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Constructing Topological Spaces in Rough Set by Using Boundary Operation

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Abstract. In order to explore use the boundary operations to establish topological spaces of rough set, in this thesis the logical reasoning method is used to experimental study on the relationship between the rough set and the topological space, according to the experimental results can reach conclusion that the approximate space of the rough set satisfies the internal operations too, defines the boundary operation on the internal operations, constructs topological rely on the boundary operation, this conclusion generalizes the existing topological research in rough set theory.

Introduction

The concept of rough sets was first proposed in literature [1] in 1982. Many important international academic conferences and Seminars have included research on rough set theory into one of the main contents of the conference and discussion sessions since 1992, which have greatly contributed to the development of the theory and its development in various fields application. In the country, there will be held rough set and soft computing academic conference from 2001 onwards each year. In addition, some studies on the test ability of rough sets are described in literature[2]. The topological space is established by using the open set theory in literature[3] - [5]. The topological space is established by using the neighborhood system theory in literature [6]. In this thesis, rough sets and topological spaces are used as the object of study,mainly discus the internal operations of rough set's approximate space, and boundary operations are defined on this basis to establish the topological space.

The relation between rough set and topological spaces[7]

Let U be a nonempty set ,there is one equivalence relation R which based on U ,note $PS(U)$ is U 's power set,then we say that the binary group (U,R) is a Pawlak approximation space. For any one $X \subseteq U$,the lower approximation $\underline{R}(X)$ and the upper approximation $\overline{R}(X)$ of approximate space (U,R) which define as the following formula (1) and (2) respectively:

$$\underline{R}(X)=\{x\in U \mid [x]_R \subseteq X\}=\bigcup\{Y\in U/R \mid Y\subseteq X\} \quad (1)$$

$$\overline{R}(X)=\{x\in U \mid [x]_R \cap X \neq \phi\}=\bigcup\{Y\in U/R \mid Y \cap X \neq \phi\} \quad (2)$$

In the formula, $[x]_R = \{y \mid (x, y) \in R\}$ is the equivalence class for x which is about R's, $U/R = \{[x]_R \mid x \in U\}$ is a set of all R's equivalents.

For any one $X \subseteq U$, we called the binary group $(\underline{R}(X), \overline{R}(X))$ rough set which belong to R. If $\underline{R}(X) = \overline{R}(X)$, then we called the binary group $(\underline{R}(X), \overline{R}(X))$ exact set which belong to R; otherwise, we called the binary group $(\underline{R}(X), \overline{R}(X))$ approximate set which belong to R. Order Z_0 is a set which composed of one element which selected from the equivalence class of each R, and note $S = \{x \in U; |[x]_R| = 1\}$.

Lemma 1[7] Let (U, R) be a Pawlak approximation space, $X, Y \subseteq U$ is a definable set which belong to R (that is, the union equitable class of which belong to R), then the (X, Y) is R's rough set if only if $X \subseteq Y$ and $(Y - X) \cap S = \phi$.

For the sake of convenience, here we define the operator which from $PS(U)$ to $PS(U)$ for any one $X \in PS(U)$ as

$$X_A = \underline{R}(X) \cup ((\overline{R}(X) - \underline{R}(X)) \cap Z_0) = \overline{R}(X) \cap \underline{R}(X) \cup Z_0$$

Where $A = (U, R)$ is an approximate space which is defined by equivalence relation of U.

Lemma 2[7] For any one $X \in PS(U)$, we have $\underline{R}(X_A) = \underline{R}(X), \overline{R}(X_A) = \overline{R}(X)$

Let $N = \{X_A, X \in PS(U)\}$, then we say that N is all rough sets which based on U. If X is a definable set of R, then there must be $X_A = X$, thus $PS(U) \subseteq N$.

Lemma 3 If (U, R) is a Pawlak approximation space, then we say that N is a topological space which based on U.

Proof:① Obviously $\phi, U \in PS(U) \subseteq N$.

② According to $\underline{R}(X) \cap \underline{R}(Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$ and $(\overline{R}(X) \cap \overline{R}(Y)) - (\underline{R}(X) \cap \underline{R}(Y)) \cap S \subseteq (\overline{R}(X) \cap \underline{R}(X)) \cup (\overline{R}(Y) \cap \underline{R}(Y)) \cap S = \phi$, thus there is $Z \in PS(U)$ which makes equation $\underline{R}(Z) = \underline{R}(X) \cap \underline{R}(Y), \overline{R}(Z) = \overline{R}(X) \cap \overline{R}(Y)$ holds, then

$$Z_A = \overline{R}(Z) \cap (\underline{R}(Z) \cup Z_0) = \overline{R}(X) \cap \overline{R}(Y) \cap ((\underline{R}(X) \cap \underline{R}(Y)) \cup Z_0)$$

$$= \overline{R}(X) \cap \overline{R}(Y) \cap (\underline{R}(X) \cup Z_0) \cap (\underline{R}(Y) \cup Z_0) = X_A \cap Y_A$$

Hence we have $X_A \cap Y_A \in N$.

f For any one $X_A^i \in W$ ($i \in I, I$ is a indicator set), according to $\bigcup_{i\in I} \underline{R}(X^i) \subseteq \bigcup_{i\in I} \overline{R}(X^i)$ and $\left(\bigcup_{i\in I} \overline{R}(X^i) - \bigcup_{i\in I} \underline{R}(X^i)\right) \cap S \subseteq \bigcup_{i\in I}\left(\left(\overline{R}(X^i) - \underline{R}(X^i)\right) \cap S\right) = \emptyset$,so there is $Z \in PS(U)$ which makes equation $\underline{R}(Z) = \bigcup_{i\in I} \underline{R}(X^i), \overline{R}(Z) = \bigcup_{i\in I} \overline{R}(X^i)$ holds,then

$$Z_A = \overline{R}(Z) \cap (\underline{R}(Z) \cup Z_0) = \left(\bigcup_{i\in I} \overline{R}(X^i)\right) \cap \left(\bigcup_{i\in I} \underline{R}(X^i) \cup Z_0\right)$$

since $\left(\bigcup_{i\in I} X_A^i\right) = \bigcup_{i\in I}\left(\overline{R}(X^i) \cap (\underline{R}(X^i) \cup Z_0)\right) \subseteq \bigcup_{i\in I} \overline{R}(X_A^i)$,

$$\left(\bigcup_{i\in I} X_A^i\right) = \bigcup_{i\in I}\left(\overline{R}(X^i) \cap (\underline{R}(X^i) \cup Z_0)\right) \subseteq \bigcup_{i\in I} \underline{R}(X^i) \cup Z_0$$

therefore $\left(\bigcup_{i\in I} X_A^i\right) \subseteq \left(\bigcup_{i\in I}\left(\overline{R}(X^i) \cap (\underline{R}(X^i) \cup Z_0)\right)\right) = \overline{R}(Z) \cap (\underline{R}(Z) \cup Z_0) = Z_A$

and because $\bigcup_{i\in I} X_A^i \supseteq \bigcup_{i\in I}\left(\underline{R}(X^i)\right), X_A = \left(\bigcup_{i\in I} \underline{R}(X^i)\right) \cup \left(\bigcup_{i\in I} \overline{R}(X^i) \cap Z_0\right) \supseteq \bigcup_{i\in I} \overline{R}(X^i) \cap Z_0$

therefore $\left(\bigcup_{i\in I} X_A^i\right) \subseteq \left(\bigcup_{i\in I} \underline{R}(X^i)\right) \cup \left(\bigcup_{i\in I} \overline{R}(X^i) \cap Z_0\right) = Z_A$,so there is $\bigcup_{i\in I} X_A^i \in N$.

Integrated the above proved①, ② and ③ of Lemma 3, we know that N is a topological space which based on U.

Internal operations of topological spaces in rough set

Definition 1 Let W be a nonempty set,and $W \subseteq N$, T is a subset family which belongs to W,if T satisfies the following conditions (3) - (5):

W、$\emptyset \in \mathrm{T}$ （3） if E、$F \in \mathrm{T}$,

then $E \cap F \in \mathrm{T}$ （4）

if $\mathrm{T}_1 \subset \mathrm{T}$, then $\bigcup_E E \in \mathrm{T}$,（note：$E \in \mathrm{T}_1$） （5）

So we called T topology which belong to W, and called （W，T） a topological space.

Definition 2 Let （W，T）be a topological space, if $E \in \mathrm{T}$, then we called E a open set which belongs to the topological space （W，T）.

Definition 3 Let W be a nonempty set, if mapping $m^*: 2^W \to 2^W$ satisfies the conditions: it satisfies the following formula(6) - (9) for $\forall E \subseteq W$、$\forall F \subseteq W$:

$$m^*(W) = W \quad (6)$$

$$m^*(E) \subset E \quad (7)$$

$$m^*(E \cap F) = m^*(E) \cap m^*(F) \quad (8)$$

$$m^*\left(m^*(E)\right)=m^*(E) \qquad (9)$$

then we called the mapping m^* internal operation which belong to W .

Lemma 4 Let m^* be the internal operation in nonempty set W ,then there is a unique topology T in W which makes the equation $m^*(E)=m(E)$ holds for $\forall E\subset W$,where $m(E)$ is the interior of E in the topological space（W，T）.

Proof: In order to facilitate the expression,here make W ’s subset family T=$\left\{E \mid m^*(E)=E, E\subset W\right\}$.

According to the formula (6) $m^*(W)=W$ which is in definition 3 ,therefore $W\in$ T;according to the formula (7) $m^*(f)\subset f$,so $m^*(f)=f$,therefore $f\in$ T,so there is W、$f\in$ T.

If E、$F\in$ T,then $m^*(E)=E, m^*(F)=F$,according to the formula (8) we know $m^*(E\cap F)=m^*(E)\cap m^*(F)=E\cap F$,so we have $E\cap F\in$ T.

Incidentally pointed out that, if $F\subset E$ ， then $m^*(F)\subset m^*(E)$ .In fact,if $F\subset E$ ， then $E\cap F=F$ ， according to the formula (8) we know $m^*(F)=m^*(E\cap F)=m^*(E)\cap m^*(F)\subset m^*(E)$.

If T$_1\subset$T, then there is $F\subset \bigcup_E E$ for any one E、$F\in$ T$_1$,so $m^*(F)\subset m^*(\bigcup_E E)$, however because $m^*(F)=F$,so there is $F=m^*(F)\subset m^*(\bigcup_E E)$,thus $\bigcup_E E\subset m^*(\bigcup_E E)$,and then we can know $\bigcup_E E=m^*(\bigcup_E E)$ from the formula (7) ,so there is $\bigcup_E E\in$ T.

In summary,the T satisfies all conditions of definition 1, thus we prove that T is the topology of W .

Let $m(E)$ be the interior of E which is in the topological space（W，T）,according to the formula (2)we know $m(E)$ is the open set which is in the topological space （W，T）,therefore we have $m(E)\in$ T, then we know $m^*\left(m(E)\right)=m(E)$ from the definition of T ；on the other hand, according to the internal properties of the topological space（W，T）we know $m(E)\subset E$,so there is $m^*\left(m(E)\right)\subset m^*(E)$,thus we have $m(E)=m^*(E)$.

According to the formula(9) $m^*\left(m^*(E)\right)=m^*(E)$ which is in definition 3 ,and we know $m^*(E)\in$ T from the definition of T ,thus $m^*(E)$ is the open set of the topological space（W， T） ,then $m\left(m^{*}(E)\right)=m^{*}(E)$,according to the formula(7) we know $m^{*}(E)\subset E$, and we know $m^{*}\left(m^{*}(E)\right)\subset m(E)$ from the internal properties of the topological space （W，T）,thus we have $m^{*}(E)\subset m(E)$.

So we can reach the conclusion: there is $m^{*}(E)=m(E)$ for $\forall E\subset W$.

Let W has another T$'$ make it also satisfies $m^{*}(E)=m'(E)$ for $\forall E\subset W$, $m'(E)$ is the interior of E which is in the topological space（W，T$'$）. Because F is the open set of topological space （W，T） for $\forall F\in$ T, therefore we have $F=m(F)$,then $F=m(F)=m^{*}(F)=m'(F)$,so F is the open set of topological space（W，T$'$） too,therefore $F\in$ T$'$， thus get the conclusion T$\subset$T$'$.

Similarly, we can prove T$'\subset$T , therefore we have T$'$=T, thus the uniqueness of the topology T is proved .

Constructing topological spaces in rough set by using boundary operation

Definition 4 Let W be a nonempty set,if mapping p^{*}： $2^{W}\to 2^{W}$ satisfies the condition: if it satisfies the following （10）-（13）formula for $\forall E\subseteq W,\forall F\subseteq W$:

$$p^{*}(\phi)=\phi \qquad (10)$$

$$p^{*}(E)=p^{*}(\sim E)\text{, where }\sim E\text{ is the rest set of }E \qquad (11)$$

$$E\cap F\cap\left(\sim p^{*}(E\cap F)\right)=E\cap F\cap\left(\sim\left(p^{*}(E)\cup p^{*}(F)\right)\right) \qquad (12)$$

$$p^{*}\left(p^{*}(E)\right)\subset p^{*}(E) \qquad (13)$$

then we called the mapping p^{*} of W's boundary operation.

Lemma 5 Let the mapping p^{*} be the boundary operation of the nonempty set W,make mapping m^{*}： $2^{W}\to 2^{W}$ for any one $E\subset W$,order $m^{*}(E)=E\cap\left(\sim p^{*}(E)\right)$,then m^{*} is the internal operation of W.

Proof: ① According to the formula （10）,（11）of definition 4 we know： $p^{*}(E)=p^{*}(\sim E)=p^{*}(\phi)=\phi$,so there is

$$m^{*}(E)=E\cap\left(\sim m^{*}(E)\right)=E\cap(\sim\phi)=E\cap E=E$$

② Since $m^{*}(E)=E\cap\left(\sim p^{*}(E)\right)$,so there is $m^{*}(E)\subset E$

f According to the formula （12）of definition 4 we know：

$$m^*(E \cap F) = (E \cap F) \cap (\sim p^*(E \cap F)) = E \cap F \cap (\sim p^*(E) \cup p^*(F))$$

$$= E \cap F \cap (\sim p^*(E)) \cap (\sim p^*(F)) = (E \cap (\sim p^*(E))) \cap (F \cap (\sim p^*(F)))$$

$$= m^*(E) \cap m^*(F)$$

④ According to the proven **f** as well as formula （11）,（13）of definition 4 ,we can inferred

$$m^*(m^*(E)) = m^*(E \cap (\sim p^*(E))) = m^*(E) \cap m^*(\sim p^*(E))$$

$$= E \cap (\sim p^*(E)) \cap (\sim p^*(E) \cap (\sim p^*(\sim p^*(E)))) = E \cap (\sim p^*(E)) \cap (\sim p^*(p^*(E)))$$

$$= E \cap (\sim (p^*(E) \cup p^*(p^*(E)))) = E \cap (\sim p^*(E)) = m^*(E)$$

so the m^* is the internal operation of W.

Theorem Let the mapping p^* be the boundary operation of the nonempty set W,then there will be a unique topology T of W which making the equation $p^*(E) = p(E)$ holds for any one $E \subset W$,where $p(E)$ is the boundary of E in the topological space （W，T）.

Proof: Make $2^W \to 2^W$ mapping of m^*：$m^*(E) = E \cap (\sim p^*(E))$ for any one $E \subset W$，according to Lemma 4 we know that m^* is the internal operation of W,thus W will has a unique topology T which making equation $m^*(E) = m(E)$ holds for any one $E \subset W$,where $m(E)$ is the boundary of E in the topological space （W，T）.

For the sake of convenience,note $p(E)$ is the boundary of E in the topological space （W，T）,according to the boundary property of the topological space,we have $p(E) = \sim (m(E) \cup m(\sim E))$,that is $\sim p(E) = m(E) \cup m(\sim E)$.

According to the formula（11）of definition 4 we know $\sim p(E) = m(E) \cup m(\sim E)$，that is

$$m(E) \cup m(\sim E) = (E \cap (\sim p^*(E))) \cup ((\sim E) \cap (\sim p^*(\sim E)))$$

$$= (E \cap (\sim p^*(E))) \cup ((\sim E) \cap (\sim p^*(E))) = (E \cup (\sim E)) \cap (\sim p^*(E)) = \sim p^*(E)$$

And because of in the topological space（W，T）,there is $m^*(E) = m(E)$ for any one $E \subset W$,thus we can get $\sim p(E) = m^*(\sim E) \cup m(E) \cup m(\sim E) = \sim p^*(E)$

thus there is $p(E) = p^*(E)$.

Next, in order to prove that the above topology T is unique,assume here may exist another topology T' of R which is in the topological space（W，T'） make equation $p'(E) = p^*(E)$ for any one $E \subset W$,where $p'(E)$ is the boundary of E which is in the topological space（W，T'）.Also let $m'(E)$ be the interior of E which is in the topological space（W，T'）,so we can get

$$m'(E) = E \cap \left(\sim p'(E)\right) = E \cap \left(\sim p^*(E)\right) = m^*(E) = m(E).$$

According to Lemma 5 we know $T=T'$,this explains the topology for make $p(E) = p^*(E)$ established is unique for any one $E \subset W$.

Conclusions

Integrated previous experimental studies have shown that,the approximate space of rough set has some important characteristics:first, meet the internal operation,second, we can define the boundary operation on the internal operation，third, we can establish the topological space relying on the boundary operation .

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