Odd Generalized Exponential Flexible Weibull Extension Distribution

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In this article we introduce a new four - parameters model called the odd generalized exponential flexible Weibull extension (OGE-FWE) distribution which exhibits bathtub-shaped hazard rate. Some of it's statistical properties are obtained including ordinary and incomplete moments, quantile and mode, the moment generating functions, reliability and order statistics. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is given. Moreover, we give the advantage of the OGE-FWE distribution by an application using real data.

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1. Introduction

The Weibull distribution is a highly known distribution due to its utility in modelling lifetime data where the hazard rate function is monotone, [27]. Recently new classes of distributions were proposed based on modifications of the Weibull distribution to provide a good fit to data set with bathtub hazard failure rate, see [25]. Among of these, Exponentiated Weibull family, [14], Modified Weibull distribution, [11, 19], Beta-Weibull distribution, [8], A flexible Weibull extension, [4], Extended flexible Weibull, [4], Generalized modified Weibull distribution, [5], Kumaraswamy Weibull distribution, [6], Beta modified Weibull distribution, [16, 22], Beta generalized Weibull distribution, [23],

A new modified weibull distribution, [2] and Exponentiated modified Weibull extension distribution, [20], among others. A good review of these models is presented in [3, 15, 17].

The flexible Weibull extension (FWE) distribution, [4] has many applications in life testing experiments, reliability analysis, applied statistics and clinical studies. For more details on this distribution, see [4].

A random variable X is said to have the Flexible Weibull Extension (FWE) distribution with parameters $\alpha, \beta > 0$ if it's probability density function (pdf) is given by

$$g(x) = \left(\alpha + \frac{\beta}{x^2}\right) \exp\left\{\alpha x - \frac{\beta}{x} - e^{\alpha x - \frac{\beta}{x}}\right\}, \, x > 0, \tag{1.1}$$

while the cumulative distribution function (cdf) is given by

$$G(x) = 1 - \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\}, \, x > 0. \tag{1.2}$$

The survival function is given by the equation

$$S(x) = 1 - G(x) = \exp\left\{-e^{\alpha x - \frac{\beta}{x}}\right\}, \, x > 0, \tag{1.3}$$

and the hazard rate function is

$$h(x) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}}. \tag{1.4}$$

Gupta and Kundu [9] proposed a generalization of the exponential distribution named as Generalized Exponential (GE) distribution. The GE distribution with parameters $\vartheta, \gamma > 0$, has the following distribution function

$$F(x; \vartheta, \gamma) = \left(1 - e^{-\vartheta x}\right)^{\gamma}, \, x > 0, \vartheta > 0, \gamma > 0. \tag{1.5}$$

Recently, a new class of univariate continuous distributions named as the odd generalized exponential (OGE) class introduced in [7, 24]. This class is flexible because of the hazard rate shapes could be increasing, decreasing, bathtub and upside down bathtub. The odd generalized exponential (OGE) class is defined as follows.

If $G(x), x > 0$ is cumulative distribution function (cdf) of a random variable X, then the corresponding survival function is $\overline{G}(x) = 1 - G(x)$ and the probability density function is $g(x)$, then we define the cdf of the OGE class by replacing x in the distribution function of generalized exponential (GE) distribution given in equation (1.5) by $\frac{G(x)}{\overline{G}(x)}$ leading to

$$F(x; \vartheta, \gamma) = \left[1 - \exp\left\{-\vartheta \frac{G(x)}{\overline{G}(x)}\right\}\right]^{\gamma}, \, x > 0, \vartheta > 0, \gamma > 0. \tag{1.6}$$

The probability density function corresponding to (1.6) is given by

$$f(x; \vartheta, \gamma) = \frac{\vartheta \gamma g(x)}{\overline{G}(x)^2} \exp\left\{-\vartheta \frac{G(x)}{\overline{G}(x)}\right\} \left[1 - \exp\left\{-\vartheta \frac{G(x)}{\overline{G}(x)}\right\}\right]^{\gamma-1}, \tag{1.7}$$

where $x > 0, \vartheta > 0, \gamma > 0$. In this article we present a new distribution depending on flexible Weibull extension distribution referred to as the odd generalized exponential flexible Weibull extension

(OGE-FWE) distribution by using the class of univariate distributions defined above.

This paper can be organized as follows, we define the cumulative, density and hazard functions of the odd generalized exponential flexible Weibull extension (OGE-FWE) distribution in Section 2. In Sections 3, we present some statistical properties including, quantile function and median, the mode, *rth* moment, skewness and kurtosis. In Sections 4, we introduce the moment generating function. The distribution of the order statistics is expressed in Section 5. The maximum likelihood estimation of the parameters is determined in Section 6. We use real data sets and analyzed it by an application in Section 7 and the results are compared with existing distributions. Finally, we present a conclusion in Section 8.

2. The Odd Generalized Exponential Flexible Weibull Extension Distribution

In this section we studied the four parameters odd generalized exponential flexible Weibull extension OGE-FWE $(\vartheta, \gamma, \alpha, \beta)$ distribution. Using $G(x)$ from Eq. (1.2) and $g(x)$ from Eq. (1.1) to obtain the cdf and pdf of Eqs. (1.6) and (1.7), respectively. The cumulative distribution function cdf of the odd generalized exponential flexible Weibull extension distribution (OGE-FWE) is given by

$$F(x;\vartheta,\gamma,\alpha,\beta) = \left[1 - e^{-\vartheta\left(e^{e^{\alpha x - \frac{\beta}{x}}} - 1\right)}\right]^{\gamma}, \quad x > 0, \vartheta, \gamma, \alpha, \beta > 0, \tag{2.1}$$

The pdf corresponding to Eq. (2.1) is given by

$$f(x;\vartheta,\gamma,\alpha,\beta) = \vartheta\gamma\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x} + e^{\alpha x - \frac{\beta}{x}}} e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}\left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}\right]^{\gamma - 1}, \tag{2.2}$$

where $x > 0$ and, $\alpha, \beta > 0$ are two additional shape parameters.
The survival function $S(x)$, hazard rate function $h(x)$ and reversed hazard rate function $r(x)$ of $X \sim$ OGE-FWE $(\vartheta, \gamma, \alpha, \beta)$ are given by

$$S(x;\vartheta,\gamma,\alpha,\beta) = 1 - \left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}\right]^{\gamma}, \quad x > 0, \tag{2.3}$$

$$h(x;\vartheta,\gamma,\alpha,\beta) = \frac{\vartheta\gamma\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x} + e^{\alpha x - \frac{\beta}{x}}} e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}\left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}\right]^{\gamma - 1}}{1 - \left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}\right]^{\gamma}}, \tag{2.4}$$

$$r(x;\vartheta,\gamma,\alpha,\beta) = \frac{\vartheta\gamma\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x} + e^{\alpha x - \frac{\beta}{x}}} e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}}{1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}} - 1)}}, \tag{2.5}$$

respectively, $x > 0$ and $\vartheta, \gamma, \alpha, \beta > 0$.
Figures (1-3) display the cdf, pdf, survival, hazard rate and reversed hazard rate function of the OGE-FWE $(\vartheta, \gamma, \alpha, \beta)$ distribution for some parameter values.
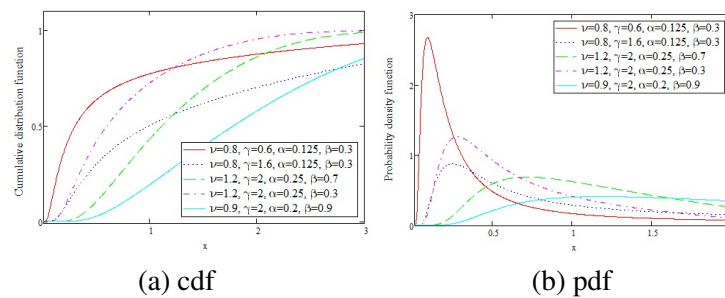
Fig. 1. The cdf and pdf of the OGE-FWE for different values of parameters.
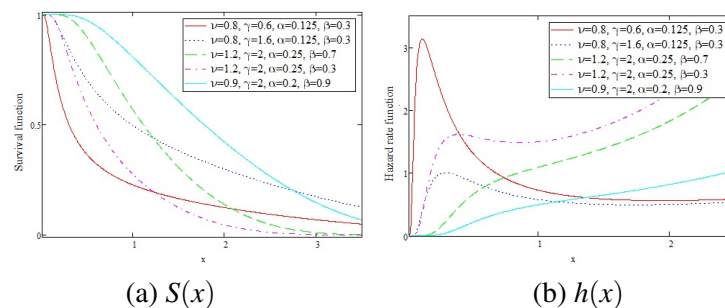

Fig. 2. The survival and hazard rate functions of the OGE-FWE for different values of parameters.
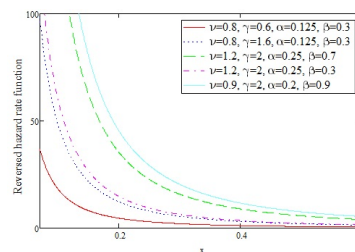

Fig. 3. The reversed hazard rate function of the OGE-FWE for different values of parameters.

3. Statistical Properties

In this section, we will study some statistical properties for the OGE-FWE distribution, specially quantile function and simulation median, the mode, moments, skewness and kurtosis.

3.1. *Quantile and median*

We determine the explicit formulas of the quantile and simulation median of the OGE-FWE distribution. The quantile x_q of the OGE-FWE$(\vartheta, \gamma, \alpha, \beta)$ distribution is given by

$$F(x_q) = P[x_q \leq q] = q, \ 0 < q < 1. \tag{3.1}$$

From Eq. (2.1), we have

$$\left[1 - e^{-\vartheta(e^{e^{\alpha x_q - \frac{\beta}{x_q}}} - 1)}\right]^{\gamma} = q, \tag{3.2}$$

we obtain x_q by solving the following equation.

$$\alpha x_q^2 - k(q)x_q - \beta = 0, \tag{3.3}$$

where

$$k(q) = \ln\left\{\ln\left[1 - \frac{\ln(1 - q^{\frac{1}{\gamma}})}{\vartheta}\right]\right\}.$$

So, the simulation of the OGE-FWE random variable is straightforward. If U is a uniform random variable on unit interval $(0,1)$. Then, by means of the inverse transformation method, we can obtain the random variable X as follows

$$X = \frac{k(u) \pm \sqrt{k(u)^2 + 4\alpha\beta}}{2\alpha}. \tag{3.4}$$

Since the median of OGE-FWE distribution can be obtain by setting $q = 0.5$ in Eq. (3.3).

3.2. *The mode*

In this subsection, we can obtain the mode of the OGE-FWE distribution by differentiating its probability density function pdf with respect to x and equaling it to zero. The mode is the solution the following equation

$$f'(x) = 0. \tag{3.5}$$

Since

$$f(x; \vartheta, \gamma, \alpha, \beta)) = h(x; \vartheta, \gamma, \alpha, \beta)S(x; \vartheta, \gamma, \alpha, \beta).$$

Then from Eq. (3.5), we have

$$\left[h'(x; \vartheta, \gamma, \alpha, \beta) - h^2(x; \vartheta, \gamma, \alpha, \beta)\right]S(x; \vartheta, \gamma, \alpha, \beta) = 0, \tag{3.6}$$

where $h(x; \vartheta, \gamma, \alpha, \beta)$ is hazard rate function of OGE-FWE distribution Eq. (2.4), and $S(x; \vartheta, \gamma, \alpha, \beta)$ is survival function of OGE-FWE Eq. (2.3).
It is difficult to get an analytic solution in x to Eq. (3.6) in the general case. So, it has to be obtained by numerically methods.

3.3. *Skewness and Kurtosis*

In this subsection, we obtain the skewness and kurtosis based on quantile measures. The Moors Kurtosis is given by, [13]

$$K_u = \frac{q_{(0.875)} - q_{(0.625)} - q_{(0.375)} + q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}}, \tag{3.7}$$

and the Bowely's skewness based on quartiles is given by, [10]

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}},$$

(3.8)

where $q_{(.)}$ is quantile function.

3.4. *The Moments*

We derive the rth moment for the OGE-FWE distribution in Theorem (3.1)

Theorem 3.1. *If X has OGE-FWE $(\vartheta, \gamma, \alpha, \beta)$ distribution, then The rth moments of random variable X, is given by*

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+m}\gamma\vartheta^{j+1}(i+1)^j(j-k+1)^{\ell}\beta^m}{j!\ell!m!(\ell+1)^{r-2m-1}\alpha^{r-m-1}} \times$$
$$\binom{\gamma-1}{i}\binom{j}{k}\left[\frac{\Gamma(r-m+1)}{\alpha(\ell+1)^2} + \beta\Gamma(r-m-1)\right].$$

(3.9)

Proof. We start with the well known distribution of the rth moment of the random variable X with probability density function $f(x)$ given by

$$\mu'_r = \int_0^{\infty} x^r f(x;\vartheta,\gamma,\alpha,\beta)dx.$$

(3.10)

Substituting from Eq. (2.2) into Eq. (3.10) we get

$$\mu'_r = \int_0^{\infty} x^r \vartheta\gamma\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} e^{e^{\alpha x - \frac{\beta}{x}}} e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}}-1)}\left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}}-1)}\right]^{\gamma-1} dx,$$

since $0 < \left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}}-1)}\right] < 1$ for $x > 0$, the binomial series expansion of $\left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}}-1)}\right]^{\gamma-1}$ yields

$$\left[1 - e^{-\vartheta(e^{e^{\alpha x - \frac{\beta}{x}}}-1)}\right]^{\gamma-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\gamma-1}{i}e^{-\vartheta i(e^{e^{\alpha x - \frac{\beta}{x}}}-1)},$$

then we get

$$\mu'_r = \sum_{i=0}^{\infty}(-1)^i\binom{\gamma-1}{i}\vartheta\gamma\int_0^{\infty} x^r\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}}e^{e^{\alpha x - \frac{\beta}{x}}}e^{-\vartheta(i+1)(e^{e^{\alpha x - \frac{\beta}{x}}}-1)}dx,$$

using series expansion

$$e^{-\vartheta(i+1)(e^{e^{\alpha x - \frac{\beta}{x}}}-1)} = \sum_{j=0}^{\infty}\frac{(-1)^j\vartheta^j(i+1)^j}{j!}\left(e^{e^{\alpha x - \frac{\beta}{x}}}-1\right)^j,$$

we obtain

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{\gamma-1}{i}\frac{\gamma\vartheta^{j+1}(i+1)^j}{j!}\int_0^{\infty} x^r\left(\alpha + \frac{\beta}{x^2}\right)e^{\alpha x - \frac{\beta}{x}}e^{e^{\alpha x - \frac{\beta}{x}}}\left[e^{e^{\alpha x - \frac{\beta}{x}}}-1\right]^j dx.$$

Using series expansion

$$\left(e^{e^{\alpha x - \frac{\beta}{x}}} - 1 \right)^j = \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} e^{(j-k)e^{\alpha x - \frac{\beta}{x}}},$$

hence

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\gamma-1}{i} \binom{j}{k} \frac{(-1)^{i+j+k} \gamma \vartheta^{j+1} (i+1)^j}{j!} \int_0^{\infty} x^r \left(\alpha + \frac{\beta}{x^2} \right) e^{\alpha x - \frac{\beta}{x}} e^{(j-k+1)e^{\alpha x - \frac{\beta}{x}}} dx,$$

using series expansion

$$e^{(j-k+1)e^{\alpha x - \frac{\beta}{x}}} = \sum_{\ell=0}^{\infty} \frac{(j-k+1)^{\ell} e^{\ell(\alpha x - \frac{\beta}{x})}}{\ell!},$$

we obtain

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \binom{\gamma-1}{i} \binom{j}{k} \frac{(-1)^{i+j+k} \gamma \vartheta^{j+1} (i+1)^j (j-k+1)^{\ell}}{j! \ell!} \int_0^{\infty} x^r \left(\alpha + \frac{\beta}{x^2} \right) e^{(\ell+1)(\alpha x - \frac{\beta}{x})} dx.$$

Using series expansion

$$e^{-(\ell+1)\frac{\beta}{x}} = \sum_{m=0}^{\infty} \frac{(-\beta)^m (\ell+1)^m}{m!} x^{-m},$$

gives

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+m} \gamma \vartheta^{j+1} (i+1)^j (j-k+1)^{\ell} \beta^m (\ell+1)^m}{j! \ell! m!} \binom{\gamma-1}{i} \binom{j}{k} \times$$

$$\int_0^{\infty} x^{r-m} \left(\alpha + \frac{\beta}{x^2} \right) e^{(\ell+1)\alpha x} dx,$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+m} \gamma \vartheta^{j+1} (i+1)^j (j-k+1)^{\ell} \beta^m (\ell+1)^m}{j! \ell! m!} \binom{\gamma-1}{i} \binom{j}{k} \times$$

$$\left[\int_0^{\infty} \alpha x^{r-m} e^{(\ell+1)\alpha x} dx + \int_0^{\infty} \beta x^{r-m-2} e^{(\ell+1)\alpha x} dx \right],$$

by using the definition of gamma function in the form, [26],

$$\Gamma(z) = x^z \int_0^{\infty} e^{tx} t^{z-1} dt, \ z, x, > 0.$$

Finally, we obtain the rth moment of OGE-FWE distribution in the form

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+m} \gamma \vartheta^{j+1} (i+1)^j (j-k+1)^{\ell} \beta^m (\ell+1)^m}{j! \ell! m!} \binom{\gamma-1}{i} \binom{j}{k} \times$$

$$\left[\frac{\Gamma(r-m+1)}{\alpha^{r-m}(\ell+1)^{r-m+1}} + \frac{\beta \Gamma(r-m-1)}{\alpha^{r-m-1}(\ell+1)^{r-m-1}} \right].$$

This completes the proof. □

4. The Moment Generating Function

The moment generating function (mgf) of the OGE-FWE distribution is given by Theorem (4.1).

Theorem 4.1. *The moment generating function (mgf) of OGE-FWE distribution is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+m}\gamma\vartheta^{j+1}(i+1)^j(j-k+1)^{\ell}\beta^m t^r}{j!\ell!m!r!\alpha^{r-m-1}(\ell+1)^{r-2m-1}}$$
$$\times \binom{\gamma-1}{i}\binom{j}{k}\left[\frac{\Gamma(r-m+1)}{\alpha(\ell+1)^2} + \beta\Gamma(r-m-1)\right]. \tag{4.1}$$

Proof. The moment generating function $M_X(t)$ of the random variable X with probability density function $f(x)$ is given by

$$M_X(t) = \int_0^{\infty} e^{tx} f(x; \vartheta, \gamma, \alpha, \beta)dx, \tag{4.2}$$

Using series expansion of e^{tx}, we obtain

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^{'} \tag{4.3}$$

Substituting from (3.9) into (4.3), we obtain

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j+k+m}\gamma\vartheta^{j+1}(i+1)^j(j-k+1)^{\ell}\beta^m t^r}{j!\ell!m!r!\alpha^{r-m-1}(\ell+1)^{r-2m-1}}$$
$$\times \binom{\gamma-1}{i}\binom{j}{k}\left[\frac{\Gamma(r-m+1)}{\alpha(\ell+1)^2} + \beta\Gamma(r-m-1)\right]. \tag{4.4}$$

This completes the proof. □

5. Order Statistics

In this section, we derive closed form expressions for the PDFs of the rth order statistic of the OGE-FWE distribution. Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ denote the order statistics obtained from a random sample $X_1, X_2, \cdots, X_n$ which taken from a continuous population with cumulative distribution function cdf $F(x; \varphi)$ and probability density function pdf $f(x; \varphi)$, then the pdf of $X_{r:n}$ is given as follows

$$f_{r:n}(x; \varphi) = \frac{1}{B(r, n-r+1)}[F(x; \varphi)]^{r-1}[1 - F(x; \varphi)]^{n-r} f(x; \varphi), \tag{5.1}$$

where $f(x; \varphi)$ and $F(x; \varphi)$ are the pdf and cdf of OGE-FWE(φ) distribution given by Eq. (2.2) and Eq. (1.7) respectively, $\varphi = (\vartheta, \gamma, \alpha, \beta)$ and $B(., .)$ is the Beta function, also we define first order statistics $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$ and the last order statistics as $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$. Since

$0 < F(x; \varphi) < 1$ for $x > 0$, we can use the binomial expansion of $[1 - F(x; \varphi)]^{n-r}$ as follows

$$[1 - F(x; \varphi)]^{n-r} = \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x; \varphi)]^i. \tag{5.2}$$

Substituting from Eq. (5.2) into Eq. (5.1), we obtain

$$f_{r:n}(x; \vartheta, \gamma, \alpha, \beta) = \sum_{i=0}^{n-r} \frac{(-1)^i n!}{i!(r-1)!(n-r-i)!} f(x; \varphi)[F(x; \varphi)]^{i+r-1}. \tag{5.3}$$

Substituting from Eq. (2.1) and Eq. (2.2) into Eq. (5.3), we obtain the probability density function for rth order statistics.

Relation (5.3) show that $f_{r:n}(x; \varphi)$ is the weighted average of the OGE-FWE distribution with different shape parameters.

6. Parameters Estimation

In this section, point and interval estimation of the unknown parameters of the OGE-FWE distribution are derived by using the maximum likelihood method based on a complete sample.

6.1. *Maximum Likelihood Estimation:*

Let $x_1, x_2, \cdots, x_n$ denote a random sample of complete data from the OGE-FWE distribution. The Likelihood function is given as

$$L = \prod_{i=1}^{n} f(x_i; \vartheta, \gamma, \alpha, \beta), \tag{6.1}$$

substituting from (2.2) into (6.1), we have

$$L = \prod_{i=1}^{n} \vartheta \gamma \left(\alpha + \frac{\beta}{x_i^2} \right) e^{\alpha x_i - \frac{\beta}{x_i}} e^{e^{\alpha x_i - \frac{\beta}{x_i}}} e^{-\vartheta[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1]} \left[1 - e^{-\vartheta[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1]} \right]^{\gamma - 1}.$$

The log-likelihood function is

$$\mathcal{L} = n \ln(\vartheta \gamma) + \sum_{i=1}^{n} \ln \left(\alpha + \frac{\beta}{x_i^2} \right) + \sum_{i=1}^{n} \left(\alpha x_i - \frac{\beta}{x_i} \right) + \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} - \vartheta \sum_{i=1}^{n} \left(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1 \right)$$

$$+ (\gamma - 1) \sum_{i=1}^{n} \ln \left[1 - e^{-\vartheta (e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} \right]. \tag{6.2}$$

The maximum likelihood estimation of the parameters $(\vartheta, \gamma, \alpha, \beta)$ are obtained by differentiated the log-likelihood function $\mathcal{L}$ with respect to the parameters $\vartheta, \gamma, \alpha$ and β and setting the result to zero, we have the following normal equations.

$$\frac{\partial \mathscr{L}}{\partial \vartheta} = \frac{n}{\vartheta} - \sum_{i=1}^{n} \left(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1 \right) + (\gamma - 1) \sum_{i=1}^{n} \frac{e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1}{e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1} = 0 \tag{6.3}$$

$$\frac{\partial \mathscr{L}}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \ln \left(1 - e^{-\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} \right) = 0 \tag{6.4}$$

$$\frac{\partial \mathscr{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{x_i^2}{\beta + \alpha x_i^2} + \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} - \vartheta \sum_{i=1}^{n} x_i \mathscr{D}_i + \vartheta(\gamma - 1) \sum_{i=1}^{n} \frac{x_i \mathscr{D}_i}{e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1} = 0 \tag{6.5}$$

$$\frac{\partial \mathscr{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{\beta + \alpha x_i^2} - \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}} + \vartheta \sum_{i=1}^{n} \frac{\mathscr{D}_i}{x_i} - \vartheta(\gamma - 1) \sum_{i=1}^{n} \frac{\mathscr{D}_i}{x_i \left[e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1 \right]} = 0, \tag{6.6}$$

where $\mathscr{D}_i = \exp \left\{ \alpha x_i - \frac{\beta}{x_i} + e^{\alpha x_i - \frac{\beta}{x_i}} \right\}$. The MLEs can be obtained by solving the equations, (6.3) – (6.6), numerically for $\vartheta, \gamma, \alpha$ and β.

6.2. *Asymptotic confidence bounds*

In this section, we derive the asymptotic confidence intervals of these parameters when $\vartheta, \gamma, \alpha > 0$ and $\beta > 0$ as the MLEs of the unknown parameters $\vartheta, \gamma, \alpha > 0$ and $\beta > 0$ can not be obtained in closed forms, by using variance covariance matrix I^{-1} see [12], where I^{-1} is the inverse of the observed information matrix which defined as follows

$$\mathbf{I}^{-1} = \begin{pmatrix} -\frac{\partial^2 \mathscr{L}}{\partial \vartheta^2} & -\frac{\partial^2 \mathscr{L}}{\partial \vartheta \partial \gamma} & -\frac{\partial^2 \mathscr{L}}{\partial a \partial \alpha} & -\frac{\partial^2 \mathscr{L}}{\partial \vartheta \partial \beta} \\ -\frac{\partial^2 \mathscr{L}}{\partial \gamma \partial \vartheta} & -\frac{\partial^2 \mathscr{L}}{\partial \gamma^2} & -\frac{\partial^2 \mathscr{L}}{\partial \gamma \partial \alpha} & -\frac{\partial^2 \mathscr{L}}{\partial \gamma \partial \beta} \\ -\frac{\partial^2 \mathscr{L}}{\partial \alpha \partial \vartheta} & -\frac{\partial^2 \mathscr{L}}{\partial \alpha \partial \gamma} & -\frac{\partial^2 \mathscr{L}}{\partial \alpha^2} & -\frac{\partial^2 \mathscr{L}}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \mathscr{L}}{\partial \beta \partial \vartheta} & -\frac{\partial^2 \mathscr{L}}{\partial \beta \partial \gamma} & -\frac{\partial^2 \mathscr{L}}{\partial \beta \partial \alpha} & -\frac{\partial^2 \mathscr{L}}{\partial \beta^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\hat{\vartheta}) & cov(\hat{\vartheta}, \hat{\gamma}) & cov(\hat{\vartheta}, \hat{\alpha}) & cov(\hat{\vartheta}, \hat{\beta}) \\ cov(\hat{\gamma}, \hat{\vartheta}) & var(\hat{\gamma}) & cov(\hat{\gamma}, \hat{\alpha}) & cov(\hat{\gamma}, \hat{\beta}) \\ cov(\hat{\alpha}, \hat{\vartheta}) & cov(\hat{\alpha}, \hat{\gamma}) & var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\beta}) \\ cov(\hat{\beta}, \hat{\vartheta}) & cov(\hat{\beta}, \hat{\gamma}) & cov(\hat{\beta}, \hat{\alpha}) & var(\hat{\beta}) \end{pmatrix}, \tag{6.7}$$

where

$$\frac{\partial^2 \mathscr{L}}{\partial \vartheta^2} = -\frac{n}{\vartheta^2} - (\gamma - 1) \sum_{i=1}^{n} e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} \mathscr{A}_i^2, \qquad \frac{\partial^2 \mathscr{L}}{\partial \vartheta \partial \gamma} = \sum_{i=1}^{n} \mathscr{A}_i \tag{6.8}$$

$$\frac{\partial^2 \mathscr{L}}{\partial \vartheta \partial \alpha} = -\sum_{i=1}^{n} x_i \mathscr{D}_i + (\gamma - 1) \sum_{i=1}^{n} x_i \mathscr{B}_i, \qquad \frac{\partial^2 \mathscr{L}}{\partial \vartheta \partial \beta} = \sum_{i=1}^{n} \frac{\mathscr{D}_i}{x_i} - (\gamma - 1) \sum_{i=1}^{n} \frac{\mathscr{B}_i}{x_i} \tag{6.9}$$

$$\frac{\partial^2 \mathscr{L}}{\partial \gamma^2} = -\frac{n}{\gamma^2}, \qquad \frac{\partial^2 \mathscr{L}}{\partial \gamma \partial \alpha} = \vartheta \sum_{i=1}^{n} \frac{x_i \mathscr{D}_i}{e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1} \tag{6.10}$$

$$\frac{\partial^2 \mathscr{L}}{\partial \gamma \partial \beta} = -\vartheta \sum_{i=1}^{n} \frac{\mathscr{D}_i}{x_i \left[e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1 \right]} \tag{6.11}$$

$$\frac{\partial^2 \mathscr{L}}{\partial \alpha^2} = -\sum_{i=1}^{n} \frac{x_i^4}{(\beta + \alpha x_i^2)^2} + \sum_{i=1}^{n} x_i^2 e^{\alpha x_i - \frac{\beta}{x_i}} - \vartheta \sum_{i=1}^{n} x_i^2 \mathscr{D}_i [e^{\alpha x_i - \frac{\beta}{x_i}} + 1] + \vartheta(\gamma - 1) \sum_{i=1}^{n} x_i^2 \mathscr{K}_i \tag{6.12}$$

$$\frac{\partial^2 \mathscr{L}}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} \frac{x_i^2}{(\beta + \alpha x_i^2)^2} - \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} + \vartheta \sum_{i=1}^{n} \mathscr{D}_i [e^{\alpha x_i - \frac{\beta}{x_i}} + 1] - \vartheta(\gamma - 1) \sum_{i=1}^{n} \mathscr{K}_i \tag{6.13}$$

$$\frac{\partial^2 \mathscr{L}}{\partial \beta^2} = -\sum_{i=1}^{n} \frac{1}{(\beta + \alpha x_i^2)^2} + \sum_{i=1}^{n} \frac{e^{\alpha x_i - \frac{\beta}{x_i}}}{x_i^2} - \vartheta \sum_{i=1}^{n} \frac{\mathscr{D}_i [e^{\alpha x_i - \frac{\beta}{x_i}} + 1]}{x_i^2} + \vartheta(\gamma - 1) \sum_{i=1}^{n} \frac{\mathscr{K}_i}{x_i^2} \tag{6.14}$$

where

$$\mathscr{A}_i = \left[e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1 \right] \left[e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1 \right]^{-1},$$

$$\mathscr{B}_i = \mathscr{D}_i \left[e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} \left(1 - \vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1) \right) - 1 \right] \left[e^{\vartheta[e^{\alpha x_i - \frac{\beta}{x_i}} - 1]} - 1 \right]^{-2},$$

$$\mathscr{K}_i = \mathscr{D}_i \left[(e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1)(e^{\alpha x_i - \frac{\beta}{x_i}} + 1) - \vartheta \mathscr{D}_i e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} \right] \left[e^{\vartheta(e^{e^{\alpha x_i - \frac{\beta}{x_i}}} - 1)} - 1 \right]^{-2}.$$

We can obtain the $(1 - \delta)100\%$ confidence intervals of the parameters $\vartheta, \gamma, \alpha$ and β by using variance matrix as in the following forms

$$\hat{\vartheta} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\vartheta})}, \quad \hat{\gamma} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\gamma})}, \quad \hat{\alpha} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{var(\hat{\beta})},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})$-th percentile of the standard normal distribution.

7. Application

In this section, we will analysis of a real data set using the OGE-FWE $(\vartheta, \gamma, \alpha, \beta)$ model and compare it with the other fitted models like a flexible Weibull extension distributions using Kolmogorov Smirnov (K-S) statistic, as well as Akaike information criterion(AIC), [?], Akaike Information Citerion with correction (AICC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Schwarz information criterion (SIC) values, [21]. The data have been obtained from [18], it is for the time between failures (thousands of hours) of secondary reactor pumps, Table 1.

Table 1. Time between failures of secondary reactor pumps.

2.160	0.746	0.402	0.954	0.491	6.560	4.992	0.347
0.150	0.358	0.101	1.359	3.465	1.060	0.614	1.921
4.082	0.199	0.605	0.273	0.070	0.062	5.320	

Table 2. MLEs and K–S of parameters for secondary reactor pumps.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\vartheta}$	$\hat{\gamma}$	K-S
OGE-FWE	0.2380	2.0700	–	0.069	0.113	0.0760
Flexible Weibull	0.0207	2.5875	–	–	–	0.1342
Weibull	0.8077	13.9148	–	–	–	0.1173
Modified Weibull	0.1213	0.7924	0.0009	–	–	0.1188
Reduced Additive Weibull	0.0070	1.7292	0.0452	–	–	0.1619
Extended Weibull	0.4189	1.0212	10.2778	–	–	0.1057

Table 3. Log-likelihood, AIC, AICC, BIC, HQIC and SIC values of models fitted.

Model	L	AIC	AICC	BIC	HQIC	SIC
OGE-FEW	-29.2980	66.5960	68.8182	71.1380	10.5590	71.1380
Flexible Weibull	-83.3424	170.6848	171.2848	172.9558	12.5416	172.9558
Weibull	-85.4734	174.9468	175.5468	177.2178	12.5915	177.2178
Modified Weibull	-85.4677	176.9354	178.1986	180.3419	12.6029	180.3419
Reduced additive Weibull	-86.0728	178.1456	179.4088	181.5521	12.6168	181.5521
Extended Weibull	-86.6343	179.2686	180.5318	182.6751	12.6296	182.6751

Table 2 gives MLEs of parameters of the OGE-FWE and K-S Statistics. The values of the log-likelihood functions, AIC, AICC, BIC, HQIC, and SIC are in Table 3.

Substituting the MLEs of the unknown parameters $\vartheta, \gamma, \alpha, \beta$ into (6.7), we obtain estimation of the variance covariance matrix as the following

$$
I_0^{-1} = \begin{pmatrix}
6.773 \times 10^{-3} & 4.189 \times 10^{-4} & -3.945 \times 10^{-3} & -1.65 \times 10^{-3} \\
4.189 \times 10^{-4} & 4.308 \times 10^{-4} & -4.871 \times 10^{-4} & -0.013 \\
-3.945 \times 10^{-3} & -4.871 \times 10^{-4} & 3.343 \times 10^{-3} & 0.017 \\
-1.65 \times 10^{-3} & -0.013 & 0.017 & 0.806
\end{pmatrix}.
$$

The approximate 95% two sided confidence intervals of the unknown parameters $\vartheta, \gamma, \alpha$ and β are $[0, 0.230], [0.072, 0.154], [0.125, 0.351]$ and $[0.31, 3.83]$, respectively.

The nonparametric estimate of the survival function $S(x)$ using the Kaplan-Meier method and its fitted parametric estimations when the distribution is assumed to be OGE-FWE, FW, W, MW, RAW and EW are computed and plotted in Figure 4.
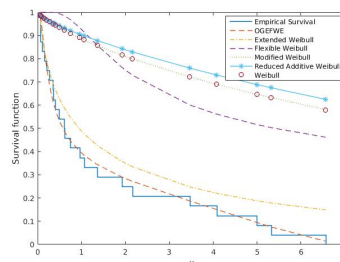

Fig. 4. The Kaplan-Meier estimate of the survival function for the data.

Figure 5 gives the form of the hazard rate $h(x)$ and cumulative density function cdf for the OGE-FWE, FW, W, MW, RAW and EW which are used to fit the data after the unknown parameters included in each distribution are replaced by their MLEs.
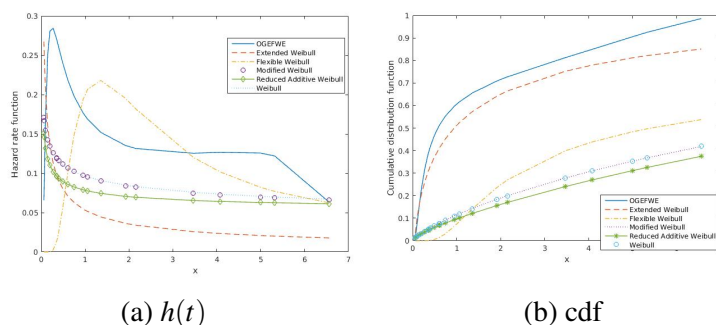

(a) $h(t)$ (b) cdf

Fig. 5. The Fitted hazard rate and cumulative distribution function for the data.

We find that the OGE-FWE distribution with the four - parameters provides a better fit than the previous new modified a flexible Weibull extension distribution(FWE) which was the best in [4]. It has the largest likelihood, and the smallest AIC, AICC, BIC, HQIC and SIC values among those considered in this paper.

8. Conclusions

We proposed a new distribution, based on odd generalized exponential family distributions, this distribution is named the odd generalized exponential flexible Weibull extension OGE-FWE distribution. Some its statistical properties are studied. The maximum likelihood method is used for estimating the parameters model. Finally, we introduce an application using real data. We show that the OGE-FWE distribution fits certain well known data sets better than existing modifications of the Weibull and flexible Weibull extension distributions.

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