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An Improved Finite Element Method for Solving a Kind of Nonlinear Volterra-Fredholm Integral Equation

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***Abstract*—In this paper we extend the idea of interpolated coefficients for a kind of nonlinear Volterra-Fredholm integral equation to the improved finite element method. we introduce this numerical approximation method and Newton iterative scheme for this integral equation. Then we derive convergence estimate for exact solution and approximation solution of the equation.**

Keywords-component; nonlinear volterra-fredholm integral equations; finite element method; newton iterative method; convergence analysis

I. Introduction

Integral equations of Volterra-Fredholm are often involved in various fields such as physics, biology and engineering, and numerical methods for solving integral equations have been studied extensively in the literature, see [1-9] for details. The finite element method with interpolated coefficients is an economic and graceful method. This method was introduced and analyzed for semilinear parabolic problems in Zlamal [13]. Xiong at. al. put the excellent interpolating coefficients idea into the finite element method for solving nonlinear differential equations [10-12].

In this paper, we shall take the interpolating coefficients idea into numerical method of the integral equations and are concerned with the improved finite element method for solving a kind of mixed nonlinear Volterra-Fredholm integral equation as

$$u(x)=f(x)+\lambda_1\int_a^x k_1(x,y)F(u(y))\mathrm{d}y + \lambda_2\int_a^b k_2(x,y)G(u(y))\mathrm{d}y, \quad a\le x\le b, \tag{1}$$

where the function f, k_1, k_2 and F, G are known continuous functions defined on $[a,b], [ab]\times[a,b]$ and $\mathbf{R}$, respectively, $u(x)$ is the unknown function, $k_1, k_2 \in \mathbf{R}$ and the parameters satisfy

$$k_1^2+k_2^2\neq 0.$$

In this paper, for some results about the convergence and stability of the present numerical method, we will assume the following conditions

(i) $k_1, k_2 \in C^1([a,b])$ and $F, G\in C^1$.

(ii) $|k_1|\gamma_1 M_1 + |k_2|\gamma_2 M_2 \le \frac{1}{b-a}$, where

$$\gamma_1=\max|k_i|, \qquad i=1,2,$$

and

$$M_1=\max|F'(u)|, \qquad M_2=\max|G'(u)|.$$

II. Method of Solution

In this section, we give a new method to compute numerical solution of mixed nonlinear Volterra-Fredholm integral equation by improved finite element method. Firstly we let J_h be a partition of the interval $[a,b]$ such that

$$J_h: a=x_0<x_1<x_2<\cdots x_n=b.$$

Assume that J_h is quasi-uniform, i.e., there is a constant $C>0$ such that

$$h\le Ch_j=C(x_j-x_{j-1}), \qquad j=1,2,\cdots,n.$$

Then we give respectively the approximation functions for $u(x), F(u(x)), G(u(x))$ as follows

$$u(x) \approx u_h(x) = \sum_{j=0}^{n} u_j \varphi_j(x), \quad a \le x \le b, \tag{2}$$

$$F(u(x)) \approx F_h(u(x)) = \sum_{j=0}^{n} F(u_j)\varphi_j(x), \quad a \le x \le b, \tag{3}$$

$$G(u(x)) \approx G_h(u(x)) = \sum_{j=0}^{n} G(u_j)\varphi_j(x), \quad a \le x \le b, \tag{4}$$

where $\varphi_j(x)$ are node basis functions defined by

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \frac{x_{j=1} - x}{x_j - x_{j-1}}, & x \in [x_j, x_{j+1}], \\ 0, & \text{otherwise}, \end{cases} \quad j = 0,1,2\cdots n.$$

The node basis functions $\varphi_j(x)$ satisfy

$$\sum_{j=0}^{n} \varphi_j(x) = 1, \tag{5}$$

and

$$0 \le \varphi_j(x) \le 1, \qquad j = 0,1,2\cdots n. \tag{6}$$

Substitute (2)-(4) into (1) yields

$$u_h(x) = f(x) + \lambda_1 \int_a^x k_1(x,y) F_h(u(y))\mathrm{d}y + \lambda_2 \int_a^b k_2(x,y) G_h(u(y))\mathrm{d}y, \tag{7}$$

which implies

$$\sum_{j=0}^{n} u_j \varphi_j(x) = f(x) + \lambda_1 \sum_{j=1}^{n} F(u_j) \int_a^x k_1(x,y)\varphi_j(y)\mathrm{d}y + \lambda_2 \sum_{j=1}^{n} G(u_j) \int_a^b k_2(x,y)\varphi_j(y)\mathrm{d}y, \tag{8}$$

Taking $x = x_i, i = 1,2,\cdots,n$ in (8) respectively, we have a system of equations from functional integral equation

$$u_i = f(x_i) + \lambda_1 \sum_{j=1}^{i} F(u_j) \int_a^{x_i} k_1(x_i,y)\varphi_j(y)\mathrm{d}y + \lambda_2 \sum_{j=1}^{n} G(u_j) \int_a^b k_2(x_i,y)\varphi_j(y)\mathrm{d}y, \tag{9}$$

where $i = 1,2,\cdots,n$.

Letting

$$\mathbf{A} = \left(\int_a^{x_i} k_1(x_i,y)\varphi_j(y)\mathrm{d}y \right)_{(n+1)\times(n+1)},$$

$$\mathbf{B} = \left(\int_a^{b} k_2(x_i,y)\varphi_j(y)\mathrm{d}y \right)_{(n+1)\times(n+1)},$$

$$\mathbf{U} = (u_0, u_1, \cdots, u_n)^T,$$
$$\mathbf{f} = (f(x_0), f(x_1), \cdots, f(x_n))^T,$$

and

$$\mathbf{F(U)} = (F(u_0), F(u_1), \cdots, F(u_n))^T,$$
$$\mathbf{G(U)} = (G(u_0), G(u_1), \cdots, G(u_n))^T,$$

the nonlinear system of (9) is rewritten in the vector form as follows

$$\mathbf{H(U)} = \mathbf{U} - \mathbf{f} - \lambda_1 \mathbf{AF(U)} - \lambda_2 \mathbf{BG(U)} = \mathbf{0}. \tag{10}$$

From (10), we obtain corresponding Newton iterative algorithm scheme

$$\mathbf{U}^{k+1} = \mathbf{U}^k - [\mathbf{DH(U}^k\mathbf{)}]^{-1}\mathbf{H(U}^k\mathbf{)}, \quad k = 0,1,2,\cdots \tag{11}$$

where $\mathbf{D}$ denotes differential with respect to vector $\mathbf{U}$.

III. Convergence Analysis

This section we analyze the error of the finite element method. To start our analysis, first define discrete norm by

$$\|u(x)-u_h\|_{h,\infty} = \max_{0\le j\le n}|u(x_j)-u_j|.$$

For our convergence analysis, we need a lemma as follows.

Lemma 1. Assume that $a,b>0, 4ab<1$ *and* $x\ge 0$, $x\le b+ax^2$, *then* $x<ab$.

From the basic mathematical analysis, it is obvious that the lemma is established.

Now we state the convergence error estimate result as follow.

Theorem 1. Assume that $f(x)$ *is a function defined on* $[a,b]$ *, and that* $F(t),G(t)$ *in* $R=(-\infty,\infty)$ *and* $k_1(x,t),k_2(x,t)$ *in* $[a,b]\times[a,b]$ *are sufficiently smooth continuous and arbitrary differentiable,* $u(x)$ *is an exact solution of (1),* $u_h(x)$ *is the finite element solution of (9), then*

$$\begin{aligned}\|e\|_{h,\infty} &= \|u(x)-u_h(x)\|_{h,\infty}\\ &\le \frac{\frac{1}{4}(M_{1,2}+M_{2,2})h^2}{1-(|\lambda_1|\gamma_1 M_{1,1}+|\lambda_2|\gamma_2 M_{2,1})(b-a)}.\end{aligned} \tag{12}$$

Proof. Subtracting (7) from (1) and taking $x=x_i$ gives

$$\begin{aligned}e_i &= \lambda_1\int_a^{x_i} k_1(x_i,y)\left[F(u(y))-F_h(u(y))\right]\mathrm{d}y\\ &+\lambda_2\int_a^{b} k_2(x_i,y)\left[G(u(y))-G_h(u(y))\right]\mathrm{d}y,\end{aligned} \tag{13}$$

where $e_i=u(x_i)-u_i$. One can easily find that

$$\begin{aligned}&F(u(y))-F_h(u(y))\\ &=F(u(y))-F_n(u(y))+F_n(u(y))-F_h(u(y))\\ &=R_F+\sum_{j=0}^{n}\left[F(u(y_j))-F(u_j)\right]\varphi_j(y)\\ &=R_F+\sum_{j=0}^{n}\left[F'(u_j)e_j+\tfrac{1}{2}F''(\xi_j)e_j^2\right]\varphi_j(y),\end{aligned} \tag{14}$$

and similarly

$$\begin{aligned}&G(u(y))-G_h(u(y))\\ &=R_G+\sum_{j=0}^{n}\left[G'(u_j)e_j+\tfrac{1}{2}G''(\eta_j)e_j^2\right]\varphi_j(y),\end{aligned} \tag{15}$$

where

$$F_n(u(y))=\sum_{j=0}^{n}F(u(y_j)), \tag{16}$$

$$G_n(u(y))=\sum_{j=0}^{n}G(u(y_j)) \tag{17}$$

and

$$\begin{aligned}R_F &= F(u(y))-F_n(u(y)),\\ R_G &= G(u(y))-G_n(u(y))\end{aligned}$$

are the remainder of interpolation corresponding to the finite element, respectively. By use of interpolation polynomial error estimation, we have

$$|R_F|\le \tfrac{1}{8}M_{1,2}h^2,\qquad |R_G|\le\tfrac{1}{8}M_{2,2}h^2,$$

where

$$\begin{aligned}M_{1,2}&=\max|F''(u)|,\\ M_{2,2}&=\max|G''(u)|.\end{aligned}$$

Substituting (14)-(17) into (13), we find

$$\begin{aligned}|e_i| &\le \tfrac{1}{8}(M_{1,2}+M_{2,2})h^2\\ &+\sum_{j=1}^{n}[|\lambda_1|\gamma_1(M_{1,1}|e_j|+\tfrac{1}{2}M_{1,2}|e_j|^2)\\ &+|\lambda_2|\gamma_2(M_{2,1}|e_j|+\tfrac{1}{2}M_{2,2}|e_j|^2)]\int_a^b|\varphi_j(y)|\mathrm{d}y\\ &\le \tfrac{1}{8}(M_{1,2}+M_{2,2})h^2\\ &+(|\lambda_1|\gamma_1 M_{1,1}+|\lambda_2|\gamma_2 M_{2,1})(b-a)\|e\|_{h,\infty}\\ &+\tfrac{1}{2}(|\lambda_1|\gamma_1 M_{1,2}+|\lambda_2|\gamma_2 M_{2,2})(b-a)\|e\|_{h,\infty}^2.\end{aligned}$$

Using the conditions (i) and (ii), we get

$$\begin{aligned}\|e\|_{h,\infty} &\le \frac{\frac{1}{8}(M_{1,2}+M_{2,2})h^2}{1-(|\lambda_1|\gamma_1 M_{1,1}+|\lambda_2|\gamma_2 M_{2,1})(b-a)}\\ &+\frac{(|\lambda_1|\gamma_1 M_{1,2}+|\lambda_2|\gamma_2 M_{2,2})(b-a)}{2-2(|\lambda_1|\gamma_1 M_{1,1}+|\lambda_2|\gamma_2 M_{2,1})(b-a)}\|e\|_{h,\infty}^2.\end{aligned}$$

Application of Lemma 1 yields the desired estimate (12) and completes the proof of Theorem 1.

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