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Some Inferences on Skew-t Distribution of 2 Degrees of Freedom

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Abstract

Understanding the properties of Skew distributions is important in many statistical applications. The Student-t distribution is widely used in statistics. Skew-t distribution is a useful model to describe data with heavy tails. In this paper several distributional properties of Skew-t distribution are given. Based on the distributional properties some characterizations of the Skew-t distribution are given.

Keywords: Skew-t Distribution; Percentile Points; Characterizations.

1. Introduction

A random variable X is said to have the skew-t distribution if its probability density function (pdf) is of the form $f_{ST}(x) = 2\, g(x)\, G(\alpha x)$ where $g(x)$ and $G(x)$ are respectively the pdf and the cumulative distribution function (cdf). The pdf $f_{ST}(x, \alpha)$ of skew-t distribution with degrees of freedom $v = 2$ and with α as a real number is given below.

$$f_{ST}(x,\alpha) = \frac{1}{(2+x^2)^{3/2}}\left(1+\frac{\alpha x}{\sqrt{2+\alpha^2 x^2}}\right), \quad -\infty < x < \infty. \tag{1}$$

The Skew-t distribution with 2 degrees of freedom was proposed by Azzalini [8] as a useful model to describe data with heavy tails. Jamalizadeh et al. [9] have shown a recurrence relation of the skew-t distribution of $v + 1$ degrees of freedom with the Student t-distribution of v (≥ 2) degrees of freedom. For $v = 2$, the cdf $F_{ST}(x,\alpha)$ of the skew-t is as follows.

$$F_{ST}(x,\alpha) = \frac{1}{2} - \frac{1}{\pi}\arctan\alpha + \frac{x}{\sqrt{2+x^2}}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{\alpha x}{\sqrt{2+x^2}}\right)\right), \quad -\infty < x < \infty. \tag{2}$$

The corresponding pdf is

$$f_{ST}(x,\alpha) = \frac{1}{\left(x^2+2\right)^{\frac{3}{2}}}\left(1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}\right), \tag{3}$$

The pdfs given at (1) and (3) are close to each other but not exactly the same and they coincide at $\alpha = 0$. In this paper we will consider some distributional properties and characterizations of the skew-t distribution with the pdf as given in (3).

2. Main Results

There are similarities of $f_{ST}(x, -\alpha)$ and $f_{ST}(x, \alpha)$. See figure 1 for $\alpha = -2$ and 2.

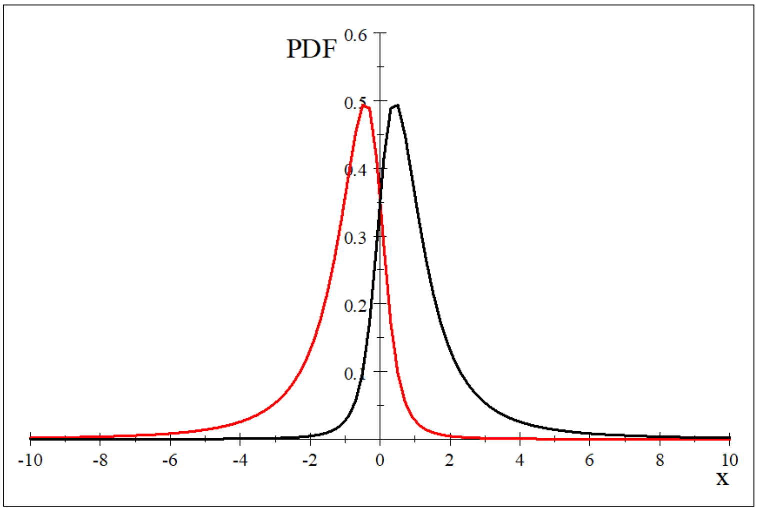

Figure 1. f_{ST}(x,-2) – Red, f_{ST}(x,2) – Black.

The two graphs are similar in shape. One is shifted to the right of the other. The percentile points of $f_{ST}(x, -\alpha)$ can easily be obtained from the percentile points of $f_{ST}(x, \alpha)$. Table 1 gives the percentile points of $f_{ST}(x, \alpha)$ for $\alpha = 0, 0.1, 0.2, 5, 125, 500$ and ∞.

Table 1. percentile points of $f_{ST}(x,\alpha)$.

p	α							
	0	0.1	0.2	1	5	125	500	∞
0.1	-1.8856	-1.7325	-1.5764	-0.6067	0.0883	0.1421	0.1421	0.1421
0.2	-1.0607	-0.94656	-0.8317	-1.2730	0.2674	0.2887	0.2887	0.2887
0.3	-0.6172	-0.51825	-0.4199	-0.8048	0.4334	0.4448	0.4448	0.4448
0.4	-0.2887	-0.19669	-0.1064	0.3783	0.6101	0.6172	0.6172	0.6172
0.5	0	0.0896	0.1763	0.6211	0.8115	0.8165	0.8165	0.8165
0.6	0.2887	0.37950	0.46639	0.8921	1.0568	1.0607	1.0607	1.0607
0.7	0.6172	0.71369	0.8048	1.2322	1.3830	1.3862	1.3862	1.3862
0.8	1.0697	1.1705	1.2730	1.7334	1.8827	1.8856	1.8858	1.8858
0.9	1.8856	2.0312	2.1652	2.7433	2.9168	2.9200	2.9200	2.9200

From the table it is evident that the percentile points remain the same for large positive α because $\frac{1}{2} - \frac{1}{\pi}$ arctan α is close to zero for large α and the cdf as given in (2) approximate the t-distribution truncated at 0 for positive x. The percentile points of $f_{ST}(x, -\alpha)$ can be obtained from the above table. For example for $p = 0.2$ and $\alpha = -5$, the percentile points of $f_{ST}(x, -5)$ will 1.8827, the one corresponding to $p = 0.8$ and $\alpha = 5$ in $f_{ST}(x, 5)$.

It can be shown that for the distribution given in (2) for $\alpha > 0$,

$$\lim_{n\to\infty}\frac{F^{-1}\left(1-\frac{1}{n}\right)-F^{-1}\left(1-\frac{2}{n}\right)}{F^{-1}\left(1-\frac{2}{n}\right)-F^{-1}\left(1-\frac{4}{n}\right)}=2^{1/2}$$

and for $\alpha < 0$,

$$\lim_{n\to\infty}\frac{F^{-1}\left(\frac{1}{n}\right)-F^{-1}\left(\frac{2}{n}\right)}{F^{-1}\left(\frac{2}{n}\right)-F^{-1}\left(\frac{4}{n}\right)}=2^{1/2}.$$

Then it follows from Ahsanullah and Nevzorov [2] that

$$\lim_{n\to\infty} P(X_{n,n} < a_n + b_n x) = e^{-x^{-2}}, \quad x \geq 0.$$

where $a_n = 0$ and $F^{-1}(1-\frac{1}{n}) \approx n^{\frac{1}{2}}$, and for $\alpha < 0$

$$\lim_{n\to\infty} P(X_{1,n} < c_n + d_n x) = 1 - e^{-(-x)^{-2}}, \quad x<0,$$
$$=1, \quad x \geq 0,$$

where c_n =0 and $d_n = |F^{-1}\left(\frac{1}{n}\right)| \approx n^{1/2}$

If $\alpha \to \infty$, then $f_{ST}(x, \alpha)$ becomes a truncated t distribution at $x = 0$. In that case

$$\lim_{n\to\infty}\frac{F^{-1}\left(1-\frac{1}{n}\right)-F^{-1}\left(1-\frac{2}{n}\right)}{F^{-1}\left(1-\frac{2}{n}\right)-F^{-1}\left(1-\frac{4}{n}\right)}=2^{1/2},$$

$$\lim_{n\to\infty} P(X_{n,n} < a_n + b_n x) = e^{-x^{-2}}, \quad x \geq 0.$$

with $a_n = 0$ and $F^{-1}(1-\frac{1}{n}) \approx n^{\frac{1}{2}}$ but

$$\lim_{n\to\infty}\frac{F^{-1}\left(\frac{1}{n}\right)-F^{-1}\left(\frac{2}{n}\right)}{F^{-1}\left(\frac{2}{n}\right)-F^{-1}\left(\frac{4}{n}\right)}=2^{-1}$$

and

$$\lim_{n\to\infty} P(X_{1,n} < c_n^* + d_n^* x) = 1- e^{-(-x)^{-1}}, \quad x < 0,$$
$$=1, \quad x \geq 0,$$

where c_n^* and d_n^* are the normalizing constants and

$$c_n^* = 0 \text{ and } d_n^* = \sqrt{\frac{2}{n}}.$$

For $\alpha = 0$, we will have the t-distribution with 2 degrees of freedom. In that case

$$\lim_{n\to\infty} P(X_{n,n} < a_n + b_n x) = e^{-x^{-2}}, \quad x \geq 0.$$

with a_n =0 and $b_n = \sqrt{(\frac{n}{2})}$ and

$$\lim_{n\to\infty} P(X_{1,n} < c_n + d_n x) = 1 - e^{-(-x)^{-2}}, \quad x < 0,$$
$$= 1, \quad x \geq 0,$$

where $c_n = 0$ and $d_n = |F^{-1}\left(\frac{1}{n}\right)| = \sqrt{(\frac{n}{2})}$.

The following two Lemmas will be used for the characterization theorems.

Lemma 1. *Suppose that X is an absolutely continuous random variable with cdf F(x) with pdf f (x). We assume f (x) is differentiable. For a continuous function g(x) on* $-\infty < x < \infty$ *with finite E(g(x)) if* $E(g(X)|X \leq x) = h_1(x)\tau(x)$, *where* $h_1(x)$ *is a differential function for all x.* $-\infty < x < \infty$ *and* $\tau(x) = \frac{f(x)}{F(x)}$, *then* $f(x) = c \exp(\int \frac{g(x)-h'_1(x)}{h_1(x)} dx)$ *and c is determined by the condition* $\int_{-\infty}^{\infty} f(x)dx = 1$.

Proof. We have

$$h_1(x) = \frac{\int_0^x g(u)f(u)du}{f(x)}$$

and

$$\int_0^x g(u)f(u)du = f(x)h_1(x).$$

Differentiating the above expression, we obtain

$$g(x)f(x) = f(x)h_1'(x) + f'(x)h_1(x)$$

On simplification, we have

$$\frac{f'(x)}{f(x)} = \frac{g(x) - h_1'(x)}{h_1(x)}$$

Integrating both sides of the above equation, we obtain

$$f(x) = c \exp(\int \frac{g(x)-h'_1(x)}{h_1(x)} dx)$$

and c is determined by the condition $\int_{-\infty}^{\infty} f(x)dx = 1$.

Lemma 2. *Suppose that X is an absolutely continuous random variable with cdf F(x) pdf f (x). We assume f (x) is differentiable. For a continuous function g(X) with finite E(g(X)) if* $E(g(X)|X \geq x) = h_2(x)r(x)$, *where* $h_2(x)$ *is a differential function in x,* $-\infty < x < \infty$ *and r(x) is the hazard function* = $\frac{f(x)}{1-F(x)}$, *then*

$$f(x) = ce^{-\int \frac{g(x)+h_2'(x)}{h_2(x)}dx}$$

and c is determined by the condition $\int_{-\infty}^{\infty} f(x)dx = 1$.

Proof. We have

$$h_2(x) = \frac{\int_x^{\infty} g(u)f(u)du}{f(x)}$$

and

$$\int_x^{\infty} g(u)f(u)du = f(x)h_2(x)$$

Differentiating the above expression, we obtain

$$-g(x)f(x)=f(x)h_2'(x)+f'(x)h_2(x).$$

On simplification, we have

$$\frac{f'(x)}{f(x)}=-\frac{g(x)+h_2'(x)}{h_2(x)}$$

Integrating both sides of the above equation with respect to x, we obtain

$$f(x)=ce^{-\int\frac{g(x)+h_2'(x)}{h_2(x)}dx}$$

and c is determined by the condition $\int_{-\infty}^{\infty}f(x)dx=1$. It is easy to show that variance of the skew t-distribution with the pdf as given in (3) does not exist but $E(X)=\frac{\alpha\sqrt{2}}{\sqrt{(1+\alpha^2)}}$. We use the characterization theorems using truncated first moment.

Theorem 1. *Suppose that X is an absolutely continuous random variable with cdf F(x) with pdf f(x). We assume that f'(x) exists for all x, $-\infty<x<\infty$ and E(X) exists. Then $E(X|X\le x)=g(x)\tau(x)$ where*

$$\iota(x)=\frac{f(x)}{F(x)}$$

and

$$g(x)=\frac{\left(x^2+2\right)^{\frac{3}{2}}p(x)}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}$$

$$p(x)=\frac{-\alpha}{\sqrt{2(1+\alpha^2)}}+\frac{1}{\sqrt{2(1+\alpha^2)}}\arctan\left(\sqrt{\frac{1+\alpha^2}{2}}x\right)-\frac{1}{\sqrt{x^2+2}}-\frac{1}{\sqrt{x^2+2}}\left(\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}\right)$$

if and only if

$$f(x)=\frac{1}{\left(x^2+2\right)^{\frac{3}{2}}}\left(1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}\right).$$

Proof. We have

$$f(x)g(x)=\int_{-\infty}^{x}\frac{u}{\left(u^2+2\right)^{\frac{3}{2}}}\left(1+\frac{2}{\pi}\arctan\frac{\alpha u}{\sqrt{u^2+2}}+\frac{2}{\pi}\frac{\alpha u\sqrt{u^2+2}}{(1+\alpha^2)u^2+2}\right)du$$

$$=\frac{-\alpha}{\sqrt{2(1+\alpha^2)}}+\frac{1}{\sqrt{2(1+\alpha^2)}}\arctan\left(\sqrt{\frac{1+\alpha^2}{2}}x\right)-\frac{1}{\sqrt{x^2+2}}-\frac{2}{\pi}\frac{1}{\sqrt{x^2+2}}\arctan\frac{\alpha x}{\sqrt{x^2+2}}.$$

$$=p(x).$$

Thus

$$g(x)=\frac{\left(x^2+2\right)^{\frac{3}{2}}p(x)}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}.$$

Suppose

$$g(x)=\frac{\left(x^2+2\right)^{\frac{3}{2}}p(x)}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}},$$

using the relation $p'(x) = x\,f(x)$ and $p(x)\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{p(x)}{f(x)}\frac{d}{dx}\ln f(x)$, we have

$$g'(x)=x-g(x)\left(\frac{-3x}{x^2+2}+\frac{\frac{4}{\pi}\frac{\alpha}{\sqrt{x^2+2}\left(x^2\alpha^2+x^2+2\right)}+\frac{2}{\pi}\frac{\alpha}{\sqrt{x^2+2}}\frac{-2x^2\alpha^2+2x^2+4}{\left(x^2\alpha^2+x^2+2\right)^2}}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}\right).$$

On simplification, we obtain

$$\frac{x-g'(x)}{g(x)}=\frac{-3x}{x^2+2}+\frac{\frac{4}{\pi}\frac{\alpha}{\sqrt{x^2+2}\left(x^2\alpha^2+x^2+2\right)}+\frac{2}{\pi}\frac{\alpha}{\sqrt{x^2+2}}\frac{-2x^2\alpha^2+2x^2+4}{\left(x^2\alpha^2+x^2+2\right)^2}}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}.$$

By Lemma 1, we have

$$\frac{f'(x)}{f(x)}=\frac{-3x}{x^2+2}+\frac{\frac{4}{\pi}\frac{\alpha}{\sqrt{x^2+2}\left(x^2\alpha^2+x^2+2\right)}+\frac{2}{\pi}\frac{\alpha}{\sqrt{x^2+2}}\frac{-2x^2\alpha^2+2x^2+4}{\left(x^2\alpha^2+x^2+2\right)^2}}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}.$$

On integrating both sides of the equation, with respect to x, we have

$$f(x)=c\frac{1}{\left(x^2+2\right)^{\frac{3}{2}}}\left(1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}\right).$$

Using the boundary condition $\int_{-\infty}^{\infty} f(x)dx=1,$ we have $c = 1$.

We will give here a characterization of the skew-t distribution using the left truncated first moment.

Theorem 2. *Suppose that X is an absolutely continuous random variable with cdf $F(x)$ with pdf $f(x)$. We assume that $E(X)$ is finite and $f'(x)$ exists for all x, $-\infty < x < \infty$. Then $E(X|X \geq x) = m(x)\,r(x)$, where*

$$r(x)=\frac{f(x)}{1-F(x)}$$

$$m(x)=\frac{n(x)}{f(x)},\quad n(x)=\frac{\sqrt{2}\alpha}{\sqrt{\alpha^2+1}}-(\frac{-\alpha}{\sqrt{2(1+\alpha^2)}}+\frac{1}{\sqrt{2(1+\alpha^2)}}\arctan(\sqrt{\frac{1+\alpha^2}{2}}x)$$
$$-\frac{1}{\sqrt{x^2+2}}-\frac{1}{\sqrt{x^2+2}}(\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}))$$

if and only if

$$f(x)=\frac{1}{\left(x^2+2\right)^{\frac{3}{2}}}(1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}).$$

Proof. We have

$$f(x)m(x)=\int_x^\infty\frac{u}{\left(u^2+2\right)^{\frac{3}{2}}}(1+\frac{2}{\pi}\arctan\frac{\alpha u}{\sqrt{u^2+2}}+\frac{2}{\pi}\frac{\alpha u\sqrt{u^2+2}}{(1+\alpha^2)u^2+2})du$$

$$=E(X)-\int_{-\infty}^x\frac{u}{\left(u^2+2\right)^{\frac{3}{2}}}(1+\frac{2}{\pi}\arctan\frac{\alpha u}{\sqrt{u^2+2}}+\frac{2}{\pi}\frac{\alpha u\sqrt{u^2+2}}{(1+\alpha^2)u^2+2})du$$

$$=\frac{\sqrt{2}\alpha}{\sqrt{\alpha^2+1}}-(\frac{-\alpha}{\sqrt{2(1+\alpha^2)}}+\frac{1}{\sqrt{2(1+\alpha^2)}}\arctan(\sqrt{\frac{1+\alpha^2}{2}}x)$$
$$-\frac{1}{\sqrt{x^2+2}}-\frac{2}{\pi}\frac{1}{\sqrt{x^2+2}}\arctan\frac{\alpha x}{\sqrt{x^2+2}}$$

$$=n(x).$$

Thus,

$$m(x)=\frac{\left(x^2+2\right)^{\frac{3}{2}}n(x)}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}.$$

Suppose

$$m(x)=\frac{\left(x^2+2\right)^{\frac{3}{2}}n(x)}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}},$$

then

$$m'(x) = -x - m(x)\left(\frac{-3x}{x^2+2} + \frac{\frac{4}{\pi}\frac{\alpha}{\sqrt{x^2+2}\left(x^2\alpha^2+x^2+2\right)} + \frac{2}{\pi}\frac{\alpha}{\sqrt{x^2+2}}\frac{-2x^2\alpha^2+2x^2+4}{\left(x^2\alpha^2+x^2+2\right)^2}}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}\right).$$

Thus

$$\frac{x+m'x)}{m(x)} = -\frac{-3x}{x^2+2} + \frac{\frac{4}{\pi}\frac{\alpha}{\sqrt{x^2+2}\left(x^2\alpha^2+x^2+2\right)} + \frac{2}{\pi}\frac{\alpha}{\sqrt{x^2+2}}\frac{-2x^2\alpha^2+2x^2+4}{\left(x^2\alpha^2+x^2+2\right)^2}}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}$$

By Lemma 2 we have

$$\frac{f'(x)}{f(x)} = \frac{-3x}{x^2+2} + \frac{\frac{4}{\pi}\frac{\alpha}{\sqrt{x^2+2}\left(x^2\alpha^2+x^2+2\right)} + \frac{2}{\pi}\frac{\alpha}{\sqrt{x^2+2}}\frac{-2x^2\alpha^2+2x^2+4}{\left(x^2\alpha^2+x^2+2\right)^2}}{1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}}.$$

Integrating the both sides of the above equation with respect to x, we obtain

$$f(x) = c\frac{1}{\left(x^2+2\right)^{\frac{3}{2}}}\left(1+\frac{2}{\pi}\arctan\frac{\alpha x}{\sqrt{x^2+2}}+\frac{2}{\pi}\frac{\alpha x\sqrt{x^2+2}}{(1+\alpha^2)x^2+2}\right).$$

Using the boundary condition $\int_{-\infty}^{\infty} f(x)dx = 1$, we have $c = 1$.

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