

Vertex Quasiprimitive and 2-Arc Transitive Graphs of Order $2p^e$

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Abstract: A graph Γ is called 2-arc transitive graph if its automorphism group $\text{Aut}(\Gamma)$ is transitive on the set of 2-arcs. In this paper, we main classify the vertex quasiprimitive 2-arc transitive graphs of order $2p^e$.

1. Introduction

Let $\Gamma=(V,E)$ be a graph with vertex set V , edge set E . For $X \leq \text{Aut}(\Gamma)$, a group of automorphisms. Γ is called G -vertex transitive or G -edge transitive if G is transitive on V or E , respectively. The motivation for this paper came from a study of s -arc transitive graphs, $s \geq 2$. An s -arc, $s \geq 1$, in a graph Γ is a sequence $\alpha_0, \alpha_1, \dots, \alpha_s$ of $s+1$ vertices of Γ such that, for all $1 \leq i \leq s$, $\alpha_{i+1} \neq \alpha_{i-1}$. A graph Γ is said to be s -arc transitive if its automorphism group $\text{Aut}(\Gamma)$ is transitive on the set of the s -arcs, and a subgroup G of $\text{Aut}(\Gamma)$ is said to be s -arc transitive on Γ if G is transitive on the set of s -arcs of Γ . By [6], it is known that a finite, connected, s -arc transitive graph, $s \geq 2$, which is not bipartite, is a cover of an s -arc transitive graph with a subgroup of automorphisms quasiprimitive on vertices. This paper aims to characterize undirected vertex quasiprimitive 2-arc transitive graphs of order $2p^e$ with p is an odd prime. Let G be a group, H be a subgroup of G , and S be s subset of G . Then the coset graph of G with respect to H and S to be the directed graph with vertex set $[G:H]$ and such that, for any $Hx, Hy \in V$. Hx is connected to Hy if and only if $yx^{-1} \in HSH$. and denote the digraph by $\text{Cos}(G,H,HSH)$. By [9], For any $G \leq \text{Aut}(\Gamma)$, Γ is $(G,2)$ -arc transitive if and only if $\Gamma \cong \text{Cos}(G,H,HSH)$ such that $g^2 \in H$ and H acts (by coset action) 2-transitive on $[H : H \cap H^g]$. Let $N \triangleleft X$, and let \bar{B} be the set of N -orbits in V . Then the normal quotient graph Γ_N of Γ induced by N is the graph with vertex set \bar{B} such that $B, B_1 \in \bar{B}$ are adjacent in Γ to some vertex $v \in B_1$. If Γ and Γ_N have the same valency, then Γ is called a normal cover of Γ_N .

2. Vertex quasiprimitive

A permutation group $X \leq \text{Sym}(\Omega)$ is called quasiprimitive if each nontrivial normal subgroups of X is transitive on Ω . Let $N = \text{Soc}(X)$ be the socle of X , that is the product of all minimal normal subgroups of X . First, we give a characterization of quasiprimitive permutation groups of degree $2p^e$.

Lemma 2.1: Let X be a quasiprimitive permutation group on Ω of degree $2p^e$, where p is an odd prime. Then X is almost simple(AS) group or product action(PA) group.

proof : Let N be a minimal normal subgroup of X . Then $N \cong T^l$, where $l \geq 1$ and T is a simple group. Since X is quasiprimitive on Ω , N is transitive on Ω . Then for any $\alpha \in \Omega$, we have $|N : N_\alpha| = |\Omega| = p^e$. If T is abelian, then $T \cong Z_p$, and $N \cong Z_{p^e}$ is regular on Ω . If $l=1$ and N is nonabelian, then X is an almost simple group. If $l \geq 2$, then $N = T_1 \times \dots \times T_l = T^l$ with $l \geq 2$, and T is a nonabelian simple group. Since $|N : N_\alpha| = |\Omega| = 2p^e$ and $|T_1 : (T_1)_\alpha| = |N : (T_1)_\alpha| \times \dots \times |T_l|$ divides $|N : N_\alpha|$, we

conclude that $(T_1)_\alpha$ has index p -power or double p -power in T . In particular, X is type of product action.

Almost Simple case

Suppose that X is an almost simple group, then $T \leq X \leq T \cdot \text{Out}(T)$, and thus for any $\alpha \in \Omega$, T_α has index $2p^e$ in T . By [8], we have the following lemma.

Lemma 2.2: Let T be a nonabelian simple group that has a subgroup H or K of index $2p^e$ with p^e odd prime, where K is a maximal subgroup of T but H is not a maximal subgroup of T . Then (T, T_α) has been listed in the table1.

where P_1 is parabolic subgroup and $|T:H|=|T:K|=d$.

When $(T : T_\alpha) = (A_6 : A_5), (A_6, 3^2 : 4), (A_{2p^l} : A_{2p^l-1}), (M_{22} : L_3(4))$ or $(L_2(4) : D_{10})$,

By [4], T is 2-transitive on $(T : T_\alpha)$. That is $\Gamma \cong K_6, K_{10}, K_{2p^l}, K_{22}, K_6$, respectively.

When $(T : H) = (L_2(4), S_3)$, there exist an involution element $g \in T$ such that $H \cap H^g = Z_2$ and $\langle H, g \rangle = G$. Thus $\Gamma \cong \text{Cos}(G, H.HgH)$ is Petersen graph.

Table1

T	K	H	$d=2p^l$
A_{2p^l}	A_{2p^l-1}		$2p^l$
A_6	A_5		6
A_6	$3^2 : 4$		10
M_{11}		A_6	22
M_{22}	$L_3(4)$		22
$L_2(4)$	D_6		10
$L_2(4)$	D_{10}		6
$L_n(p^f)$	P_1		$\frac{q^n - 1}{q - 1}$
$L_n(p^f)$		H	
$L_n(p^f)$		$O_p(P_1)L_1R$	
$PSp(4,3)$		$2^4 : D_{10}$	$2 \cdot 3^4$

If $(T,K)=(Sp_{2m}(2^f), O^-_{2m}(2))$ with $f=1, m=2$, then $T \cong Sp_4(2) \cong S_6$ is not a simple group.

If $(T,K)=(Sp_{2m}(2^f), O^+_{2m}(2^f))$ with $fm=2$, when $f=1, m=2$, $T \cong Sp_4(2) \cong S_6$ is not a simple group;

$f=2, m=1$, $T \cong Sp_2(4) \cong L_2(4) \cong A_5$ and $K = O^+_{2}(4)$

If $(T, K) = (\Omega^-_{2m}(2^f), Sp_{2m-2}(2^f))$ with $fm=2$, then $f=1, m=2$. That is

By [8], Γ is $(G,2)$ - arc transitive if and only if $\Gamma \cong Cos(G, H, HgH)$ such that

$g^2 \in H$ and H acts (by coset action) 2-transitive on $[H : H \cap H^g]$. Next we need consider the rest of the group on the form one by one.

Lemma 2.3: Let Γ be a $(G,2)$ -arc transitive with vertex V . Assume that G is quasiprimitive on V , $|V|=22$, and $Soc(G)=T=M_{11}$. Then Γ is $(G,2)$ -transitive graph with order 22, and valency 6.

Proof: By [4], it is known that $Out(T)=1$, since G is type of AS,

$G = T \cdot \text{Out}(T) \cong M_{11}$, and $G_\alpha = T_\alpha \cdot \text{Out}(T) = A_6$. There exist an element $g^2 \in A_6$ such that

$H \cap H^g = A_5$, then T is 2-transitive on $[H : H \cap H^g]$. Thus Γ is $(G, 2)$ -

transitive graph with order 22, and valency 6.

Product Action case

Now we assume that $l \geq 2$, then X is type of product action and

$N = T_1 \times \cdots \times T_l \cong T^l$ with $l \geq 2$, and T is a nonabelian simple group. Since $|N : N_\alpha| = |\Omega| = 2p^e$ and

$|T_1 : (T_1)_\alpha| = |N : (T_1)_\alpha \times T_2 \times \cdots \times T_l| = 2p^e$ divides $|N : N_\alpha|$, we

conclude that $(T_1)_\alpha$ has index p -power or double p -power in T . We only need consider the case that

$(T_1)_\alpha$ has index p -power in T . In this case, By [8], $|T_1 : (T_1)_\alpha|$ has been listed in the following table:

Table 2

T	T_α	$ T : T_\alpha $
A_{p^r}	$A_{p^{r-1}}$	p^r
$PSL(d, q)$	P_1	$\frac{q^d - 1}{q - 1}$
M_{11}	M_{10}	11
M_{23}	M_{22}	23
$PSL(2, 11)$	A_5	11
$PSU(4, 2)$	$Z_2^4 : A_5$	162

Since $|N : N_\alpha| = |\Omega| = 2p^e$, and $|N : (T_\alpha)^l| = p^{rl}$, we

have that $|N_\alpha : (T_\alpha)^l| = 2$. Then one of the following holds:

(1) $T = M_{11}$, $T_\alpha = M_{10}$;

(2) $T = PSL(d, q)$, $T_\alpha = q^{d-1} \frac{1}{(d-1, q)} GL(d-1, q)$, where $(d-1, q-1)$ is even.

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