2	20.74	22.99	23.69	0.1936	0.2256	0.2345	-0.1452	-0.1866	-0.1987

Example 3. Cantilever beam divided by five quadrilateral elements

A cantilever beam divided by five irregular quadrilateral elements is shown in Fig.3. Two loading cases are considered: (a) pure bending under moment $M=2000$; (b) shear bending under transverse force $P=150$. The Young's modulus $E=1500$ and Poisson's ratio $\mu=0.25$. The results of the vertical defection v_A at point A and the stress σ_{xB} at point B are given in Tab. 3.
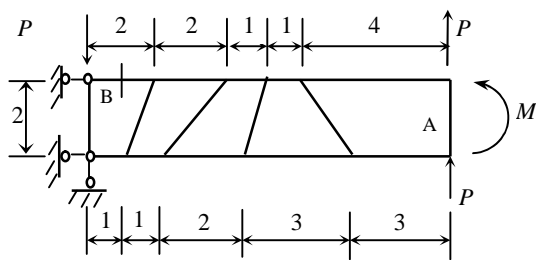
Fig.3 Cantilever beam with five irregular elements

Tab.3 Results of cantilever beam

Element	M		P	
	v_A	σ_{xB}	v_A	σ_{xB}
Q4	45.7	-1761	50.7	-2448
QM6	96.1	-2497	98.0	-3235
QC6[13]	96.1	-2439	98.1	-3339
NQ6[14]	96.1	-2439	98.0	-3294
P-S	96.2	-3001	98.1	-3899
ACQλ2	96.0	-3015	98.0	-4135
Exact	100	-3000	102.6	-4050

Conclusions

By using the second kind of quadrilateral coordinate method QACM-II, a 4-node plane element ACQλ2 was developed based on generalized conforming theory. Optimized conforming conditions were selected to make the nodal displacement field keep lowest order and fewest restrains were applied on element's sides. Additional displacement field based on internal parameters was also used to make the whole displacement field quadratic complementary.

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