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The Theoretical Limit of Sagnac Effect

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Abstract. The precision of optical gyroscope can be improved by enhancing Sagnac effect. Aiming at the accuracy of optical gyroscope restricted by the theoretical limit, shot noise limit, we firstly analyze the theoretical limit of Sagnac effect in optical regime with quantum theory, and point out the physical reasons for the precision limit. And then a quantum entanglement state, Yurke state, with photon number difference measurement is advised to break through shot noise limit for Sagnac effect enhancement. This proposed scheme can achieve the true quantum limit, Heisenberg limit, and offer a new idea for improving the performance of optical sensors based on Sagnac effect.

1 Introduction

Sagnac effect is first proposed by a French scientist Georges Sagnac in 1913 [1], which has been widely used in the field of sensing, including current sensor [2], pressure sensor [3] and temperature sensor [4], etc., and is also the physical basis of optical gyroscope [5]. The optical gyroscope, mainly including laser gyroscope and fiber optic gyroscope, is the most popular inertial navigation device at present, and plays a vital role in military field [6]. The performance of such sensors, especially the sensitivity of the optical gyroscope can be improved by enhancing Sagnac effect [7-11].

Nevertheless, there is an insurmountable theoretical limit for the accuracy improvement of optical gyroscope due to the physical property of light field [12]. By optimizing the components, structures and measurement methods continuously, the accuracy of optical gyroscope is approaching the theoretical limit [13]. It is extremely difficult to further improve the accuracy of optical gyroscope. According to wave-particle duality theory, many scholars have proposed to enhance the Sagnac effect by using electron [14], neutron [15], atom [16], and superfluid [17] etc.. However, there are few researches on how to enhance Sagnac effect in optical regime for the precision of optical gyroscope improvement.

This paper firstly analyzes the theoretical limit of Sagnac effect in optical regime with quantum theory, and explains physical reasons of the precision limit. And then a quantum entanglement state, Yurke state, with photon number difference measurement is advised to break through the classical theoretical limit which can achieve the true quantum limit. Finally, we draw a conclusion. This paper provides a new idea for further enhancing Sagnac effect in optical regime with quantum theory.

2 The Principle of Classical Sagnac Effect

Sagnac effect means that a beam of light is divided into two beams which translate along clockwise and anti-clockwise direction respectively, and will introduce a non-reciprocity phase shift as the interferometer rotates. The phase shift is proportional to the angular velocity. In particular, as shown in Fig.1, a beam of light is divided into two beams at point A through a beam splitter (BS). They respectively translate along the clockwise and anti-clockwise direction. If the interferometer doesn't rotate, these two beams will join and form interference fringes at point A through the same time $t = 2\pi r/c$ where r is the interferometer radius and c is the speed of light. If the interferometer rotates clockwise at angular rate Ω_r , the optical path difference of these two beams propagating a circle can be written as

$$\Delta L = \frac{4\pi r^2 \Omega_r}{c} \tag{1}$$

where $\vec{n}$ is the unit normal vector of the interferometer plane, $\vec{\Omega}_r$ is the rotation vector and S is the interference loop area. The equation(1) indicates that the optical path difference has nothing to do with the shape of the interferometer but depends on the flux of the rotation vector $\vec{\Omega}_r$. This flux can be effectively enhanced by using optical fiber to form a multi-loop circular light path. If the optical fiber length is L, the phase difference is

$$\theta = \frac{4\pi L r \Omega_r}{c\lambda} \tag{2}$$

where λ is the wavelength of light. It can be seen that the phase difference θ is linear to the length of the optical fiber L, the interferometer radius r and the rotation speed Ω_r. By measuring the phase difference θ, the rotation speed Ω_r can be obtained directly, of which accuracy depends on the measurement precision of phase difference θ. It only needs to improve the precision of phase difference θ for Sagnac effect enhancement.

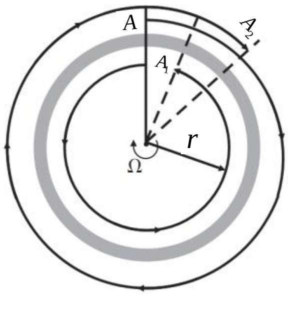

Fig.1 The Schematic Diagram of Sagnac Effect

3 The Quantum Model of Sagnac Effect

In the Sagnac interferometer shown in Fig. 2, when the two beams of light $\hat{a}_i$ and $\hat{b}_i$ pass through a beam splitter, the relative phase caused by interferometer rotation can be represented by a scattering matrix $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, where $\hat{a}_i^+(\hat{a}_i)$ and $\hat{b}_i^+(\hat{b}_i)$ are respectively generation (annihilation) operators of the input port a_i and b_i respectively. Consequently, the output light $\hat{a}_f$ and $\hat{b}_f$ of Sagnac interferometer can be expressed as

$$\begin{pmatrix}\hat{a}_f\\ \hat{b}_f\end{pmatrix}=\begin{pmatrix}t_{BS} & r_{BS}\\ r_{BS}' & t_{BS}'\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}\begin{pmatrix}t_{BS} & r_{BS}'\\ r_{BS} & t_{BS}'\end{pmatrix}\begin{pmatrix}\hat{a}_i\\ \hat{b}_i\end{pmatrix} \tag{3}$$

where $t_{BS}=t'_{BS}=1/\sqrt{2}$, $r_{BS}=r'_{BS}=-i/\sqrt{2}$ for 50:50 beam splitter and $\hat{a}_f^+(\hat{a}_f)$ and $\hat{b}_f^+(\hat{b}_f)$ are respectively generation (annihilation) operators of the output port a_f and b_f. The equation (3) can be rewritten as

$$\begin{pmatrix}\hat{a}_f\\ \hat{b}_f\end{pmatrix}=-ie^{i\theta/2}\begin{pmatrix}\sin\theta/2 & \cos\theta/2\\ \cos\theta/2 & -\sin\theta/2\end{pmatrix}\begin{pmatrix}\hat{a}_i\\ \hat{b}_i\end{pmatrix} \tag{4}$$

where $-ie^{i\theta/2}$ is a global phase and can be ignored as it does not reflect the relative phase of the two beams of light. Finally, the relationship between the input and output in the Sagnac interferometer is

$$\begin{pmatrix}\hat{a}_f\\ \hat{b}_f\end{pmatrix}=\begin{pmatrix}\sin\theta/2 & \cos\theta/2\\ \cos\theta/2 & -\sin\theta/2\end{pmatrix}\begin{pmatrix}\hat{a}_i\\ \hat{b}_i\end{pmatrix} \tag{5}$$

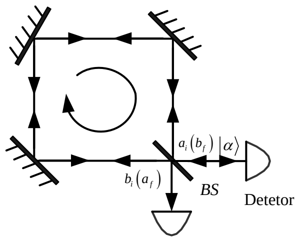

Fig.2 The Schematic Diagram of Sagnac Interferometer

4 Classical Physical Limit of Sagnac Effect

When a laser beam $|\alpha\rangle$ in coherent state enters into the Sagnac interferometer, If the relative phase $\theta=0$, destructive interference occurs at a_f and there will be no photon output. Meanwhile, constructive interference occurs at b_f where all the photons output. If $\theta=\pi$, constructive interference occurs at the output port a_f while destructive interference occurs at the output port b_f. For a macroscopic beam of laser, the intensity of the output light detected at the output port a_f is proportional to $\cos^2(\theta/2)$ and the intensity of the output light detected at the output port b_f is proportional to $\sin^2(\theta/2)$. For a single photon, $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$ indicates the probability of detecting the photon at the output port a_f and b_f. Obviously, there is a correspondence between the number of output photons and the phaseθ, and the phase information can be achieved by measuring the number of output photons. In order to investigate the theoretical limit of the Sagnac effect, we measure the difference of the output photon number for analysis.

The photon number difference between these two output ports a_f and b_f is

$$\hat{M}=\hat{n}_a-\hat{n}_b \tag{6}$$

where $\hat{n}_a$ and $\hat{n}_b$ are respectively the output photon number operators of output port a_f and b_f, $\hat{n}_a=\hat{a}_f^+\hat{a}_f$ and $\hat{n}_b=\hat{b}_f^+\hat{b}_f$ With the Sagnac input and output relationship, the equation (6) can be further written as

$$\hat{M}=\cos\theta\left(-\hat{a}_i^+\hat{a}_i+\hat{b}_i^+\hat{b}_i\right)+\sin\theta\left(\hat{a}_i^+\hat{b}_i+\hat{a}_i\hat{b}_i^+\right) \tag{7}$$

The first and second moment of the photon number difference operator $\widehat{M}$ respectively are

$$\langle \hat{M} \rangle = -\bar{n}_T \cos\theta$$
$$\langle \hat{M}^2 \rangle = \bar{n}_T + \bar{n}_T^{\,2} \cos^2\theta \tag{8}$$

Then

$$\Delta^2 \hat{M} = \langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2 = \bar{n}_T \tag{9}$$

and

$$\frac{d\langle \hat{M} \rangle}{d\theta} = \bar{n}_T \sin\theta \tag{10}$$

According to the error transformation formula [18], we can obtain the phase uncertaity

$$\delta\theta = \frac{\Delta\hat{M}}{\left| d\langle \hat{M} \rangle / d\theta \right|} = \frac{1}{\sqrt{\bar{n}_T}\sin\theta} \tag{11}$$

Obviously, the phase accuracy is relative to the phase θ, which is consistent with the phase sensitivity of the classical Sagnac effect. When $\theta = \pi/2$, there is a minimum sensitivity $\delta\theta = 1/\sqrt{\bar{n}_T}$ which is called shot noise limit (SNL). In a interferometer, the SNL is caused by the vacuum fluctuation. The input port of BS will introduce vacuum fluctuation due to the absence of input light, resulting in phase accuracy limited by the SNL.

The sensitivity of interference limited by another factor is the discrete nature of light. Assume that there are N photons in an optical system, according to the Heisenberg uncertainty principle, the number of photons N and the fluctuations of the phase θ satisfy the inequality

$$\delta N \delta\theta \geq 1 \tag{12}$$

where δN and δθ respectively represent the photon number fluctuation of N and phase θ. The ideal laser is in coherent state $|\alpha\rangle$, and the probability of detecting N photons is

$$P(N) = \frac{e^{-\bar{n}_T} \bar{n}_T^{\,N}}{N!} \tag{13}$$

where $\bar{n}_T = |\alpha|^2$ is the total average number of photons. It can be found that the probability is Poisson distribution. The second moment of the photon number is

$$\langle N^2 \rangle = \langle \alpha | \hat{a}^+ \hat{a} \hat{a}^+ \hat{a} | \alpha \rangle = \bar{n}_T^{\,2} + \bar{n}_T \tag{14}$$

Then

$$\delta N = \sqrt{\langle N^2 \rangle - \bar{n}_T^{\,2}} = \sqrt{\bar{n}_T} \tag{15}$$

Combining equation (12), we obtain

$$\delta\theta \geq 1/\sqrt{\bar{n}_T} \tag{16}$$

which is the SNL. Heisenberg uncertainty principle is the fundamental principle of quantum mechanics, which has been confirmed by numerous experiments and theories. It can be seen that the SNL is also determined by the discrete nature of electromagnetic field of which the photons has Poisson statistical properties.

Through the analysis above, we point out two factors that restrict the accuracy of phase as follows.

(1) The vacuum fluctuation introduced by beam splitter in the interferometer.

(2) The particle property of the input quantum field which obey Poisson distribution and are independent of each other.

Any one of the two factors above will cause the phase accuracy to be limited by SNL.

5 Heisenberg Limit

According to the above analyses, in order to obtain the sensitivity beyond SNL, it is necessary to feed two beams of light in to the two input ports of the BS to suppress the vacuum fluctuation, and at the same time, the two beams of light need to satisfy certain correlation inside the interferometer. We choose Yurke state as the input state and adopt the photon number difference measurement method to obtain the phase parameter. Theoretical analysis will show that this method can overcome SNL and even reach Heisenberg limit which is the true physical limit.

The Yurke state is

$$|Yurke\rangle = \frac{1}{\sqrt{2}}\left(|N_-\rangle_a |N_+\rangle_b + |N_+\rangle_a |N_-\rangle_b\right) \tag{17}$$

where $N_\pm \equiv (N \pm 1)/2$. Yurke state is a strongly correlated quantum entanglement signal which indicates the number of photons N_+ and N_- entering into the input port a_i and b_i respectively with a probability of 1/2. In particular, the event that N_- photons in the input port a_i while N_+ photons in the input port b_i and the other event that N_+ photons in the input port a_i while N_- photons in the input port b_i occur with a probability of 1/2. We are not able to know the number of photons in the two input ports without detection. Once the number of photons at one input port is known after detection, the number of photons in the other input port is also determined. Therefore, Yurke state has very strong non-local correlation.

To obtain the phase information, the photon number difference between the two output ports is measured. According to the formula(7), the first moment of the photon number difference operator $\widehat{M}$ and its fluctuation $\Delta\widehat{M}$ respectively are

$$\begin{aligned} \langle \hat{M} \rangle &= -N_+ \sin\theta \\ \Delta^2 \hat{M} &= \cos(2\theta) + (N_+ \sin\theta)^2 \end{aligned} \tag{18}$$

According to the error transformation formula, the phase uncertainty is

$$\delta\theta = \frac{\sqrt{\cos(2\theta) + (N_+ \sin\theta)^2}}{N_+ \cos\theta} \tag{19}$$

When $\theta = 0$, we can obtain the phase uncertainty $\delta\theta = 2/(N+1)$, which breaks through SNL and reaches the order of magnitude with Heisenberg limit $1/N$. It shows that SNL can be effectively broken by using the entangled state and the corresponding measurement method. The performance of optical sensors based on Sagnac effect can be improved by breaking through SNL.

6 Conclusion

As the precision of sensors based on Sagnac effect are restricted by the classical theoretical limit, SNL, we analyze the physical reasons of the theoretical limit of Sagnac effect in optical regime with quantum theory, and point out that vacuum fluctuation and particle property obeying Poisson distribution are both the physical reasons of SNL. A quantum enhanced scheme is proposed by utilizing entanglement state, Yurke state, and photon number difference measurement, which can break through SNL and reach the true quantum limit, HL. A new idea for further enhancing Sagnac effect in optical regime with quantum theory is presented in this manuscript.

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