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Derivation of Lorentz Transformation via Lorentz Invariant

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***Abstract*—We derive the Lorentz transformation matrix via the invariance of spacetime interval. The derivation is greatly simplified by using hyperbolic functions. The Lorentz scalar and the 4-vector are explained via the concept of relativistic covariance and the results are applied to relativistic mechanics. This paper helps to better understand the mathematical structure of special relativity, and may be useful for engineers working with relativistic applications such as satellite navigation and precision chronometry.**

Keywords—Special Relativity; Lorentz Transformation; Lorentz Invariant; Scalar; 4-Vector

I. INTRODUCTION

Special relativity is an important part in college physics. However, it is always difficult to impart such counterintuitive knowledge to students. Although it is easy to use special relativity to calculate typical relativistic problems, a complete understanding of the physical and mathematical structure of special relativity still costs enormous effort. Among all the usual confusions about special relativity, the derivation of Lorentz transformation is a prominent one. It appears more severe when students become to realize that almost all theoretical conclusions of special relativity can be attributed to the Lorentz transformation. However, this issue is not yet well handled in present teaching practice.

Historically, the derivation of Lorentz transformation in Einstein's original article was not an easy nut to digest, because of the profound philosophical assumptions he had used [1]. Later on Minkowski proposed that the Lorentz transformation can be derived solely from the invariance of spacetime interval [2], which is equivalent to Einstein's proposition on constant speed of light. Although the derivation is clear in this approach, it is still bypassed in most college physics textbooks.

Here we try to use the most concise procedure to derive the Lorentz transformation following the Minkowski's approach. Meanwhile, we have shown in a preceding article [3] that the rotational transformation can be derived using the rotational invariant (i.e., the length square). Readers may find the method and ideology in this article are similar to those in Reference [3]. The only significant difference is perhaps the hyperbolic functions used herein instead of the trigonometric functions. By comparing with the space rotation, the 'Lorentz boost' may be tentatively understood as a 'special rotation' in the Minkowski space. Finally, to complete the topic we further introduced the concept of Lorentz scalar and 4-vector, as well as their application in physics, in analogy to what have done for rotational transformation in References [4-7].

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II. INVARIANCE OF SPACE-TIME INTERVAL

The null result of Michelson–Morley experiment [8] indicates that the speed of light is constant and isotropic irrespective of the motion of the earth. Einstein further generalizes it to the famous postulation: the principle of invariance of light speed. That is to say, a spherical light wave observed in an inertial reference frame Σ described by

$$x^2 + y^2 + z^2 = (ct)^2 \tag{1}$$

preserves its spherical form in another inertial reference frame Σ' as

$$(x')^2 + (y')^2 + (z')^2 = (ct')^2 \tag{2}$$

Equation (1) and (2) can be rearranged as

$$(w')^2 - (x')^2 - (y')^2 - (z')^2 = w^2 - x^2 - y^2 - z^2 = 0 \tag{3}$$

where we have used $w = ct$ and $w' = ct'$ for convenience. Minkowski inspected equation (3) and made a mental jump to point out that any event with space-time coordinate (w, x, y, z) in one inertial reference frame appears with coordinate (w', x', y', z') in another inertial reference frame according to the equality

$$(w')^2 - (x')^2 - (y')^2 - (z')^2 = w^2 - x^2 - y^2 - z^2 \equiv s^2 \tag{4}$$

where s^2 is the square of *space-time interval*. The Minkowski's postulation can be summarized as the invariance of space-time interval between inertial reference frames.

III. LORENTZ TRANSFORMATION

Now we seek to reproduce the Lorentz transformation according to equation (4). To simplify the derivation we assume the Lorentz boost is along the *x*-direction, i.e., the *y* and z coordinates are untouched. In this case the equation (4) reduces to

$$(w')^2 - (x')^2 = w^2 - x^2 \tag{5}$$

Assume the transformation is linear and homogenous, i.e.,

$$\begin{cases} w' = aw + bx \\ x' = cw + dx \end{cases} \tag{6}$$

where the coefficients a, b, c, and d are to be determined. Inserting equation (6) into (5) we find

$$\begin{aligned} &(aw+bx)^2 - (cw+dx)^2 \\ &= (a^2-c^2)w^2 - (d^2-b^2)x^2 + 2(ab-cd)wx \\ &= w^2 - x^2 \end{aligned} \tag{7}$$

By equating the corresponding terms of w and x we obtain the following set of equations

$$\begin{cases} a^2 - c^2 = 1 \\ d^2 - b^2 = 1 \\ ab - cd = 0 \end{cases} \tag{8}$$

The first equation in (8) suggest that a and c can be denoted by the *hyperbolic* functions, without loss of generality, as

$$\begin{cases} a = \cosh\theta \\ c = \sinh\theta \end{cases} \tag{9}$$

where θ is an arbitrary parameter. Similarly, the second equation in (8) suggest that

$$\begin{cases} b = \sinh\xi \\ d = \cosh\xi \end{cases} \tag{10}$$

where ξ is another arbitrary parameter. Inserting the equations (9) and (10) into the last equation of (8) it follows that

$$\cosh\theta\sinh\xi - \sinh\theta\cosh\xi = 0 \tag{11a}$$

i.e.,

$$\sinh(\xi - \theta) = 0 \tag{11b}$$

which is equivalent to

$$\theta = \xi \tag{12}$$

Now the transformation (6) is simply denoted by

$$\begin{cases} w' = w\cosh\xi + x\sinh\xi \\ x' = w\sinh\xi + x\cosh\xi \end{cases} \tag{13}$$

In this formalism $\xi \in (-\infty, \infty)$ is a real number to be determined. Assume v is the Lorentz boost velocity, then the origin of the reference frame Σ' must move with velocity v in the reference frame Σ, i.e.,

$$\begin{aligned} &x' = 0 \Rightarrow w\sinh\xi + x\cosh\xi = 0 \\ &\Rightarrow \tanh\xi = -\frac{x}{ct} = -\frac{v}{c} \end{aligned} \tag{14}$$

Because $-1 \le \tanh\xi \le 1$, the speed v can not exceed the vacuum speed of light c. Furthermore, using

$$\begin{cases} \cosh\xi = \dfrac{1}{\sqrt{1-\tanh^2\xi}} \\ \sinh\xi = \cosh\xi\tanh\xi \end{cases} \tag{15}$$

and making the substitutions

$$\begin{cases} \beta \equiv -\tanh\xi \\ \gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} \end{cases} \tag{16}$$

equations (13) can be rewritten as

$$\begin{cases} w' = \gamma(w - \beta x) \\ x' = \gamma(x - \beta w) \end{cases} \tag{17}$$

Manifestly, equations (18) are the familiar form of Lorentz transformation. Finally, we can package the Lorentz transformation (13) into a 4×4 matrix form

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{18}$$

where the coordinates (w, x, y, z) are renamed as $\boldsymbol{x}=(x_0, x_1, x_2, x_3)$ for convenience. For Lorentz boost at an arbitrary direction, we can always firstly perform two 3d space rotations in the two reference frames, respectively, to turn the x and x' axes to the direction of the relative velocity, and then apply the equation (18).

IV. Lorentz Scalar and 4-Vectors in Minkowski Space

Analogous to the rotational scalar in 3d configuration space, we can define a Lorentz scalar in the Minkowski space as an invariant under the Lorentz transformation. As shown by the derivation in last section, the transformation (18) preserves the square of space-time interval

$$s^2 = x_0^{\ 2} - x_1^{\ 2} - x_2^{\ 2} - x_3^{\ 2} \tag{19}$$

therefore s^2 is a Lorentz scalar. Furthermore, if a 3d vector (A_1, A_2, A_3) can be upgraded into a 4-component quantity $\boldsymbol{A}=(A_0, A_1, A_2, A_3)$, it is regarded as a *4-vector* if it abides the same transformation as in (18) under a Lorentz boost, i.e.,

$$\begin{pmatrix} A_0' \\ A_1' \\ A_2' \\ A_3' \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{20}$$

Obviously, the space-time coordinates $\boldsymbol{x}=(x_0, x_1, x_2, x_3)$ itself is a 4-vector. Moreover, the 'Minkowski length square' of a 4-vector

$$\boldsymbol{A}^2 = A_0^{\ 2} - A_1^{\ 2} - A_2^{\ 2} - A_3^{\ 2} \tag{21}$$

is also a Lorentz scalar according to the property of Lorentz transformation.

Prepared with above knowledge, we see that the *proper time*

$$d\tau = \frac{1}{c}\sqrt{dx_0^{\ 2} - dx_1^{\ 2} - dx_2^{\ 2} - dx_3^{\ 2}} \tag{22}$$

is a Lorentz scalar. The proper time for a moving particle is best calculated in a reference frame Σ' in which the particle is at rest, i.e., $dx_1'=dx_2'=dx_3'=0$, such that

$$\begin{aligned} d\tau &= \frac{1}{c}\sqrt{c^2 dt'^2 - dx_1'^2 - dx_2'^2 - dx_3'^2} \\ &= dt' = \frac{dt}{\cosh(-\xi)} = \frac{dt}{\gamma} \end{aligned} \tag{23}$$

And we see that to be Lorentz covariant the *4-velocity* $\boldsymbol{U}$ must have the form

$$\begin{aligned} \boldsymbol{U} &\equiv (U_0, U_1, U_2, U_3) = \frac{d\boldsymbol{x}}{d\tau} \\ &= \left(\frac{dx_0}{d\tau}, \frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}\right) \\ &= \gamma\left(c, \boldsymbol{v}\right) \end{aligned} \tag{24}$$

Consequently the *4-momentum* is defined as the *mass* multiplied with the 4-velocity, i.e.,

$$\begin{aligned} \boldsymbol{P} &\equiv (P_0, P_1, P_2, P_3) = m\frac{d\boldsymbol{x}}{d\tau} \\ &= m\gamma\left(c, \boldsymbol{v}\right) = \gamma\left(mc, \boldsymbol{p}\right) \end{aligned} \tag{25}$$

where the mass m is a Lorentz scalar. Then the Minkowski length square of the 4-momentum $\boldsymbol{P}$ is a Lorentz scalar, i.e.,

$$\gamma^2 m^2 c^2 - \gamma^2 p^2 = m^2 c^2 \tag{26}$$

where the right hand term is obtained in a reference frame in which the particle is at rest. Einstein particularly inspected the zeroth component of $\boldsymbol{P}$ and pointed out that

$$P_0 = \gamma mc = \frac{1}{c}\cdot\frac{mc^2}{\sqrt{1 - v^2/c^2}} \equiv \frac{E}{c} \tag{27}$$

where E is the total *relativistic energy* of a moving particle. Moreover, he coined another terminology, the *relativistic momentum*, as

$$\tilde{p} = \gamma p \tag{28}$$

By substituting (27) and (28) into (26) we finally arrive at the famous *mass-energy equation*

$$E^2 = \tilde{p}^2 c^2 + m^2 c^4 \tag{29}$$

V. Conclusion

In this paper we have reviewed the Lorentz transformation in detail. The Lorentz transformation matrix is derived by assuming Lorentz invariance in the four-dimensinal spacetime. The derivation is shown with great simplification by applying hyperbolic functions. The invariance of Lorentz scalars and covariance of 4-vectors are discussed with physical examples to complete the topic. We hope this paper is helpful to those who are confused by the mathematical structure of special relativity. Engineers working with satellite navigation and precision chronometry may also find this paper beneficial because relativistic effects are significant in such applications.

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