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Smooth Test for Elliptical Copulas

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***Abstract*—Based on the Cholesky decomposition and spherical harmonics, we propose the smooth test for testing the elliptical copulas. The asymptotic null distribution of the transformed sample is obtained. An algorithm is given to estimate the p-value of the test statistic by Monte Carlo simulation. The maximum goodness-of-fit estimation can be used to estimate the copula parameters. An example is provided in order to illustrate the smooth test for the multivariate t- copulas.**

Keywords- elliptical copulas; spherical harmonics; smooth test; p-value; t- copulas

I. INTRODUCTION

Copula-based time series models have found many successful applications of late, notably in modern finance and insurance, where data often exhibit heavy-tail dependence. One attractive property of copulas is that the copulas are invariant under strictly increasing transformations of each coordinate. Fang et al. use elliptical copula to construct the class of meta-elliptical distributions with arbitrary given margins[1]. Goodness-of-fit tests for copulas are of importance since different copulas lead to multivariate time series models that may have very different dependence properties. Based on the empirical marginals, the testing procedure for the Gaussian copula hypothesis was suggested. It was showed that the Gaussian copula was inadequate to describe the dependence between assets[2]. Based on the kernel estimator of a copula, the chi-square test for copulas was presented [3].

In this paper, we will assume that the variables of interest are of the continuous type and the marginal distributions can be determined by Goodness-of-fit tests.

Let $Y=(Y_1,Y_2,\cdots,Y_d)^T$ be a random vector with cumulative distribution function(cdf) F and let F_i denote the marginal cdf of Y_i for $i=1,\cdots,n.$ Then Sklar's Theorem states that there exists a copula $C(\cdot,\cdots,\cdot):\ [0,1]^d \to [0,1]$ such that

$$F(y)=C(F_1(y_1),\cdots,F_d(y_d)), \qquad (1)$$

for all $y=(y_1,\cdots,y_d)^T \in R^d$.

When $F_1,F_2,\cdots,F_d$ are all continuous, $C=C(\cdot,\cdots,\cdot)$ is unique and C can be obtained by the formula

$$C(u_1,\cdots,u_d)=F(F_1^{-1}(u_1),\cdots,F_d^{-1}(u_d)), \qquad (2)$$

where F_i^{-1} denoted the inverse of the cdf F_i. Conversely, if C is a $d-$dimensional copula and $F_1,F_2,\cdots,F_d$ are all univariate, then the function F defined by (1) is a $d-$dimensional cdf with margins $F_1,F_2,\cdots,F_d$.

Let Ω_d denote the surface of a unit sphere centered at the origin in R^d and let $U(\Omega_d)$ denote the uniform distribution on Ω_d. The asymptotic null distribution of the transformed sample is $U(\Omega_d)$. Therefore, the goodness-of-fit test for elliptical copula can be translated into the goodness-of-fit test for the uniform distribution on Ω_d.

Based on the smooth test for $U(\Omega_d)$, We propose a new test for elliptical copula. The transformation based on Cholesky decomposition leads to the transformed sample whose joint distribution does not depend on the unknown parameter R of the elliptical copula. Thus, the $p-$value of the test statistic can be estimated by Monte Carlo method with the matrix $R=I_d$, where I_d denotes the $d\times d$ identity matrix.

The paper is organized as follows. In Section 2, we introduce definitions and some lemmas. In Section 3, the smooth test for elliptical copula is proposed. The asymptotic null distribution of the test statistic is obtained and the algorithm to estimate the $p-$ value is given. Some conclusions and further applications are given in Section 4. The proofs are presented in Appendix.

II. DEFINITIONS AND SOME LEMMAS

A. *Elliptical Copulas And Meta-elliptical Distributions*

Definition1[4, 10]. The $d\times 1$ random vector X is said to have a spherically symmetric distribution if

$$X \overset{d}{=} \xi U^{(d)}, \qquad (3)$$

where $\xi \geq 0$ is a random variable, $U^{(d)}$ is uniformly distributed on Ω_d and is independent of ξ. Here $\overset{d}{=}$ signifies that the two sides have the same distribution.

Remark1 The density of X in (3) is of the form $\varphi(x^T x)$. The function $\varphi(\cdot)$ of a scalar variable is called the density generator of X.

Definition2[1,4]. Let A be a $d \times d$ nonsingular matrix, $\Sigma = A^T A$ with rank $(\Sigma) = d$. The $d \times 1$ random vector Y is said to have an elliptical distribution with parameters μ and Σ if

$$Y \overset{d}{=} \mu + \xi A^T U^{(d)} \overset{d}{=} \mu + A^T X, \tag{4}$$

where X, ξ and $U^{(d)}$ are as in (3). We shall use the notation $Y \sim E_d(\mu, \Sigma, \varphi)$.

Remark2 When $\varphi(\upsilon) = (2\pi)^{-d/2} \exp(-\upsilon/2)$, Y has a $d-$dimensional normal distribution. When

$$\varphi(\upsilon) = \frac{\Gamma((d+k)/2)}{(\pi k)^{d/2}\Gamma(k/2)}(1+\upsilon/k)^{-(d+k)/2}, \tag{5}$$

Y has a $d-$dimensional $t-$distribution and is denoted as $Y \sim Mt_d(k, \mu, \Sigma)$. In particular, all the univariate marginal distributions of the $Mt_d(k, \mu, \Sigma)$ distribution are still $t-$distribution with degrees of freedom k.

Lemma1[4, 10]. If $Y \sim E_d(0, \Sigma, \varphi)$ and $\mathrm{E}(\xi^2) < \infty$,then

$$E(Y) = 0, \ \mathrm{Cov}(Y) = \frac{E(\xi^2)}{d}\Sigma. \tag{6}$$

Definition3[1] Let $Y = (Y_1, \cdots, Y_d)^T \sim E_d(0, R, \varphi)$,where $R = (\rho_{ij})_{d\times d}$ is a positive matrix with

$$\rho_{ij} = 1, i = j, \ -1 < \rho_{ij} < 1, \ \ i \neq j, \ \ \ i, j = 1, \cdots, d.$$

The copula C_φ of Y is called the elliptical copula. When φ is defined by (5), the copula of Y is called the multivariate $t-$ copula[5].

Remark3 The copula is invariant under strictly increasing transformations of the margins. Thus, Σ in $E_d(0, \Sigma, \varphi)$ can be replaced by R.

Lemma2[4, 10] If $Y = (Y_1, \cdots, Y_d)^T \sim E_d(0, R, \varphi)$, P is a $d \times d$ matrix, then

$$PY \sim E_d(0, PRP^T, \varphi). \tag{7}$$

Lemma3[1] If $Y = (Y_1, \cdots, Y_d)^T \sim E_d(0, R, \varphi)$ then all the marginal distributions of Y are identical with the same probability density function (pdf) $q_\varphi(\cdot)$ and the cdf $Q_\varphi(\cdot)$.

Definition 4[1] Let $Z = (Z_1, \cdots, Z_d)^T$ be a random vector with each component Z_i having a given continuous density $g_i(z_i)$ and the cdf $G_i(z_i)$. Let C_φ be the elliptical copula.

Z is said to have a meta-elliptical distribution, if its cdf F_Z is given by

$$F_Z(z_1, \cdots, z_d) = C_\varphi(G_1(z_1), \cdots, G_d(z_d)). \tag{8}$$

Denote $Z \sim ME_d(0, R, \varphi, G_1, \cdots, G_d)$.

The dependence structure of meta-elliptical distributions is represented by elliptical copulas.

B. Smooth Alternatives

Let $s = (s_1, s_2, \cdots, s_d)^T$ denotes a typical point in R^d. For $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_d)^T$ a multi-index, define

$$s^\alpha = s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_d^{\alpha_d}, \ D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_d^{\alpha_d}, \tag{9}$$

where $D_j^{\alpha_j}$ denotes the α_j^{th} partial derivative with respect to the j^{th} coordinate variable. The collection of all spherical harmonics of degree m will be denoted by $H_m(\Omega)$.

Lemma4 [6] Let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. If $d > 2$ then the set $\{D^\alpha \|s\|^{2-d} : |\alpha| = m \text{ and } \alpha_1 \leq 1\}$

is a vector space basis of $H_m(\Omega)$, where D^α is defined in (9) and $\|s\|$ denotes the Euclidean norm of s.

Lemma5[4, 10] If An $d \times 1$ random vector ς has a spherical distribution then

$$\varsigma / \|\varsigma\| \sim U(\Omega_d) \tag{9}$$

Lemma6[7] Let $N_{k,d} = \dim[H_k(\Omega)]$. Let $B_k = \{V_{k,j}(w) \in H_k(\Omega),\ j = 1,2,\cdots,N_{k,d}\}$

be a complete orthonormal basis(CONB) for $H_k(\Omega)$. Let $B = \{B_k : k = 0,1,2\}$. Then B is a set of orthonormal functions.

Let $\wedge = B \setminus B_0$ and let us denote $N = \#(\wedge)$, we have

$$N = d + N_{2,d}, \tag{10}$$

where # denotes cardinality. The elements of $\wedge$ are arranged with $k = 1,2$. The set $\wedge = B \setminus B_0$ can be written as $\wedge = \{\ h_i(w) : i = 1,\cdots,N\}$ with

$$h_1(w) = V_{1,1}(w),\ \cdots, h_N(w) = V_{2,N_{2,d}}(w). \tag{11}$$

Let $f(\cdot)$ be a density on Ω_d and let $f_0(\cdot)$ denote the density of $U(\Omega_d)$. Consider the null hypothesis

$$H_0 : f(w) = f_0(w). \tag{12}$$

A smooth alternative probability density function can be defined by[8]

$$g_N(w,\eta) = b(\eta)\exp\{\sum_{i=1}^{N}\eta_i h_i(w)\}, \tag{13}$$

where $\eta = (\eta_1, \eta_2, \cdots, \eta_N)^T$ and $h_1, h_2 \cdots, h_N$ are defined by (11).

Lemma7[7] (Smooth test for $U(\Omega_d)$) Let $U_1^{(d)},\cdots,U_n^{(d)}$

be a random sample from $g_N(w,\eta)$ defined in (13). Then

(a) The score statistic ψ_N for testing

$$H_0 : \eta = 0, H_1 : \eta \neq 0 \tag{14}$$

is

$$\psi_N = \sum_{i=1}^{N}\Phi_i^2, \Phi_i = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} h_i(U_j^{(d)}), \tag{15}$$

(b) Under $H_0 : \eta = 0$, ψ_N is asymptotically distributed as χ_N^2 random variable, where χ_N^2 represents chi-square distribution on N degrees of freedom.

Remark4 Smooth test consists of embedding the null density $f_0(w)$ into a large one, say $\{g_N(w,\eta),\ \ \eta \in \Theta\}$, such that $g(w,0) = f_0(w)$. Thus the hypothesis(12) is equivalent to the hypothesis(14).

Remark5[4] Theoretical analysis and power simulation showed that the smooth tests for $U(\Omega_d)$ based on spherical harmonics of degree at most $m = 2$ are generally powerful.

Lemma8[9] Let $X, X_1, X_2, \cdots$ be random $d-$ vectors defined on a probability space and g be a measurable function from (R^k, B^k) to (R^l, B^l) . Suppose that g is continuous. Then

$$X_n \xrightarrow{d} X \text{ implies } g(X_n) \xrightarrow{d} g(X),\quad n \to \infty.$$

III. A NEW TEST FOR ELLIPTICAL COPULAS

Let $Z = (Z_1, \cdots, Z_d)^T$ be the random vector with each component Z_i having the density $g_i(z_i)$ and the cdf $G_i(z_i)$ and let $Z_1^{(d)},\cdots,Z_n^{(d)}$ be a random sample from the population Z . Let C be the copula of Z and let C_φ be the $d-$dimensional elliptical copulas. We want to test the null hypothesis

$$H_0 : C = C_\varphi. \tag{16}$$

Theorem1 Let $Z = (Z_1, \cdots, Z_d)^T$, $g_i(z_i)$ and $G_i(z_i)$ be defined in Definition4. Let $Q_\varphi(\cdot)$ be defined in Lemma3. Suppose that

$$Y_i = Q_\varphi^{-1}[G_i(Z_i)],\quad i = 1,\cdots,d, \tag{17}$$

$$F_Z(z_1,\cdots,z_d) = C_\varphi(G_1(z_1),\cdots,G_d(z_d)). \tag{18}$$

Then the cdf of $Y = (Y_1, \cdots, Y_d)^T$ is given by

$$F_Y(y_1,\cdots,y_d) = C_\varphi(Q_\varphi(y_1),\cdots,Q_\varphi(y_d)),$$

i.e., $Y = (Y_1, \cdots, Y_d)^T \sim E_d(0, R, \varphi)$.

Remark6 Theorem1 indicates that the goodness-of-fit test for elliptical copula can be translated into the goodness-of-fit test for elliptical symmetry.

Define

$$Z_i^{(d)} = (Z_{i1}, \cdots, Z_{id})^T, Y_{ij} = Q_\varphi^{-1}[G_i(Z_{ij})],$$

$$Y_i^{(d)} = (Y_{i1}, \cdots, Y_{id})^T, \quad (19)$$

$i = 1, \cdots, n, \quad j = 1, \cdots, d.$

Let

$$S = \frac{1}{n}\sum_{i=1}^{n} Y_i^{(d)}[Y_i^{(d)}]^T \quad (20)$$

and let the Cholesky decomposition of R and $\hat{R}$ be

$$R = [L(R)][L(R)]^T, S = [L(S)][L(S)]^T, \quad (21)$$

respectively. The scaled residuals are defined as

$$\tilde{Y}_i^{(d)} = L^{-1}(S)Y_i^{(d)}, \quad i = 1, \cdots, n, \quad (22)$$

where L^{-1} is the inverse of L.

Remark7[10] Let $\tilde{Y} = (\tilde{Y}_1^{(d)}, \cdots, \tilde{Y}_n^{(d)})^T$ and let A be defined in (4). Then the distribution of $\tilde{Y}$ does not depend on $R = A^T A$.

Theorem2 Let $\tilde{Y}_1^{(d)}, \cdots, \tilde{Y}_n^{(d)}$ be defined in (22) and let

$$\gamma_i^{(d)} = \tilde{Y}_i^{(d)} \Big/ \left\|\tilde{Y}_i^{(d)}\right\| = (\gamma_{i1}, \cdots, \gamma_{id})^T, i = 1, \cdots, n. \quad (23)$$

Then the asymptotic distribution of $\gamma_i^{(d)}$ is $U(\Omega_d)$ and $\gamma_1^{(d)}, \cdots, \gamma_n^{(d)}$ are asymptotically independent.

Assume that $Z_1^{(d)}, \cdots, Z_n^{(d)}$ be a random sample from the population $Z \sim ME_d(0, R, \varphi, G_1, \cdots, G_d)$. Let

$$\tilde{\psi}_N = \sum_{i=1}^{N} \tilde{\Phi}_i^2, \tilde{\Phi}_i = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} h_i(\gamma_j^{(d)}), \quad (24)$$

where $\gamma_1^{(d)}, \cdots, \gamma_n^{(d)}$ are defined in (23). The elliptical copula is rejected for small $p-$value of $\tilde{\psi}_N$.

The algorithm to estimate the $p-$value of $\tilde{\psi}_N$ consists of the following steps:

1. Generate $Y_{1*}^{(d)}, \cdots, Y_{n*}^{(d)}$ from the elliptical distribution $E_d(0, I_d, \varphi)$.

2. Compute

$$S_* = \frac{1}{n}\sum_{i=1}^{n} Y_{i*}^{(d)}[Y_{i*}^{(d)}]^T,$$

$$\tilde{Y}_{i*}^{(d)} = L^{-1}(S_*)Y_{i*}^{(d)}, \quad i = 1, \cdots, n,$$

where $S_* = [L(S_*)][L(S_*)]^T$ (the Cholesky decomposition).

3. Compute $\gamma_{i*}^{(d)} = \tilde{Y}_{i*}^{(d)} \Big/ \left\|\tilde{Y}_{i*}^{(d)}\right\|, \quad i = 1, \cdots, n.$

4. Compute

$$\tilde{\psi}_* = \sum_{i=1}^{N} \tilde{\Phi}_{i*}^2, \tilde{\Phi}_{i*} = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} h_i(\gamma_{j*}^{(d)}),$$

5. Doing steps 1-4 M times gives a sample of replicates $\tilde{\psi}_{1*}, \cdots, \tilde{\psi}_{M*}$. Let $\tilde{\psi}_N$ be defined by (24), the estimated $p-$value of $\tilde{\psi}_N$ is given by

$$\hat{p}^* = \frac{\#_{j=1}^{M}\{\tilde{\psi}_{j*} \geq \tilde{\psi}_N\}}{M}. \quad (25)$$

Remark8 Since the distribution of $\tilde{\psi}_*$ does not depend on A in (4) so that in step 1 we can take $A = I_d$, where I_d is the $d \times d$ identity matrix.

Example1 (Smooth test for $t-$copula)

Let φ and $Y_i^{(d)} = (Y_{i1}, \cdots, Y_{id})^T (i \leq n)$ be defined by (5) and (19), respectively. Then $Y_{11}, Y_{21}, \cdots, Y_{n1}$ are i.i.d. according to the $t(k)$ distribution, where $t(k)$ denotes the $t-$ distribution with k degrees of freedom.

The maximum goodness-of-fit estimators(GFE) of the parameters of the distribution can be obtained by minimizing the empirical distribution function(EDF) statistics with respect to the unknown parameters [11].

Anderson-Darling(AD) statistic is one of the classical EDF statistics. The AD test is more sensitive to departures from the hypothesized distribution in the tail of the distribution. Let $Y_{(11)},\cdots,Y_{(n1)}$ be the order statistics of $Y_{11},\cdots,Y_{n1}$ and let

$$t_{(i)} = F_{t,k}(Y_{(i1)}),\quad i=1,\cdots,n, \tag{26}$$

where $F_{t,k}(\cdot)$ denotes the cdf of $t(k)$ distribution. The computational formula for AD of $F_{t,k}(\cdot)$ is given by

$$AD_k = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\ln t_{(i)} + \ln(1-t_{(n+1-i)})]. \tag{27}$$

The GFE of the copula parameter k can be obtained by

$$\hat{k} = \min\{k : AD_k, k=1,\cdots,30\}. \tag{28}$$

The estimated $\hat{p}^*$ is defined by (25), but with k replaced by $\hat{k}$ in (28). The null hypothesis

$$H_0 : C \text{ is a } t-\text{copula}$$

is rejected when $\hat{p}^* \le \alpha$, where α is a size of the test.

Remark9 Let $Y=(Y_1,\cdots,Y_d)^T \sim Mt_d(k,0,R)$, then $Y_j \sim t(k), j\le d$. Since the limit distribution of $t(k)$ is $N(0,1)$ distribution, thus we can take $1\le k\le 30$ in (28).

IV. CONCLUSIONS

The meta-elliptical distribution is built by coupling a elliptical copula with arbitrarily given margins. The class of meta-elliptical distributions is a natural extension of the elliptical distributions. The meta-Gaussian and the meta-t distributions belong to this class.

The copula allows to capture the full dependence with in multivariate time series without specifying a shape of the marginal distributions. The modified AD statistic can be used to test whether the marginal distribution is a specific distribution[12]. If it is difficult to determine the cdf $G_i(z_i)$, the transformed sample Y_i can be calculated with the cdf $G_i(z_i)$ replaced by the empirical cdf $\hat{G}_i(z_i)$ in (17).

$\tilde{Y}_1^{(d)},\cdots,\tilde{Y}_n^{(d)}$ in (22) are also known as the spherized data. Based on the scaled residuals, the smooth test for elliptical copulas is equivalent to the smooth test for the uniform distribution on Ω_d. The smooth test for elliptical copulas can be extended to testing the meta-elliptical distribution of the innovations in the VAR and GARCH models.

V. APPENDIX

Proof of theorem 1. Let $y=(y_1,\cdots,y_d)^T \in R^d$.By (17) and (18), we have

$$\begin{aligned}F_Y(y_1,\cdots,y_d) &= P(Y_1\le y_1,\cdots,Y_d\le y_d)\\ &= P(G_1(Z_1)\le Q_\varphi(y_1),\cdots,G_d(Z_d)\le Q_\varphi(y_d))\\ &= P(Z_1\le G_1^{-1}[Q_\varphi(y_1)],\cdots,Z_d\le G_d^{-1}[Q_\varphi(y_d)])\\ &= C_\varphi(Q_\varphi(y_1),\cdots,Q_\varphi(y_d)).\end{aligned}$$

Thus $Y=(Y_1,\cdots,Y_d)^T \sim E_d(0,R,\varphi)$.

Proof of theorem 2. Let $\sigma_0^2 = E(\xi^2)/d$. By (6) and the law of large numbers, we have

$$S \xrightarrow{p} \sigma_0^2 R,\quad n\to\infty, \tag{29}$$

where $\xrightarrow{p}$ denotes convergence in probability as $n\to\infty$. Let the Choleski decomposition of R and S be

$$R=[L(R)][L(R)]^T, S=[L(S)][L(S)]^T \tag{30}$$

respectively. By (29) and (30), we have

$$L(S) \xrightarrow{p} \sigma_0 L(R),\quad n\to\infty. \tag{31}$$

By (31) we have

$$\tilde{Y}_i^{(d)} = L^{-1}(S)Y_i^{(d)} \xrightarrow{d} \breve{Y}_i = \sigma_0^{-1}L^{-1}(R)Y_i^{(d)},\quad n\to\infty, \tag{32}$$

where $\xrightarrow{d}$ denotes convergence in distribution as $n\to\infty$. $Y_1^{(d)},\cdots,Y_n^{(d)}$ are independent $E_d(0,R,\varphi)$ random vectors, thus $\tilde{Y}_1^{(d)},\cdots,\tilde{Y}_n^{(d)}$ are asymptotically independent. Since

$$\sigma_0^{-1}L^{-1}(R)R[\sigma_0^{-1}L^{-1}(R)]^T = \sigma_0^{-2}I_d,$$

we have by (7), $\breve{Y}_i \sim E_\varphi(0,\sigma_0^{-2}I_d)$, i.e., $\breve{Y}_i$ has a spherical distribution. By Lemma5,

$$\breve{Y}_i / \|\breve{Y}_i\| \sim U(\Omega_d).$$

Thus by Lemma8, we have

$$\gamma_i^{(d)} = \frac{\tilde{Y}_i^{(d)}}{\left\| \tilde{Y}_i^{(d)} \right\|} \xrightarrow{d} U(\Omega_d), n \to \infty. \quad (33)$$

(33) implies that the transformed sample $\gamma_1^{(d)}, \cdots, \gamma_n^{(d)}$ are asymptotically independent.

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