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Perturbed Iterative Algorithms for Split General Mixed Variational Inequality Problem

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***Abstract*—In this paper, we introduce a split general mixed variational inequality problem, which is a natural extension of a split variational inequality problem, split general quasi-variational inequality problem in Hilbert spaces. Using the resolvent operator technique, we propose two classes of perturbed iterative algorithms for the split general mixed variational inequality problem. Further, we discuss the convergence criteria of the iterative algorithms. The results presented here extend and improve many previously known results in this area.**

Keyword-split general mixed variational inequality problem; split general quasi-variational inequality problem; perturbed iterative algorithm;convergence

I. INTRODUCTION

In a recent paper [1], Kazmi has developed an iterative algorithm for finding approximate solution for a new split general quasi-variational inequality problem in Hilbert spaces. The aim of this work is to extend his idea to more general problem. Throughout the paper unless stated otherwise, for each $i \in \{1,2\}$, let H_i be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, let $f_i : H_i \to H_i, g_i : H_i \to H_i$ be continuous mappings with $\operatorname{Im} g_i \cap dom\varphi_i \neq \phi$, Let $A: H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. We consider the following problem: Find $x_1^* \in H_1$ such that $g_1(x_1^*) \in dom\varphi_1$ and

$$\left\langle f_1(x_1^*), x_1 - g_1(x_1^*) \right\rangle \geq \varphi_1(g_1(x_1^*)) - \varphi_1(x_1), \quad \forall x_1 \in H_1, \tag{1}$$

and such that $x_2^* = Ax_1^* \in H_2$, $g_2(x_2^*) \in dom\varphi_2$ solves

$$\left\langle f_2(x_2^*), x_2 - g_2(x_2^*) \right\rangle \geq \varphi_2(g_2(x_2^*)) - \varphi_2(x_2), x_2 \in H_2. \tag{2}$$

We call problem (1)-(2)the split general mixed variational inequality problem (in short, SpGMVIP). SpGMVIP(1)-(2) amounts to saying: find a solution of general mixed variational inequality problem (1) image under a given bounded linear operator is a solution of general mixed variational inequality problem (2). For convenience, we denote the solution set of SpGMVIP(1)-(2) by

$$\Gamma = \left\{ x_1^* \in H_1 \middle| x_1^* \, solves (1) \; and \; Ax_1^* \in H_2 \; solves \; (2) \right\}.$$

Next, we give some special cases of SpGMVIP (1)-(2).

If we set $g_i = I_i$, where I_i is an identity operator on H_i, then SpGMVIP(1)-(2) is reduced to the following split mixed variational inequality problem (In short, SpMVIP): Find $x_1^* \in H_1$ such that

$$\left\langle f_1(x_1^*), x_1 - x_1^* \right\rangle \geq \varphi_1(x_1^*) - \varphi_1(x_1), \forall x_1 \in H_1, \tag{3}$$

and such that $x_2^* = Ax_1^* \in H_2$ solves

$$\left\langle f_2(x_2^*), x_2 - x_2^* \right\rangle \geq \varphi_2(x_2^*) - \varphi_2(x_2), \forall x_2 \in H_2, \tag{4}$$

which appears to be new one.

If we set $\varphi_i(\cdot) = \delta_{C_i}(\cdot - m_i(x_i^*)) = \delta_{C_i + m_i(x_i^*)}(\cdot)$, $m_i : H_i \to H_i$ is a single-valued mapping, where $C_i(x_i^*) = C_i + m_i(x_i^*)$, and C_i is a closed convex subset of H_i, then SpMVIP (1)-(2) is reduced to the following split general quasi-variational inequality problem (in short, SpGQVIP): Find $x_1^* \in H_1$, such that $g_1(x_1^*) \in C_i(x_1^*)$ and

$$\left\langle f_1(x_1^*), x_1 - g_1(x_1^*) \right\rangle \geq 0, \forall x_1 \in C_1(x_1^*), \tag{5}$$

and such that $x_2^* = Ax_1^*$, $g_2(x_2^*) \in C_2(x_2^*)$ solves

$$\left\langle f_2(x_2^*), x_2 - g_2(x_2^*) \right\rangle \geq 0, \forall x_2 \in C_2(x_2^*). \tag{6}$$

This problem was introduced and studied by Kazmi in [1] and he exhibited split quasi-variational inequality problem, split general variational inequality problem and quasi-variaeional inequality problem as special cases of SpGQVIP (5)-(6). For details, see reference[1].

If $\varphi_i = \delta_{C_i}$ the indicator function of a closed convex set $C_i \subset H_i$, $g_i = I_i$ the identity mapping H_i, then SpGMVIP (1)-(2) is reduced to the following split general variational inequality problem (in short, SpVIP): Find $x_1^* \in H_1$,such that

$$\left\langle f_1(x_1^*), x_1 - x_1^* \right\rangle \geq 0, \forall x_1 \in C_1, \tag{7}$$

and $x_2^* = Ax_1^* \in H_2$ solves

$$\left\langle f_2(x_2^*), x_2 - x_2^* \right\rangle \geq 0, \forall x_2 \in C_2, \tag{8}$$

which has been introduced and studied by Censor, Gibali and Reich[2]. It is worth noting that SpGMVIP(1)-(2) is quite general and includes as special cases split minimization between two spaces so that the image of a minimize of a given function, under a bounded linear operator, is a minimizer of a given function, under a bounded linear operator, is a minimizer of another function, split zero problem and the split feasibility problem which have already been studied and used in practice as a model in the intensity-moducated radiation therapy planning, see[3, 4, 5].

In a word, SpGMVIP is more general, which is one of our motivations to write this paper. By using the resolvent operator technique about the maximal monotone mapping, we propose two classes of perturbed iterative algorithms taking into account a possible in exact computation for SpGMVIP (1)-(2) and discuss the convergence criteria of these iterative algorithms. The results presented here extend and improve the previously known results in this area.

II. PERTURBED ITERATIVE ALGORITHMS

To begin with, let us transform SpGMVIP (1)-(2) into fixed point problems.

Lemma 2.1. $x_1^* \in \Gamma$ if and only if x_1^* satisfies the following relations

$$g_1(x_1^*) = J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)), \tag{9}$$

$$g_2(Ax_1^*) = J_{\rho_2}^{\partial\varphi_2}(g_2(Ax_1^*) - \rho_2 f_2(Ax_1^*)), \tag{10}$$

where $\rho_i > 0$ is a constant and $J_{\rho_i}^{\partial\varphi_i} := (I + \rho_i \partial\varphi_i)^{-1}$ is the resolvent operator of the maximal monotone mapping $\partial\varphi_i$,

noting that $\partial\varphi_i$ denotes the subdifferential of a proper, convex and lower semi-continuous function $\varphi_i : H \to R \cup \{+\infty\}$.

Proof From definition of $J_{\rho_i}^{\partial\varphi_i}$, It follows from (9)that

$$g_1(x_1^*) - \rho_1 f_1(x_1^*) \in g_1(x_1^*) + \rho_1 \partial\varphi_1(g_1(x_1^*)),$$

then $-f_1(x_1^*) \in \partial\varphi_1(g_1(x_1^*))$, definition of $\partial\varphi$ implies

$$\varphi_1(x_1) \ge \varphi_1(g_1(x_1^*)) + \langle -f_1(x_1^*), x_1 - g_1(x_1^*) \rangle, \forall x_1 \in H_1.$$

this is,

$$\langle f_1(x_1^*), x_1 - g_1(x_1^*) \rangle \ge \varphi_1(g_1(x_1^*)) - \varphi_1(x_1), \forall x_1 \in H_1.$$

thus x_1^* is a solution of (1). Similarly, it is easy to know Ax_1^* solves(2), hence $x_1^* \in \Gamma$. The converse relation is obvious, so is omitted, completing the proof.

Based on Lemma 2.1, we can propose the following perturbed iterative algorithms for approximating a solution to SpGMVIP(1)-(2). Let $\{\alpha^n\} \subseteq (0,1)$ be a sequence such that $\sum_{n=1}^{\infty} \alpha^n = \infty$ and let ρ_1, ρ_2, γ be the parameters with positive values.

Algorithm 2.1. Given $x_1^0 \in H_1$, compute the iterative sequence $\{x_1^n\}$ defined by the iterative schemes

$$g_1(y^n) = J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^n) - \rho_1 f_1(x_1^n)), \tag{11}$$

$$g_2(z^n) = J_{\rho_2}^{\partial\varphi_2^n}(g_2(Ay^n) - \rho_2 f_2(Ay^n)), \tag{12}$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)] + \alpha^n e^n, \tag{13}$$

for all $n = 0,1,2,\cdots, \rho_1, \rho_2, \gamma > 0$ and take into account a possible inexact computation, an $\|e^n\| \to 0 (n \to \infty)$. error e^n is added in the right hand side of(13) withMoreover, we consider other perturbations by replacing in (11) and (12) φ_i by φ_i^n ,where the sequence $\{\varphi_i^n\}$ approximates φ_i , $\{\varphi_i^n\}$ is a collection of proper convex semi-continuous functions on H_i.

If $\varphi_i(\cdot) = \delta_{C_i(x_i^*)}(\cdot)$, where $C_i(x_i^*)$ is same as the above, $e^n = 0$, then Algorithm 2.1 is reduced to the following algorithm for SpGQVIP:

Algorithm 2.2. Given $x_1^0 \in H_1$, compute the iterative sequence $\{x_1^n\}$ defined by the iterative schemes

$$g_1(y^n) = P_{C_1(x_1^n)}(g_1(x_1^n) - \rho_1 f_1(x_1^n)), \tag{14}$$

$$g_2(z^n) = P_{C_2(Ay^n)}(g_2(Ay^n) - \rho_2 f_2(Ay^n)), \tag{15}$$

$$x_1^{n+1} = (1 - \alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)], \tag{16}$$

for all $n = 0,1,2,\cdots, \rho_1, \rho_2, \gamma > 0$, where P_{C_i} is the metric projection of H_i on to C_i, and it is well known that P_{C_i} is a nonexpansive mapping. Algorithm 2.2 was proposed by Kazmi [1] for SpGQVIP.

Observe that (9)and (10) can change into the following :

$$x_i^* = x_i^* - g_i(x_i^*) + J_{\rho_i}^{\partial\varphi_i}(g_i(x_i^*) - \rho_i f_i(x_i^*)), i = 1,2,$$

where $x_2^* = Ax_1^*$, $\rho_i > 0$ is a constant. In view of the above equations, we can propose another perturbed iterative algorithm for SpGMVIP.

*A*lgorithm 2.3. Given $x_1^0 \in H$, compute the iterative sequence $\{x_1^n\}$ defined by the iterative schemes

$$y^n = x_1^n - g_1(x_1^n) + J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^n) - \rho_1 f_1(x_1^n)), \tag{17}$$

$$z^n = Ay^n - g_2(y^n) + J_{\rho_2}^{\partial\varphi_2^n}(g_2(Ay^n) - \rho_2 f_2(Ay^n)), \tag{18}$$

$$x_1^{n+1} = (1-\alpha^n)x_1^n + \alpha^n[y^n + \gamma A^*(z^n - Ay^n)] + \alpha^n e^n, \tag{19}$$

for all $n = 0,1,2,\cdots,\rho_1,\rho_2,\gamma > 0$, e^n is an error term and $\|e^n\| \to 0 (n \to \infty)$.

In order to obtain our main results, we need the following definition, Assumption and lemmas.

Definition 2.1. A nonlinear mapping $f: H_1 \to H_1$ is said to be

(i) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle f(x) - f(y), x-y \rangle \geq \alpha \|x-y\|^2, \forall x, y \in H_1.$$

(ii) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\| f(x) - f(y) \| \leq \beta \|x-y\|, \forall x, y \in H_1.$$

Remark 2.1. It is easy to know that if $f: H_1 \to H$ is α-strongly monotone and β-Lipschitz continuous then $\alpha \leq \beta$.

Assumption 2.2. *For* $i \in \{1,2\}$, *let* $\varphi_i : H_i \to \overline{R} = R \cup \{+\infty\}$ *be a proper, convex and lower semi-continuous function,* $\{\varphi_i^n\}$ *approximate* φ_i *and satisfies the condition:*

$$\lim_{n\to\infty} \left\| J_{\rho_i}^{\partial\varphi_i^n}(v_i) - J_{\rho_i}^{\partial\varphi_i}(v_i) \right\| = 0, \forall v_i \in H_i.$$

Lemma 2.3([6]). *Let* $\{a_k\}$ *be a sequence of nonnegative real numbers satisfying the condition*

$$a_{k+1} \leq (1-m_k)a_k + m_k \delta_k, \forall k \geq 0.$$

where $\{m_k\}$, $\{\delta_k\}$ are sequences of real numbers such that

(i) $\{m_k\} \subset [0,1]$ and $\sum_{k=0}^{\infty} m_k = \infty$, or, equivalently,

$$\prod_{k=0}^{\infty}(1-m_k) := \lim_{k\to\infty}\prod_{j=0}^{k}(1-m_j) = 0;$$

(ii) $\limsup_{k\to\infty} \delta_k \leq 0$, or (ii)' $\sum_{k=0}^{\infty} \delta_k m_k$ is convergent.

Then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.4. *Let* H *be a real Hilbert space, for all* $x, y \in H$, *the following hold:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle,$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2.$$

III. Main Results

Theorem 3.1. For each $i \in \{1,2\}$, *let* $g_i : H_i \to H_i$ *be* δ_i *-Lipschitz continuous such that* $(g_i - I_i)$ *is* δ_i *-strongly monotone, where* I_i *is the identity operator on* H_i. *Let* $f_i : H_i \to H_i$ *be* α_i *-strongly monotone with respect to* g_i *and* β_i *-Lipschitz continuous. Let* $A: H_1 \to H_2$ *be a bounded linear operator and let* A^* *be its adjoint operator. Suppose* $x_1^* \in \Gamma$ *and Assumption 2.2 holds. Then the sequence* $\{x_1^n\}$ *generated by Algorithm 2.1 converges strongly to* x_1^* *provided that the constant* ρ_i *and* γ *satisfy the conditions*

$$\left|\rho_1 - \frac{\alpha}{\beta_1^2}\right| < \frac{\sqrt{\tau_1^2 + \alpha_1^2 - \delta_1^2}}{\beta_1^2}, \quad \delta_1 < \sqrt{\tau_1^2 + \alpha_1^2},$$

$$\gamma \|A\|^2 \theta_1\theta_2 < 1 - \theta_1, \gamma \in (0, \frac{2}{\|A\|^2}),$$

$$\theta_i = \tau_i \sqrt{\delta_i^2 - 2\rho_i\alpha_i + \rho_i^2\beta_i^2}, \tau_i = \frac{1}{\sqrt{1+2\sigma_i}}, \rho_i > 0, i = 1,2.$$

Proof Since $x_1^* \in \Gamma$, *then* $x_i^* \in H_i$ *such that* $g_i(x_i^*) \in dom\varphi_i(g_i(x_i))$ and

$$g_1(x_1^*) = J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)), \tag{20}$$

$$g_2(Ax_1^*) = J_{\rho_2}^{\partial\varphi_2}(g_2(Ax_1^*) - \rho_2 f_2(Ax_1^*)), \tag{21}$$

for $\rho_i > 0$ and $x_2^* = Ax_1^*$. From Algorithm 2.1(11), Assumption 2.2 and (20), we have

$$\begin{aligned}
&\|g_1(y^n) - g_1(x_1^*)\| \\
&= \left\| J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^n) - \rho_1 f_1(x_1^n)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) \right\| \\
&\leq \left\| J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^n) - \rho_1 f_1(x_1^n)) - J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) \right\| \\
&\quad + \left\| J_{\rho_1}^{\partial\varphi n_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) \right\| \\
&\leq \left\| g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*)) \right\| + \varepsilon_1^n,
\end{aligned} \tag{22}$$

where $\varepsilon_1^n = \left\| J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) \right\|$ and $\lim_{n\to\infty} \varepsilon_1^n = 0$ owns to Assumption 2.2. Now, using the facts that f_1 is α_1-strongly monotone with respect to g_1 and β_i-Lipschitz continuous, and g_1 is δ_i-Lipschitz continuous,we have

$$\begin{aligned}
&\left\| g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*)) \right\|^2 = \left\| g_1(x_1^n) - g_1(x_1^*) \right\|^2 \\
&- 2\rho_1 \langle f_1(x_1^n) - f_1(x_1^*), g_1(x_1^n) - g_1(x_1^*) \rangle + \rho_1^2 \left\| f_1(x_1^n) - f_1(x_1^*) \right\|^2 \\
&\leq (\delta_1^2 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2) \left\| x_1^n - x_1^* \right\|^2.
\end{aligned} \tag{23}$$

Combing (22)and(23), we have

$$\|g_1(y^n)-g_1(x_1^*)\|\le\sqrt{\delta_1^2-2\rho_1\alpha_1+\rho_1^2\beta_1^2}\,\|x_1^n-x_1^*\|+\varepsilon_1^n. \quad (24)$$

Since (g_1-I_i) is σ_1-strongly monotone, we have

$$\|y^n-x_1^*\|^2\le\|g_1(y^n)-g_1(x_1^*)\|^2-2\langle(g_1-I_1)y^n-(g_1-I_1)x_1^*,y^n-x_1^*\rangle$$
$$\le\|g_1(y^n)-g_1(x_1^*)\|^2-2\sigma_1\|y^n-x_1^*\|^2,$$

which implies

$$\|y^n-x_1^*\|\le\tau_1\|g_1(y^n)-g_1(x_1^*)\|, \quad (25)$$

where $\tau_1=\frac{1}{\sqrt{1+2\sigma_1}}$. From(24)and(25), we get

$$\|y^n-x_1^*\|\le\theta_1\|x_1^n-x_1^*\|+\tau_1\varepsilon_1^n, \quad (26)$$

where $\theta_1=\tau_1\sqrt{\delta_1^2-2\rho_1\alpha_1+\rho_1^2\beta_1^2}$. Similarly, from Algorithm 2.1(12), Assumption 2.2 and (21) and using the facts that f_2 is α_2-strongly monotone with respect to g_2 and β_2-Lipschitz continuous, (g_2-I_2) is σ_2-strongly monotone, and g_2 is δ_2-Lipschitz continuous, we have

$$\|g_2(z^n)-g_2(Ax_1^*)\|\le\sqrt{\delta_2^2-2\rho_2\alpha_2+\rho_2^2\beta_2^2}\,\|Ay^n-Ax_1^*\|, \quad (27)$$

and

$$\|z^n-Ax_1^*\|\le\theta_2\|Ay^n-Ax_1^*\|+\tau_2\varepsilon_2, \quad (28)$$

where

$$\tau_2=\frac{1}{\sqrt{2\sigma_2+1}},\quad \theta_2=\tau_2\sqrt{\delta_2^2-2\rho_2\alpha_2+\rho_2^2\beta_2^2},$$

$$\varepsilon_2^n=\left\|J_{\rho_2}^{\partial\varphi_2^n}(g_2(Ax_1^*)-\rho_2f_2(Ax_1^*))-J_{\rho_2}^{\partial\varphi_2}(g_2(Ax_1^*)-\rho_2f_2(Ax_1^*))\right\|,$$

and $\lim_{n\to\infty}\varepsilon_2^n=0$ owns to Assumption 2.2. From Algorithm 2.1(13), we obtain

$$\|x_1^{n+1}-x_1^*\| \quad (29)$$
$$\le(1-\alpha^n)\|x_1^n-x_1^*\|+\alpha^n\left[\|y^n-x_1^*-\gamma A^*(Ay^n-Ax_1^*)\|+\gamma\|A^*(z^n-Ax_1^*)\|\right]+\alpha^n\|e^n\|.$$

Further, using the definition of A^*, the face that A^* is a bounded linear operator with $\|A^*\|=\|A\|$, and the given condition on γ, we have

$$\|y^n-x_1^*-\lambda A^*(Ay^n-Ax_1^*)\|^2 \quad (30)$$
$$=\|y^n-x_1^*\|^2-2\gamma\langle y^n-x_1^*,A^*(Ay^n-Ax_1^*)\rangle+\gamma^2\|A^*(Ay^n-Ax_1^*)\|^2$$
$$\le\|y^n-x_1^*\|^2-\gamma(2-\gamma\|A\|^2)\|Ay^n-Ax_1\|^2\le\|y^n-x_1^*\|^2,$$

and, using(3.9), we have

$$\|A^*(z^n-Ax_1^*)\|\le\|A\|\|z^n-Ax_1^*\|\le\theta_2\|A\|\|Ay^n-Ax_1^*\|+\|A\|\tau_2\varepsilon_2^n \quad (31)$$
$$\le\theta_2\|A\|^2\|y^n-x_1^*\|+\|A\|\tau_2\varepsilon_2^n.$$

It follows from (29)-(31), we obtain

$$\|x_1^{n+1}-x_1^*\|$$
$$\le(1-\alpha^n)\|x_1^n-x_1^*\|+\alpha^n[(1+\gamma\|A\|^2\theta_2)\|y^n-x_1^*\|]+\alpha_n\gamma\|A\|\tau_2\varepsilon_2^n+\alpha^n\|e^n\|$$
$$\le(1-\alpha^n)\|x_1^n-x_1^*\|+\alpha^n\theta_1(1+\gamma\|A\|^2\theta_2)\|x_1^n-x_1^*\|$$
$$+\alpha^n[(1+\gamma\|A\|^2\theta_2)\tau_1\varepsilon_1^n+\gamma\|A\|\tau_2\varepsilon_2^n+\|e^n\|]$$
$$=\left[1-\alpha^n(1-\theta)\right]\|x_1^n-x_1^*\|+\alpha^n\left[(1+\gamma\|A\|^2\theta_2)\tau_1\varepsilon_1^n+\gamma\|A\|\tau_2\varepsilon_2^n+\|e^n\|\right] \quad (32)$$

where $\theta=\theta_1(1+\gamma\|A\|^2\theta_2)$. It follows from the conditions on ρ_1, ρ_2 and γ that $\theta\in(0,1)$. Letting

$$a_n=\|x_1^n-x_1^*\|, m_n=\alpha^n(1-\theta),$$

$\delta_n=\frac{1}{1-\theta}[(1+\gamma\|A\|^2\theta_2)\tau_1\varepsilon_1^n+\gamma\|A\|\tau_2\varepsilon_2^n+\|e^n\|],\forall n\ge0.$ By virtue of (3.13), we have $a_{n+1}\le(1-m_n)a_n+m_n\delta_n$. Moreover the conditions (i) and (ii) of Lemma 3.1 are satisfied. It follows that $\{x_1^n\}$ converges strongly to x_1^* as $n\to\infty$.

Since A is continuous, it follows from (25), (26)-(28) that $y^n\to x_1^*$, $g_1(y^n)\to g_1(x_1^*)$, $Ay^n\to Ax_1^*$, $z^n\to Ax_1^*$ and $g_2(z^n)\to g_2(Ax_1^*)$ as $n\to\infty$. This completes the proof.

In the following, we consider the convergence of Algorithm 2.3 for SpGMVIP.

Theorem 3.2. For each $i\in\{1,2\}$, let $g_i:H_i\to H_i$ be σ_i-strongly monotone and δ_i-Lipschitz continuous. Let $f_i:H_i\to H_i$ be α_i-strongly monotone with respect to g_i and β_i-Lipschitz continuous. Let $A:H_1\to H_2$ be a bounded linear operator and A^* be its adjoint operator. Suppose $x_1^*\in\Gamma$ and Assumption 2.2 holds. Then the sequence $\{x_1^n\}$ generated by Algorithm 2.3 converges strongly to x_1^* provided that the constants ρ_i and γ satisfy the conditions

$$\left|\rho_1-\frac{\alpha_1}{\beta_1^2}<\frac{\sqrt{\alpha_1^2-\beta_1^2k_1(2-k_1)}}{\beta_1^2}\right|,\quad \alpha_1>\beta_1\sqrt{k_1(2-k_1)},\quad k_1<1,$$

$$k_i=\sqrt{1-2\sigma_i+\delta_i^2},\ i=1,2,\ \gamma\|A\|^2\theta_1\theta_2<1-\theta,\ \gamma\in(0,\frac{2}{\|A\|^2}),$$

$$\theta_i=k_i+\sqrt{1-2\rho_i\alpha_i+\rho_i^2\beta_i^2},\quad \rho_i>0,\ i=1,2.$$

Proof Since $x_1^*\in\Gamma$, then $x_1^*\in H_1$ is such that $g_i(x_i^*)\in dom\varphi_i$ and

$$x_1^*=x_1^*-g_1(x_1^*)+J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*)-\rho_1f_1(x_1^*)), \quad (33)$$

$$Ax_1^* = Ax_1^* - g_2(Ax_1^*) + J_{\rho_2}^{\partial\varphi_2}(g_2(Ax_1^*) - \rho_2 f_2(Ax_1^*)), \quad (34)$$

for $\rho_i > 0$. From Algorithm 2.3(19), Assumption2.2and（33）, we have

$$\begin{aligned}&\|y^n - x_1^*\|\\ &= \|x_1^n - x_1^* - (g_1(x_1^n) - g_1(x_1^*)) + J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^n) - \rho_1 f_1(x_1^n)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*))\|\\ &\le \|x_1^n - x_1^* - (g_1(x_1^n) - g_1(x_1^*))\| + \|g_1(x_1^n) - g_1(x_1^*) - \rho_1(f_1(x_1^n) - f_1(x_1^*))\|\\ &+ \|J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*))\|.\end{aligned} \quad (35)$$

Note that g_1 is σ_1-strongly monotone and δ_1-Lipschitz continuous, and f_1 is α_1-strongly monotone with respect to g_1 and β_1-Lipschitz continuous, we have

$$\|x_1^n - x_1^* - (g_1(x_1^n) - g_1(x_1^*))\| \le k_1\|x_1^n - x_1^*\|, \quad (36)$$

where $k_1 = \sqrt{1 - 2\sigma_1 + \delta_1^2}$,

$$\|g_1(x_1^n) - g_1(x_1^*) - \rho_1 f_1(x_1^n) -_1 f_1(x_1^*)\| \le \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2}\|x_1^n - x_1^*\|, \quad (37)$$

from (33)-(35), we have

$$\|y^n - x_1^n\| \le \theta_1\|x_1^n - x_1^*\| + \varepsilon_1^n, \quad (38)$$

where

$$\theta_1 = k_1 + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2},$$
$$\varepsilon_1^n = \|J_{\rho_1}^{\partial\varphi_1^n}(g_1(x_1^*) - \rho_1 f_1(x_1^*)) - J_{\rho_1}^{\partial\varphi_1}(g_1(x_1^*) - \rho_1 f_1(x_1^*))\|.$$

Similary ,from Algorithm 2.3(18), Assumption 2.2 and (34),and using the fact that f_2 is α_2-strongly monotone with respect to g_2 and β_2-Lipschitz continuous, g_2 is σ_2-strongly monotoneand δ_2-Lipschitz continuous, we have

$$\|z^n - Ax_1^n\| \le \theta_2\|Ay^n - Ax_1^n\| + \varepsilon_2^n, \quad (39)$$

where $\theta_2 = k_2 + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2}$, $k_2 = \sqrt{1 - 2\sigma_2 + \delta_2^2}$,

$$\varepsilon_2^n = \|J_{\rho_2}^{\partial\varphi_2^n}(g_2(Ax_1^*) - \rho_2 f_2(Ax_1^*)) - J_{\rho_2}^{\partial\varphi_2}(g_2(Ax_1^*) - \rho_2 f_2(Ax_1^*))\|.$$

Combining (29)-(31), (38), (39), we obtain

$$\begin{aligned}&\|x_1^{n+1} - x_1^*\|\\ &\le [1 - \alpha^n(1-\theta)]\|x_1^n - x_1^*\| + \alpha^n[(1 + \gamma\|A\|^2\theta_2)\varepsilon_1^n + \gamma\|A\|\varepsilon_2^n + \|e^n\|],\end{aligned} \quad (40)$$

where $\theta = \theta_1 + \gamma\|A\|^2\theta_1\theta_2$. It follows from the conditions on ρ_1, ρ_2 and γ that $\theta \in (0,1)$. Letting

$$a_n = \|x_1^n - x_1^*\|, \ m_n = \alpha^n(1-\theta),$$

$$\delta_n = \frac{1}{1-\theta}[(1 + \gamma\|A\|^2\theta_2)\varepsilon_1^n + \gamma\|A\|\varepsilon_2^n) + \|e^n\|], \ \forall n \ge 0.$$

Then (40) implies

$$a_{n+1} \le (1 - m_n)a_n + m_n\delta_n, \ \forall n \ge 0.$$

Noting the conditions (i) and (ii) of Lemma 3.1 are satisfied and it follows that $\{x_1^n\}$ converges strongly to x_1^* as $n \to \infty$. The rest of argument is same as in Theorem 3.3, so is omitted, which is completed the proof.

Remark 3.3. Algorithm 2.2 is a special case of Algorithm 2.1 and Theorem 3.1 extends the corresponding results in [1].

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