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Thoughts on Mathematical Ideology of "Linear Space"

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Abstract. This paper is about the author's thoughts on mathematical thinking method while teaching "linear space" of Advanced Algebra. The ideas of finite and infinite, transformation, unity and isomorphism of algebra and geometry are expounded. If we bring mathematical thinking method to teaching process to share our cognition, feelings and ideas with students, they can profoundly understand the logic of math and appreciate the beauty of math, so as to better learn math, use math, and love math.

1. Introduction

Starting from our school-University of Science and Technology, its name contains two meanings: science and technology. Science mainly studies the question of why, whose purpose is to know the world while technology focuses on the problem of what and how to do, whose purpose is to change the world. Only the unity of science and technology can form a powerful force. We have a similar issue in the course of teaching, which is about training skills and imparting ideas. The skill here refers to solving problem. Students need to prepare for the final examination and postgraduate entrance examination, which demands skill of solving problems. Both teachers and students have spent a lot of effort in this respect. Facing various examinations, we are so focused on numerous exercises that sometimes we neglect the influence of mathematical thoughts. While setting syllabus and the goal of teaching, we invariably write: "To cultivate students' ability to innovate, to analyze and solve problems, and to think logically." The cultivation of these abilities can not be achieved by only training the skills of solving problems. This is particularly important for mathematics majors.

This paper is about the author's thoughts on mathematical thinking method while teaching "linear space" of Advanced Algebra. It includes the following five aspects:

- Linear space -- a new leap in mathematical thinking;
- Base -- finite and infinite;
- Transition matrix between two bases -- transformation idea;
- Coordinates -- the unity of algebra and geometry;
- Dimension -- the essential feature of the finite dimensional linear space (isomorphism).

2. Linear space -- a new leap in mathematical thinking

Linear space is the first abstract concept that students encounter from the beginning of mathematics' learning. It is the core and soul of modern mathematics. If the norm is given on the basis of the linear space, the normed space will be obtained; if the inner product is given, the inner product space will be obtained; if the uniform space is given, the Hilbert space will be obtained. They form the framework of modern mathematics. As an algebraic system $\langle V, P, +, \bullet \rangle$, the elements in a linear space are abstract, and it can be an array with n elements, a function, or a matrix. Addition operation and multiplication operation are also abstract, as long as the corresponding 8 rules of operation are satisfied. So the

concept of linear space has a high abstraction.

Because of its high abstraction, linear space has a wide range of applications. While talking about this problem on class, some students laughed and said: " What's use of such abstract thing?" I asked them: " Is $a^2+b^2=c^2$ or $3^2+4^2=5^2$ more useful?" They became silent. In fact, linear space has a deep application in both engineering and economics.

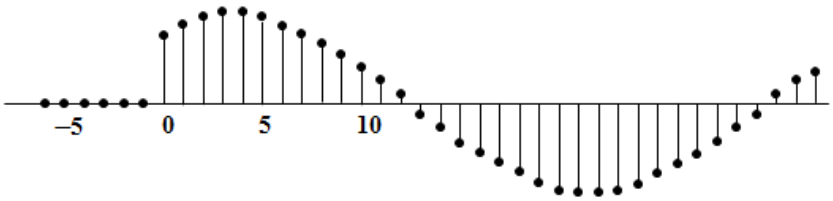
Here is a simple introduction to the application of linear space in Engineering.

Example 2.1 Assuming that S is a space made up of a series of numerical sequences with infinity at both ends, written as:

$$\{x_n\}=\{\cdots,x_{-2},x_{-1},x_0,x_1,x_2,\cdots\},\ x_i\in R.$$

If $\{y_n\}$ is one element of S, $\{x_n\}+\{y_n\}$ is a new sequence from the addition of $\{x_n\}$ and corresponding $\{y_n\}$, and quantity product $\lambda\{y_n\}$ is sequence $\{\lambda y_n\}$. As the addition and multiplication operation of the sequence is the operation of regular real numbers, and it is closed to the operation(for new sequence is a numerical sequence with infinite at both ends), the operation of S satisfies the rule of linear space, so S makes a linear space for the defined operations.

Elements in S often appear in engineering, such as the measurement or sampling of signals on a discrete time period. The signal can be electronic, mechanical, optical, etc. The central control system of space shuttle uses discrete signals. For the convenience of use, S is called (discrete time) signal space. It can be directly reflected as following:



In addition, the concept of linear space also fully embodies the freedom of thought. Its system is based on an abstract axiomatic definition. Sometimes students are told: "You can also take some elements, define an operation, do some restrictive requests for operations, form an algebraic system, and try to deduce some conclusions." In fact, in the course of the development of mathematics, Euclidean geometry and projective geometry are built in this way. Mathematics provides space and platform for creativity and novelty. It can satisfy people's all imagination of freedom.

3. Base -- finite and infinite

Definition 3.1 In a n-dimensional linear space V, n linearly independent vector $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$, are called a group of bases of V.

There are an infinite elements in other nonzero spaces. It is impossible to know the infinite elements certainly one by one. So we try finding a group of representative elements to help us know those infinite elements. We succeed and that is base. Every element α in linear space V can be linearly presented by base $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$, which can be written as the following:

$$\alpha=x_1\varepsilon_1+x_2\varepsilon_2+\cdots+x_n\varepsilon_n,\ x_i\in P,\ i=1,2,\cdots,n$$

The introduction of base realizes the goal of using finite to present infinite, and it also embodies the thought of the unity of finite and infinite opposites.

4. Transition matrix between base and base -- transformation idea

In linear space, the base is not unique, so it is necessary to consider the relationship between different bases.

Definition 4.1 Assuming that V is n-dimensional linear space of number field P, $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ and $\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_n'$ are two groups of base of V, if

$$\left(\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_n'\right) = \left(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\right)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

then matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is the transition matrix from base $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ to $\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_n'$.

It is because of the existence of transition matrix that when we discuss problems, we can choose a set of appropriate bases to transform the discussed problems into a more concise form.

Here is an example.

Example 4.1 In linear space $R[x]_3$, please find out the transition matrix from base $x+1, x+x^2, x^2$ to $1, x^2-\mathrm{x}, x^2+x$, and the coordinates of the vector $1+2x+x^2$ under the two groups of bases.

We don't encourage students to try number one by one. Usually such topics can not be tried out so we need a general approach. The two groups of bases in this problem are kind of complex. There is a group of simple bases in $R[x]_3$: $1, x, x^2$. Every polynomial in $R[x]_3$ can be presented by this group of bases. There is a transition matrix A from base $1, x, x^2$ to $x+1, x+x^2, x^2$, and a transition matrix B from base $1, x, x^2$ to $1, x^2-\mathrm{x}, x^2+x$. Through their relationship, it can be found out that the transition matrix from base $x+1, x+x^2, x^2$ to $1, x^2-\mathrm{x}, x^2+x$ is $A^{-1}B$. The coordinates of vector $1+2x+x^2$ under these two groups of bases can be obtained in the same way. We often use such way of thinking in mathematics: When it is difficult to solve a problem directly, we can build a bridge to solve the problem indirectly. As the saying goes, "The nature of gentlemen is not different, but they are good at using tools."

5. Coordinates -- the unity of algebra and geometry

Assuming that $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ is a group of base of linear space V, $\alpha \in V$, if $\alpha = x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n,\ x_i \in P,\quad i = 1, 2, \cdots, n$, then array $x_1, x_2, \cdots, x_n$ is called α' coordinates under base $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, written as $\left(x_1, x_2, \cdots, x_n\right)'$. Thus, it becomes the problem of geometric space to map the abstract elements in linear space to the n-element array in the n-dimensional vector space. We can explain and understand the problems in linear space by means of geometric language, and we can also solve geometric problems by means of algebra. The introduction of coordinates has realized the unity of algebra and geometry.

6. Dimension -- the essential feature of the finite dimensional linear space (isomorphism)

It is the introduction of base and coordinates that constructs isomorphic relationship between the abstract linear space and the specific n-dimensional vector space. The following conclusions can be obtained:

(1) Every n-dimensional linear space in number field P is isomorphic to P^n;

(2) The sufficient and necessary condition for two finite dimensional linear space V_1, V_2 on number field P to be isomorphic is $\dim(V_1) = \dim(V_2)$.

In this way, we not only develop from abstract to concrete, but also classify numerous linear spaces by putting those with equal dimensions to a group, whose representative is n-dimensional vector space P^n.

The above are some thoughts on several points in linear space. If we bring mathematical thinking method to teaching process to share our cognition, feelings and ideas with students, they can profoundly understand the logic of math and appreciate the beauty of math, so as to better learn math, use math, and love math.

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References

[1] Zhu Shuilin and Ying Zhiyi, Philosophy of Mathematics, *Beijing: The Commercial Press*, 2003.

[2] Zheng Yuxin and Lin Zeng, Mathematical Logic and Philosophy, *Hubei: Hubei People's Press*, 1987.

[3] Wang Efang & Shi Shengming, Advanced Algebra, *Beijing: Higher Education Press*, 2003.

[4] Ren Zhanghui, Theory of Mathematics Thinking, *Nanning: Guangxi Education Publishing House*, 1990.

[5] Wang Ai ling, Jiang Feng, Explore the Ways of Learning Advanced Algebra, 2013 International Conference on Education and Teaching, Wuhan, HuBei,China, 15th March, pp. 539-544, 2013.

[6] LIU Landong and LIU Mei, Some Philosophical Thinking of Linear Algebra, *Journal of Educational Institute of Jilin Province*, vol.349, pp. 148-150, 2014.

[7] LI Xiao-hong, Discussion on Philosophy Thought in Linear Algebra, *Education Teaching Forum*, vol.39, pp. 219-220, 2017.