Tail dependence coefficient of generalized hyperbolic distribution

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The tail dependence describes the limiting proportion of exceeding one margin over a certain threshold given that the other margin has already exceeded that threshold. In this paper, we obtain the limit tail dependence coefficient for the generalized hyperbolic distribution.

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1. Introduction

The generalized hyperbolic distribution was introduced in [3], and has been developed by many authors, see among others [17], [12], [16], [2], especially in relation with several applications in the Finance, see [15], [6]. This family contains and generalizes many familiar distributions such as the Student t, Gaussian, variance gamma, Cauchy and others.

The aim of this paper is to investigate the upper tail dependence coefficient in the case of the generalized hyperbolic distribution. More precisely, a bivariate random variable $X = (X_1, X_2)$ follows a generalized hyperbolic distribution $GH(\lambda, \alpha, \beta, \delta, \gamma)$ if

$$X_1 = \mu_1 + \theta_1 W_1 + \sqrt{W_1} Z_1, \qquad X_2 = \mu_2 + \theta_2 W_2 + \sqrt{W_2} Z_2, \qquad (1.1)$$

where $Z = (Z_1, Z_2) \sim N_2(0, \Sigma)$, is the bivariate normal distribution with mean 0 and a correlation matrix Σ and $W = (W_1, W_2) \sim GIG(\lambda, \delta, \gamma)$, is the generalized inverse gaussian with probability density function

$$f_W(x) = \begin{cases} \frac{(\frac{\gamma}{\delta})^\lambda}{2k_\lambda(\delta\gamma)} x^{\lambda-1} \exp(-\frac{1}{2}(\gamma^2 x + \frac{\delta^2}{x})) & \text{if } x > 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

where $\gamma = \sqrt{\alpha^2 - \theta^2}$, $\lambda \in \mathbb{R}, \delta > 0$ and $\gamma \geq 0$. The probability density function of $GH(\lambda, \alpha, \beta, \delta, \mu)$ is given by

$$f_X(x) = \frac{(\alpha^2 - \theta^2)^{\frac{\lambda}{2}} k_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2}) \exp(\theta(x-\mu))}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}} k_\lambda(\delta\sqrt{\alpha^2 - \beta^2})\delta^\lambda(\sqrt{\delta^2 + (x-\mu)^2})^{-\lambda+\frac{1}{2}}},$$

where K_λ is the modified Bessel function of the third kind, given by

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty \exp(-\frac{x}{2}(y + \frac{1}{y}))y^{\lambda-1} du, \qquad \text{for} \quad x > 0, \lambda \in \mathbb{R}, \tag{1.2}$$

see for instance, [7], [11]. [1] have given general results for upper tail dependence of skewed grouped t-distributions in the case where its degrees of freedom are different, *i.e.* for a random variable $X = (X_1, X_2)$, we can write

$$X_1 = \frac{\theta_1}{G_\mu^{-1}(U)/\mu} + \frac{Z_1}{\sqrt{G_\mu^{-1}(U)/\mu}}, \qquad X_2 = \frac{\theta_2}{G_\nu^{-1}(U)/\nu} + \frac{Z_2}{\sqrt{G_\nu^{-1}(U)/\nu}}, \tag{1.3}$$

where $Z = (Z_1, Z_2)$ is a bivariate normal distribution with mean 0 and correlation matrix $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, with correlation coefficient ρ and Z is independently distributed of the standard uniform random variance U. The parameter $\theta = (\theta_1, \theta_2)' \in \mathbb{R}$ controls the asymmetry of the distribution and τ, ν are positive constants. The function $G_\eta^{-1}(.)$ is the inverse distribution function of $\Gamma(\frac{\eta}{2}, \frac{\eta}{2})$ with $\eta > 0$.

[9] have particularly studied a special case of the skewed grouped t-distributions, see [1], when $\eta = \nu = \tau$ in (1.3), which gives the case of the skew t-distribution, introduced in [5]. This model can be defined for $X = (X_1, X_2)$ by

$$X_1 = \frac{\theta_1}{G_\tau^{-1}(U)} + \frac{Z_1}{\sqrt{G_\tau^{-1}(U)}}, \qquad X_2 = \frac{\theta_2}{G_\nu^{-1}(U)} + \frac{Z_2}{\sqrt{G_\nu^{-1}(U)}},$$

where Z_1, Z_2, Σ, τ and ν are defined previously.

The coefficient of the lower tail dependence for a bivariate random vector $X = (X_1, X_2)$, with marginal distributed functions F_1 and F_2 is defined as $\lambda_L = \lim_{u \to 0} \mathbb{P}\left(X_1 \leq F_1^{(-1)}(u) | X_2 \leq F_2^{(-1)}(u)\right)$ if this limit exists, where F_1^{-1} and F_2^{-1} are the generalized inverse functions of F_1 and F_2, see [10]. The coefficient of the upper tail dependence of a random vector X can be defined similarly as $\lambda_U = \lim_{u \to 1} \mathbb{P}\left(X_1 \geq F_1^{(-1)}(u) | X_2 \geq F_2^{(-1)}(u)\right)$ or equivalently

$$\lambda_U = \lim_{x \to +\infty} \mathbb{P}\left(X_1 \geq F_1^{(-1)}(F_2(x)) | X_2 \geq x\right). \tag{1.4}$$

2. Main result

The aim of this paper is to provide an analytic result of the tail dependence coefficient of the generalised hyperbolic model. We will consider the upper tail dependence for X in the model (1.1), using a method deriving from an equivalent form of (1.4) for λ_U:

$$\lambda_U = \lim_{x \to +\infty} \mathbb{P}\left(X_1 \geq F_1^{(-1)}(F_2(x))|X_2 = x\right) + \lim_{x \to +\infty} \mathbb{P}\left(X_2 \geq F_2^{(-1)}(F_1(x))|X_1 = x\right),$$

see [13].

Theorem 2.1. *The upper tail coefficient of the vector* (X_1, X_2) *defined in* (1.1) *is given by:*
1) If $\theta_1 = \theta_2 = 0$*, then*

$$\lambda_U = 1 - \int_0^\infty g_{\lambda,\gamma,\delta}(t)dt,$$

2) If $\theta_1, \theta_2 > 0$*, then*

$$\lambda_U = 1,$$

3) If $\theta_1 = \theta_2 = \theta < 0$*, then*

$$\lambda_U = 1.$$

Proof. Let $\theta_1 = \theta_2 = 0$. For $x \to a$, we use the notation $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ and $f(x) \propto g(x)$ if there exists $c > 0$, such as $g(x) = cf(x)$. Let $h(x) = F_1^{-1} \circ F_2(x)$, $g(x) = \sqrt{Q_{\lambda_1,\delta_1,\gamma_1}^{-1} \circ Q_{\lambda_2,\delta_2,\gamma_2}(x^2)}$ and $S^2 = Q_{\lambda_2,\delta_2,\gamma_2}^{-1}(U)$.
Equation (1.1) can be written as

$$X_1 = \mu_1 + \frac{\theta_1}{(g(S))^2} + \frac{Z_1}{g(S)}, \quad X_2 = \mu_2 + \frac{\theta_2}{S^2} + \frac{Z_2}{S},$$

where μ_1, μ_2, Z_1 and Z_2 are defined previously. The upper tail dependence coefficient can be rewritten as

$$\lambda_U = \lim_{x \to \infty} \frac{\mathbb{P}((X_1 \geq h(x), X_2 \geq x)}{\mathbb{P}(X_2 \geq x)},$$

$$= \lim_{x \to \infty} \frac{\int_{h(x)}^\infty \int_x^\infty f_{X_1,X_2}(s,t)dsdt}{\mathbb{P}(X_2 \geq x)},$$

We use the l'Hopital's rule to obtain

$$\lambda_U = \lim_{x \to \infty} \frac{-\int_{h(x)}^\infty f_{X_1,X_2}(s,x)ds - h'(x)\int_x^\infty f_{X_1,X_2}(h(x),t)dt}{-f_2(x)},$$

$$= \lim_{x \to \infty} \left(\mathbb{P}(X_1 \geq h(x)/X_2 = x) + h'(x)\frac{f_1(x)}{f_2(x)}\mathbb{P}(X_2 \geq x/X_1 = h(x))\right). \tag{2.1}$$

The distribution function Q_{W_2} of the generalized inverse gaussian is given by

$$Q_{W_2}(x) = \int_\infty^x \frac{(\frac{\gamma}{\delta})^\lambda}{2k_\lambda(\delta\lambda)}t^{\lambda-1}\exp(-\frac{1}{2}(\gamma^2 t - \frac{\delta^2}{t}))dt.$$

It follows that the corresponding distribution function Q_{W_2} satisfies, as $x \to \infty$, $1 - Q_{W_2}(x) \sim c_{\lambda,\gamma,\delta} \frac{x^\lambda}{\lambda} \exp(-\frac{1}{2}\gamma^2 x)$, where $c_{\lambda,\gamma,\delta} = \frac{(\frac{\gamma}{\delta})^\lambda}{2k_\lambda(\delta\lambda)}$. Then

$$Q_{W_2}(x) = 1 - c_{\gamma,\delta} \frac{x^\lambda}{\lambda} \cdot \exp(-\frac{1}{2}\gamma^2 x).$$

We use Lemma 3.1, see [1], to invert the last equation and obtain, as $u \to 1$,

$$Q_{W_2}^{-1}(u) = -\frac{\log(1-u)}{\gamma^2/2}.$$

Hence for $x \to \infty$

$$g(x) = m_{\gamma_2,\delta_2} x^{2\lambda_2} \exp(-\frac{1}{2}\gamma_2^2 x^2), \qquad \text{where} \quad m_{\lambda_2,\gamma_2,\delta_2} = 2\frac{c_{\lambda_2,\gamma_2,\delta_2}}{\lambda_2\gamma_2^2}.$$

The density f_1 of the generalized hyperbolic distribution with parameters $\lambda, \alpha, \theta, \delta, \gamma$ is given by

$$f_1(x) = \ell_\lambda(\alpha,\theta,\delta)k_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})(\sqrt{\delta^2+(x-\mu)^2})^{\lambda-\frac{1}{2}}\exp(\theta(x-\mu)),$$

where $\ell_\lambda(\alpha,\theta,\delta) = \frac{(\alpha^2-\theta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda k_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$. It follows that the corresponding distribution function F_1 satisfies, as $x \to \infty$,

$$1 - F_1(x) \sim \int_x^\infty \ell_\lambda(\alpha,\theta,\delta)t^{\lambda-1/2}k_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(t-\mu)^2})\exp(\theta(t-\mu))dt,$$
$$\sim d_\lambda x^\lambda \exp(-(\alpha-\theta)x),$$

where $d_\lambda = \frac{\pi}{2\lambda\alpha}\ell_\lambda(\alpha,\theta,\delta)$. Using the same lemma as above, for $u \to 1$, we can invert F_1 and obtain $F_1^{-1}(u) = -\frac{\log(1-u)}{\alpha-\theta}$. It follows that the strictly increasing function $h(x) = F_1^{-1} \circ F_2(x)$ satisfies, as $x \to \infty$,

$$h(x) \sim (\alpha-\theta)x.$$

The derivative of h is equal to $h' = \frac{f_2}{f_1 \circ h}$, and hence, as $x \to \infty$,

$$\frac{f_1(x)}{f_2(x)}h'(x) = \frac{f_1(x)}{f_1 \circ h(x)} \sim (\alpha-\theta)^{1-\lambda}\exp(-(\alpha-\theta)x(1-(\alpha-\theta))).$$

Note that this quantity tends to zero for $\alpha < 1+\theta$. Hence the second term in (2.1) goes to zero. It remains to calculate the first term of the equation (2.1). The conditional density of S^2 given X_2 satisfies, by Bayes rule,

$$p_{S^2|X_2=x_2}(t) \propto p_{X_2|S^2=t}(x_2)p_{S^2}(t),$$
$$\propto t^{(\lambda-1/2)-1}\exp(-\frac{1}{2}(\gamma^2 t + \frac{\delta^2+x^2}{t})).$$

Then $S^2|X_2 = x_2 \sim GIG(\lambda-1/2, \gamma, \sqrt{\delta^2+x^2})$. Now, we compute the second term of (2.1).

$$\mathbb{P}(X_1 \geq x_1|X_2=x_2) = \int_0^\infty \mathbb{P}(X_1 \geq x_1|X_2=x_2, s^2=t)f_{S^2|X_2=x_2}(t)dt,$$
$$\geq = \int_0^\infty \mathbb{P}(\frac{1}{g(\sqrt{t})}(\rho\sqrt{t}x_2+(1-\rho^2)W) \geq x_1)g_{\lambda',\gamma,\delta'}(t)dt,$$

where $\lambda' = \lambda - 1/2$, $\delta' = \sqrt{\delta^2 + x^2}$ and $W \sim \mathcal{N}(0,1)$. We can re-express the right side of (2.1) as $\lim_{x \to \infty} \int_0^\infty \mathbb{P}((1 - \rho^2)W \geq f(x)g(t) - \rho\sqrt{t}x)g_{\lambda',\gamma,\delta'}(t)dt$. Using the dominated convergence theorem, we obtain $\lambda_U = 1 - \int_0^\infty g_{\lambda',\gamma,\delta'}(t)dt$.

Let $\theta_1, \theta_2 > 0$ or $\theta_1, \theta_2 < 0$, For our quantitative developement we need the modified Bessel function of the third kind, see [14], with index $\lambda \in \mathbb{R}$:

$$K_\lambda(x) = \sqrt{\frac{\pi}{2x}} \exp(-x) \left(1 + o(\frac{1}{x})\right), \tag{2.2}$$

According to the equation (1.2), we have

$$\frac{K_{\lambda+1}(x)}{K_\lambda(x)} = 1 + \frac{8\lambda + 4}{8x} + \frac{32\lambda^2 - 8}{(8x)^2} + o\left(\frac{1}{x^3}\right), \tag{2.3}$$

$$\frac{K_{\lambda+1}(x)K_{\lambda-1}(x)}{(K_\lambda(x))^2} = 1 + \frac{1}{x} + \frac{64 - 256\lambda^2}{(8x)^3} + o\left(\frac{1}{x^4}\right). \tag{2.4}$$

The marginal density of X_1 is defined by

$$f_{X_1}(x_1) = \frac{k_{\lambda-\frac{1}{2}}\left(\delta^2 + (x_1 - \mu_1)^2, \alpha^2\right)}{\sqrt{2\pi}k_\lambda\left(\delta^2, \alpha^2 - \theta_1^2\right)} e^{\theta_1(x_1 - \mu_1)}, \tag{2.5}$$

where $K_\lambda(\delta^2, \alpha^2 - \beta^2) = \int_0^\infty x^{\lambda-1} e^{-\frac{1}{2}\left((\alpha^2 - \beta^2)x + \frac{\delta^2}{x}\right)} dx$. The conditional density of $X_{2.1} = X_2|X_1 = x_1$ is equal to

$$\begin{aligned} f_{X_2|X_1}(x_2|x_1) &= \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} \\ &= \frac{k_{\lambda-1}\left(\delta^2 + a(x_1,x_2), \alpha^2\right)}{\sqrt{2\pi}k_{\lambda-\frac{1}{2}}\left(\delta^2 + (x_1 - \mu_1)^2, \alpha^2\right)} e^{\theta_2(x_2 - \mu_2)}, \end{aligned}$$

where $a(x_1, x_2) = \frac{1}{(1-\rho^2)}\left((x_1 - \mu_1)^2 + 2(x_1 - \mu_1)(x_2 - \mu_2) + (x_2 - \mu_2)^2\right)$. Note that $X_{2.1}$ has a generalised hyperbolic distribution. It is known that the conditional distribution of $X_{2.1}|W = w$ is normal $N(\mu_2 + \theta_2 w, w)$, where $W \sim GIG\left(\lambda - \frac{1}{2}, \delta^2 + (x_1 + \mu_1)^2, \alpha^2 + \theta_2^2\right)$.

The proof we propose here requires the use of the Chebyshev's inequality, so we compute the first two moments. We have

$$\begin{aligned} \mathbb{E}(X_{2.1}) &= \mu + \theta_2 \mathbb{E}(W) \\ &= \mu + \theta_2 \frac{\delta^2 + (x_1 + \mu_1)^2}{\alpha^2 + \theta_2^2} \frac{K_{\lambda+\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}{K_{\lambda-\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}, \end{aligned}$$

You can easily compute the variance of $X_{2.1}$ from the variance decomposition formula and obtain

$$\begin{aligned} Var(X_{2.1}) &= \theta_2^2 \left(\frac{\delta^2 + (x_1 + \mu_1)^2}{\alpha^2 + \theta_2^2}\right)^2 \left[\frac{K_{\lambda+\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}{K_{\lambda-\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}\right]^2 \\ &\quad \times \left(\frac{K_{\lambda+\frac{3}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}{K_{\lambda-\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))} - 1\right) \\ &\quad + \theta_2 \frac{\delta^2 + (x_1 + \mu_1)^2}{\alpha^2 + \theta_2^2} \frac{K_{\lambda+\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}{K_{\lambda-\frac{1}{2}}((\alpha^2 + \theta_2^2)(\delta^2 + (x_1 + \mu_1)^2))}, \end{aligned}$$

By the Chebyshev's Inequality, for all positive ε,

$$
\begin{aligned}
\mathbb{P}\left(|X_{2.1} - \mathbb{E}(X_{2.1})| \geq \varepsilon |x_1|\right) &\leq \frac{Var(X_{2.1})}{\varepsilon^2 |x_1|^2} \\
&= \frac{1}{\varepsilon^2 |x_1|^2} \left[(\delta^2 + (x_1 - \mu_1)^2) \left(\frac{\alpha^2 + 2\theta_2^2}{(\alpha^2 + \theta_2^2)^3} \right) \right. \\
&\quad \left. + \lambda \left(\frac{\alpha^2 + 3\theta_2^4}{(\alpha^2 + \theta_2^2)^4} \right) + o\left(\frac{1}{|x_1|} \right) \right] = o\left(\frac{1}{|x_1|} \right).
\end{aligned}
$$

We will combine the variance of $X_{2.1}$ with (2.3) and (2.4), as $x_1 \to -\infty$. This gives

$$
\mathbb{P}\left(\mathbb{E}(X_{2.1}) - \varepsilon |x_1| \leq X_{2.1} \leq \mathbb{E}(X_{2.1}) + \varepsilon |x_1| \right) \geq 1 - o\left(\frac{1}{|x_1|} \right), \text{ as } x_1 \to -\infty,
$$

Hence $\lim_{x_1 \to -\infty} \mathbb{P}\left(X_2 \leq F_2^{-1}(F_1(y)) | X_1 = x_1 \right) = 0$ if

$$
\begin{aligned}
F_2^{-1}(F_1(x_1)) &< \mathbb{E}(X_{2.1}) - \varepsilon |x_1| \\
&= \left(\mu_2 - \frac{\theta_2}{(\alpha^2 + 2\theta_2^2)^2} + \varepsilon \right) x_1 + \lambda \frac{\theta_2}{(\alpha^2 + 2\theta_2^2)^2} + o\left(\frac{1}{|x_1|} \right),
\end{aligned}
$$

We use the expectation of the random variable $X_{2.1}$ and (2.3), as $x_1 \to -\infty$, to show

$$
F_1(x_1) < F_2\left(\left(\mu_2 - \frac{\theta_2}{(\alpha^2 + 2\theta_2^2)^2} + \varepsilon \right) x_1 + \lambda \frac{\theta_2}{(\alpha^2 + 2\theta_2^2)^2} + o\left(\frac{1}{|x_1|} \right) \right). \tag{2.6}
$$

Hence for $i = 1, 2$,

$$
\begin{aligned}
F_i(x_1) &= \mathbb{P}(X_i \leq x_1) \\
&= \frac{x^{\lambda - \frac{3}{2}} e^{(\theta_i^2 + \alpha^2 + \theta_i) x_1}}{2 K_\lambda \left(\delta^2, \alpha^2 - \theta_1^2 \right) \left(\theta_i^2 + \alpha^2 \right)^{(\lambda - 1/2)/2} \left(\theta_i^2 + \alpha^2 + \theta_i \right)} \left(1 + o\left(\frac{1}{|x_1|} \right) \right). \tag{2.7}
\end{aligned}
$$

By combining (2.7) and (2.6), we have

$$
\left(\theta_1^2 + \alpha^2 + \theta_1 \right) > \left(\theta_2^2 + \alpha^2 + \theta_2 \right) \left(\mu_2 - \frac{\theta_2}{\sqrt{\theta_1^2 + b^2}} + \varepsilon \right).
$$

as $\varepsilon > 0$ is arbitrary, we obtain $\frac{\theta_1^2 + \alpha^2 + \theta_1}{\theta_2^2 + \alpha^2 + \theta_2} - \left(\mu_2 - \frac{\theta_2}{(\alpha^2 + 2\theta_2^2)^2} \right) > 0$. If $\theta_1 = \theta_2 = \theta \in \mathbb{R}^*$, then $1 - \mu_2 + \frac{\theta}{(\alpha^2 + 2\theta^2)^2} > 0$. Then $\lim_{x_1 \to +\infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(y)) | X_1 = x_1 \right) = 1$. Similarly, one can show that $\lim_{x_1 \to +\infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(y)) | X_2 = x_1 \right) = 0$ and results in $\lambda_U = 1$. $\square$

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