Generalized Semi Exponential Type Estimator under Systematic Sampling

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Abstract

In sample surveys, collection of auxiliary information together with the main variable of interest is very important to increase the efficiency of the estimators of population parameters of interest. Regression and ratio estimation are very popular and are widely used methods that benefit from the use of auxiliary information for the estimation of population parameters like mean, total, variance, proportion etc. A generalized semi-exponential type estimator is proposed in this paper using two auxiliary variables under the framework of systematic sampling. The expressions of approximate bias and mean square error of the proposed estimator have been derived. Algebraic conditions have been obtained under which the proposed estimator is more efficient than the competing estimators considered here. An empirical study has been carried out to show the improvement in efficiency of the proposed estimator as compared to the existing estimators.

Keywords: Auxiliary information, Generalized Semi-Exponential type estimator, Systematic Sampling, Mean Square Error, Percentage Relative Efficiency.

1. Introduction

In survey sampling, no single estimation procedure will always work the best. Various sampling designs are available for different populations under different situations. The natural variation among real life situations requires development of different types of estimators for different sampling techniques under different situations for different populations.

Systematic sampling is often advantageous over the simple random sampling for being easy to implement and often providing increased precision in estimates of population parameters of interest. Under this design, only first sampling unit is selected randomly and the subsequent units are then selected by according to certain rules. According to W.G. Madow and L.H. Madow (1944),"Systematic sampling is most commonly used probability design for the estimation of finite population parameters, due to its simplicity. "Cochran (1946) declared that apart from easy to implement, systematic sampling often provides more efficient estimators as compared to the simple random sampling (*SRS*) or stratified random sampling for different types of populations under different situations.

Auxiliary information is commonly used together with the main variable of interest to improve the estimates of population parameter like the mean, total and variance, etc. Ratio and regression estimation methods use auxiliary information in many ways to obtain better estimation results in terms of minimum mean square errors. Various authors have used auxiliary information to improve the estimators in terms of relative efficiency under systematic sampling. For details, readers may refer to Quenouille (1956) and Hansen *et al.* (1946). Swain (1964), Shukla (1971) and Singh (1967) have proposed the classical ratio, product and ratio-cum-product-type estimators respectively under the framework of systematic sampling. Srivastava and Jhajj (1983) used multi-auxiliary variables to propose a new family of estimators. Kushwaha and Singh (1989), Banarasi *et al.* (1993) and Singh and Singh (1998) suggested different modified ratio, product and difference type estimators under systematic sampling. For more recent work on systematic sampling including some exponential type estimators using auxiliary information, one can refer to Singh *et al.* (2011), Singh and Solanki (2012), Singh and

Jatwa (2012),Tailor *et al.*(2013), Khan and Singh (2015) and Khan (2016).

In this paper, a generalized semi-exponential type estimator is proposed with two auxiliary variables with the expectation that this estimator will be more efficient than some of the other competing estimators considered in this paper under systematic sampling. Some existing estimators along with the methodology and useful notations of systematic sampling are given in Section 2. The expressions for approximate bias and mean square error of the proposed estimator have been derived in Section 3. Some special cases are also given in the same Section in which the proposed estimator reduces to many other exponential and non-exponential type estimators. Theoretical comparisons with some other existing estimators are addressed in Section 3. An empirical study that shows percentage relative efficiency of the proposed estimator with respect to the mean per unit estimator under systematic sampling is carried out in Section 4. Some concluding remarks are given in Section 5.

2. Methodology of Systematic Sampling with Associated Estimators

In this section, we introduce the following terminology that is needed in this paper. Let y is the study variable and x and z be the auxiliary variables defined on a finite population P consisting of N distinct but identifiable units, $P = (P_1, P_2, \ldots, P_N)$, numbered in some specific order. A random sample of size n is selected from the first k units and then every k^{th} units is selected corresponding to each unit in the sample from the first k units. So there will be total k samples, each of size n, such that $N = nk$, where n and k are positive integers. Let (y_{ij}, x_{ij}, z_{ij}) for $i=1,2,3,\ldots, k$ and $j=1,2,3,\ldots,n$ denote the values of jth unit in the ith sample. The systematic sample means of the variable of interest and the auxiliary variables to be estimated as, $\bar{y}_{sys} = n^{-1}\sum_{j=1}^{n} y_{ij}$, $\bar{x}_{sys} = n^{-1}\sum_{j=1}^{n} x_{ij}$ and $\bar{z}_{sys} = n^{-1}\sum_{j=1}^{n} z_{ij}$ are unbiased estimators of the corresponding population means $\bar{Y}$, $\bar{X}$ and $\bar{Z}$ respectively.

We also denote the following error terms and the other notations:

$$\bar{y}_{sys} = \bar{Y}\left(1+e_y\right) \text{ and } \bar{z}_{sys} = \bar{Z}\left(1+e_z\right)$$

$$\text{such that } E\left(e_y\right) = E\left(e_x\right) = E\left(e_z\right) = 0$$

$$E\left(e_y^2\right) = \theta\left(1+(n-1)\rho_y\right)C_y^2 = C_0^2,$$

$$E\left(e_x^2\right) = \theta\left(1+(n-1)\rho_x\right)C_x^2 = C_1^2$$

$$E\left(e_z^2\right) = \theta\left(1+(n-1)\rho_z\right)C_z^2 = C_2^2$$

$$E\left(e_z^2\right) = \theta\left(1+(n-1)\rho_z\right)C_z^2 = C_2^2$$

$$E\left(e_y e_x\right) = \theta\left(1+(n-1)\rho_y\right)^{1/2}\left(1+(n-1)\rho_x\right)^{1/2}\rho_{yx}C_yC_x = C_0C_1$$

$$E\left(e_y e_z\right) = \theta\left(1+(n-1)\rho_y\right)^{1/2}\left(1+(n-1)\rho_z\right)^{1/2}\rho_{yz}C_yC_z = C_0C_2$$

$$E\left(e_x e_z\right) = \theta\left(1+(n-1)\rho_x\right)^{1/2}\left(1+(n-1)\rho_z\right)^{1/2}\rho_{xz}C_xC_z = C_1C_2$$
(1)

$$\rho_{yx}^* = \frac{\left(1+(n-1)\rho_y\right)}{\left(1+(n-1)\rho_y\right)}, \rho_{yz}^* = \frac{\left(1+(n-1)\rho_y\right)}{\left(1+(n-1)\rho_z\right)} \text{ and } \rho_{xz}^* = \frac{\left(1+(n-1)\rho_x\right)}{\left(1+(n-1)\rho_z\right)}$$

$$\rho_y = \frac{E\left(y_{ij}-\bar{Y}\right)\left(y_{ij'}-\bar{Y}\right)}{E\left(y_{ij}-\bar{Y}\right)^2}, \rho_x = \frac{E\left(x_{ij}-\bar{X}\right)\left(x_{ij'}-\bar{X}\right)}{E\left(x_{ij}-\bar{X}\right)^2}$$

$$\text{and } \rho_z = \frac{E\left(z_{ij}-\bar{Z}\right)\left(z_{ij'}-\bar{Z}\right)}{E\left(z_{ij}-\bar{Z}\right)^2}$$

$$\theta = (N-1)/Nn, \rho_{ij}^* = \left(1+(n-1)\rho_i\right)/\left(1+(n-1)\rho_j\right)$$

$$\rho_{ij} = S_{ij}/S_iS_j \text{ and } H_{ij} = \rho_{ij}C_i/C_j \text{ where } \begin{matrix} i = x,y,z \\ j = x,y,z \end{matrix} i \neq j$$

where ρ_y, ρ_x and ρ_z are the intraclass correlation coefficients for study variable y and both auxiliary variables x and z, respectively and ρ_{ij} is the correlation coefficient between study variable and auxiliary variables. Also the quantities C_y, C_x and C_z are population coefficients of variation of the study variable and the auxiliary variables respectively.

The unbiased mean estimator, without using auxiliary information, together with the expression for the variance in systematic sampling is defined as:

$$t_0 = \bar{y}_{sys},$$
(2)

$$var\left(t_0\right) = \bar{Y}^2 C_0^2.$$
(3)

Swain (1964) and Shukla (1971) proposed classical ratio and product-type estimators under systematic sampling is given by

$$t_1 = \frac{\bar{y}_{sys}}{\bar{x}_{sys}}\bar{X},$$
(4)

$$t_2 = \frac{\bar{y}_{sys}}{\bar{Z}}\bar{z}_{sys}.$$
(5)

The expressions for the mean square error for the estimators t_1 and t_2 upto the first order of approximation are given respectively by

$$MSE(t_1) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1-2H_{yx}\sqrt{\rho_{yx}^*}\right)\right),$$
(6)

$$MSE(t_2) = \bar{Y}^2\left(C_0^2 + C_2^2\left(1+2H_{yz}\sqrt{\rho_{yz}^*}\right)\right).$$
(7)

The traditional regression estimator for population mean under systematic sampling is given by

$$t_3 = \bar{y}_{sys} + b_{yx}\left(\bar{X}-\bar{x}_{sys}\right),$$
(8)

where b_{yx} is the sample regression coefficient between y and x. The expression for the mean square error for the estimator t_3, up to the first order approximation is given as

$$MSE(t_3) = \bar{Y}^2 C_0^2\left(1-\rho_{yx}^2\right).$$
(9)

Singh *et al.* (2011) proposed exponential ratio and product-type estimators for finite population mean under systematic sampling. The proposed estimators are given by

$$t_4 = \bar{y}_{sys}\exp\left(\frac{\bar{X}-\bar{x}_{sys}}{\bar{X}+\bar{x}_{sys}}\right),$$
(10)

$$t_5 = \bar{y}_{sys}\exp\left(\frac{\bar{z}_{sys}-\bar{Z}}{\bar{Z}+\bar{z}_{sys}}\right).$$
(11)

The expressions for mean square error for the estimators t_4 and t_5 using first order approximation given respectively by

$$MSE(t_4) = \bar{Y}^2\left(C_0^2 + 0.25C_1^2\left(1-4H_{yx}\sqrt{\rho_{yx}^*}\right)\right),$$
(12)

$$MSE(t_5) = \bar{Y}^2\left(C_0^2 + 0.25C_2^2\left(1+4H_{yz}\sqrt{\rho_{yz}^*}\right)\right).$$
(13)

Tailor *et al*. (2013) proposed the following ratio-cum-product type estimator for finite population mean under systematic sampling:

$$t_6 = \bar{y}_{sys}\left(\frac{\bar{X}}{\bar{x}_{sys}}\right)\left(\frac{\bar{z}_{sys}}{\bar{Z}}\right). \tag{14}$$

The expression for the mean square error of estimator t_6, up to first order approximation, is given by:

$$MSE(t_6) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*} - 2H_{xz}\sqrt{\rho_{xz}^*}\right)\right). \tag{15}$$

Khan (2016) proposed a generalized class of exponential estimators for the estimation of finite population mean under systematic sampling. The estimator with the expression for the mean square error is given by:

$$t_7 = \bar{y}_{sys}\exp\left(\alpha\frac{\bar{X} - \bar{x}_{sys}}{\bar{X} + (a-1)\bar{x}_{sys}} + \beta\frac{\bar{Z} - \bar{z}_{sys}}{\bar{Z} + (b-1)\bar{z}_{sys}}\right), \tag{16}$$

$$MSE(t_6) = \bar{Y}^2\left(\begin{array}{l} C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) \\ + KC_2^2\left(K - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \end{array}\right), \tag{17}$$

where

$$a = \frac{\alpha}{J}, \qquad J = \sqrt{\rho_{yx}^*}\frac{H_{yx} - H_{yz}H_{zx}}{\left(1 - \rho_{xz}^2\right)},$$

and

$$b = \frac{\beta}{M}, \qquad M = \sqrt{\rho_{yz}^*}\frac{H_{yz} - H_{yx}H_{xz}}{\left(1 - \rho_{xz}^2\right)}.$$

3. Generalized Semi-Exponential Type Estimator

In this section, a generalized semi-exponential type mean estimator is proposed making use of two auxiliary variables under the framework of systematic sampling. The proposed estimator is:

$$t_{GE} = \bar{y}_{sys}\left(\frac{\bar{X}}{\bar{x}_{sys}}\right)^{v_1}\exp\left(\alpha\frac{\bar{Z} - \bar{z}_{sys}}{\bar{Z} + (v_2-1)\bar{z}_{sys}}\right), \tag{18}$$

where v_1 and v_2 ($v_2 > 0$) are constants that need to be optimized and estimated for the expression of minimum value of the mean square error of the proposed estimator t_{GE}. A generalized constant α can assume values -1, 0 and 1 to give various special cases of the proposed estimators.

In order to obtain the expression of mean square error of the proposed estimator, we expand the proposed estimator expression using the notations given in (1), and get.

$$t_{GE} = \bar{Y}(1+e_y)\left(\frac{\bar{X}}{\bar{X}(1+e_x)}\right)^{v_1}\exp\left(\alpha\frac{\bar{Z} - \bar{Z} - \bar{Z}e_z}{\bar{Z} + \bar{Z}(v_2-1)(1+\bar{Z}e_z)}\right). \tag{19}$$

Applying Taylor series and ignoring the terms beyond the second order of approximation and taking expectation on both sides of (19), we get

$$E(t_{GE} - \bar{Y}) = \bar{Y}E\left(e_y - e_x - v_1e_ye_x + v_1^2e_x^2\right)$$
$$\left(1 - \left\{-\frac{\alpha}{v_2}e_z + \frac{\alpha}{v_2}e_z^2 - \frac{\alpha}{v_2^2}e_z^2\right\} + \frac{\alpha^2}{v_2^2}e_z^2\right). \tag{20}$$

After simplification of (20), we have

$$bias(t_{GE}) = \bar{Y}\left(\begin{array}{l} v_1C_1^2\left(1 - H_{yx}\sqrt{\rho_{yx}^*}\right) + \frac{\alpha}{2a^2}C_2^2 \\ \left\{2(a-1) + \alpha - H_{yz}\sqrt{\rho_{yz}^*} - v_1H_{zx}\sqrt{\rho_{zx}^*}\right\} \end{array}\right). \tag{21}$$

In order to obtain the expression for the mean square error of the proposed estimator, applying Taylor series and ignoring the terms up to the first order approximation of (19), we have

$$t_{GE} = \bar{Y}\left(1 + e_y - v_1e_x\right)\exp\left(-\frac{\alpha}{v_2}e_z\right). \tag{22}$$

Squaring and taking expectation on both sides of (22), we get

$$E(t_{GE} - \bar{Y})^2 = \bar{Y}^2E\left(e_y - v_1e_x - \frac{\alpha}{v_2}e_z\right)^2. \tag{23}$$

The expression for the mean square error of the proposed estimator is given as

$$MSE(t_{GE}) = \bar{Y}^2\left(\begin{array}{l} C_0^2 + v_1C_1^2\left(v_1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + \frac{\alpha}{v_2}C_2^2 \\ \left\{\frac{\alpha}{v_2} - 2H_{yz}\sqrt{\rho_{yz}^*} + 2v_1H_{xz}\sqrt{\rho_{xz}^*}\right\} \end{array}\right). \tag{24}$$

The optimum values of v_1 and v_2 are

$$v_1 = \sqrt{\rho_{yx}^*} \, \frac{H_{yx} - H_{yz}H_{zx}}{\left(1 - \rho_{xz}^2\right)} = J \quad \text{and} \quad v_{2(opt)} = \frac{\alpha}{M},$$

where

$$M = \sqrt{\rho_{yz}^*} \, \frac{H_{yz} - H_{yx}H_{xz}}{\left(1 - \rho_{xz}^2\right)}.$$

The expression of minimum mean square error of the proposed estimator t_{GE} is given as:

$$MSE_{min}\left(t_{sysGE}\right) = \bar{Y}^2 \left(\frac{C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2}{\left\{M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right\}} \right). \quad (25)$$

It is noticed that for different values of v_1, v_2 and α we may get various forms of exponential and semi exponential type estimators as new families of t_{GE}, as given in Table 1 (Appendix A).

4. Relative Performance of Proposed Estimator Compared to other Estimators

In this section, the theoretical comparisons of the proposed estimator are given with some relevant competing estimators.

i. The proposed generalized semi exponential type estimator t_{GE} will be more precise estimator then the unbiased mean per unit estimator given in (3) when

$$MSE_{min}\left(t_{GE}\right) \leq Var\left(t_0\right),$$

$$\bar{Y}^2 \left(\begin{array}{c} C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + \\ MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \end{array} \right) \leq \bar{Y}^2 C_0^2,$$

$$MC_2^2\left(2H_{yz}\sqrt{\rho_{yz}^*} - M - 2JH_{xz}\sqrt{\rho_{xz}^*}\right) - JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) \geq 0.$$

ii. The proposed estimator t_{GE} will be more precise estimator than the classical ratio estimator given in (6) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_1\right),$$

$$\bar{Y}^2 \left(\begin{array}{c} C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + \\ MC_2^2\left\{M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right\} \end{array} \right) \leq \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right)\right),$$

$$C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) - JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right)$$
$$- MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \geq 0.$$

iii. The proposed estimator t_{GE} will be more precise estimator than the product estimator given in (7) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_2\right).$$

$$AC_1^2\left(A - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2AH_{xz}\sqrt{\rho_{xz}^*}\right)$$
$$\leq C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*}\right),$$

$$C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*}\right) - AC_1^2\left(A - 2H_{yx}\sqrt{\rho_{yx}^*}\right)$$
$$- MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2AH_{xz}\sqrt{\rho_{xz}^*}\right) \geq 0.$$

iv. The proposed estimator t_{GE} will be more precise estimator than the regression estimator given in (9) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_3\right),$$

$$\bar{Y}^2 \left(\begin{array}{c} C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2 \\ \left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \end{array} \right) \leq \bar{Y}^2 C_0^2\left(1 - \rho_{yx}^2\right),$$

$$\rho_{yx}^2 \geq \left(\begin{array}{c} JC_1^2\left(2H_{yx}\sqrt{\rho_{yx}^*} - J\right) + MC_2^2 \\ \left(2H_{yz}\sqrt{\rho_{yz}^*} - M - 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \end{array} \right) / C_0^2.$$

v. The proposed estimator t_{GE} will be more precise estimator than the exponential ratio type estimator given in (12) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_4\right),$$

$$\bar{Y}^2\left(C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right)\right)$$
$$\leq \bar{Y}^2\left(C_0^2 + 0.25C_1^2\left(1 - 4H_{yx}\sqrt{\rho_{yx}^*}\right)\right),$$

$$C_1^2\left(0.25 - 4H_{yx}\sqrt{\rho_{yx}^*}\right) - JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right)$$
$$-MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \geq 0.$$

vi.   The proposed estimator t_{GE} will be more precise estimator than the exponential product-type estimator given in (13) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_5\right),$$

$$\bar{Y}^2\left(C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right)\right)$$
$$\leq \bar{Y}^2\left(C_0^2 + 0.25C_2^2\left(1 + 4H_{yz}\sqrt{\rho_{yz}^*}\right)\right),$$

$$C_2^2\left(0.25 + 4H_{yz}\sqrt{\rho_{yz}^*}\right) - JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right)$$
$$-MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \geq 0.$$

vii.   The proposed estimator t_{GE} will be more precise estimator than the ratio-cum-product type estimator given in (15) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_6\right),$$

$$\bar{Y}^2\left(C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right)\right)$$
$$\leq \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*} - 2H_{xz}\sqrt{\rho_{xz}^*}\right)\right),$$

$$C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*} - 2H_{xz}\sqrt{\rho_{xz}^*}\right) - JC_1^2$$
$$\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) - MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right) \geq 0.$$

viii.   The proposed estimator t_{GE} will be more precise estimator than the generalized exponential-cum-exponential type estimator given in (17) when

$$MSE_{min}\left(t_{GE}\right) \leq MSE\left(t_7\right),$$

$$\bar{Y}^2\left(C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2\left(M - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right)\right)$$
$$\leq \bar{Y}^2\left(C_0^2 + JC_1^2\left(J - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + KC_2^2\left(K - 2H_{yz}\sqrt{\rho_{yz}^*} + 2JH_{xz}\sqrt{\rho_{xz}^*}\right)\right),$$
$$C_1^2\left(J - H_{yx}\sqrt{\rho_{yx}^*}\right) + MC_2^2 H_{xz}\sqrt{\rho_{xz}^*} \geq 0.$$

5.   Empirical Study

Evaluation of the proposed estimator is based on the percentage relative efficiencies (*PRE's*) compared to the traditional unbiased mean estimator. So the value

greater than one hundred indicates that the competing estimators are more efficient than the usual mean estimator. The *PRE's* of all the estimators over the mean per unit estimator can be obtained from the following mathematical formula:

$$PRE = \frac{\text{var}(t_0)}{MSE\left(t_i\right)} \times 100,$$

$$i = 0,1,2,...,7,G1,G2,...,G10.$$

Here, we consider a population data set from the literature given in Table 2, to examine the performance of the proposed estimator over the other competing estimators at optimum conditions.

Table2: Population (Source: Tailor et al. (2013)

N	15	n	3	-	-
$\bar{Y}$	80	C_y	0.56	ρ_{yx}	0.9848
$\bar{X}$	44.47	C_x	0.28	ρ_{yz}	-0.9760
$\bar{Z}$	48.40	C_z	0.43	ρ_{xz}	-0.9539
S_{yx}	538.57	S_y^2	2000	ρ_y	0.6652
S_{yz}	-902.87	S_x^2	149.55	ρ_x	0.7070
S_{xz}	-241.06	S_z^2	427.83	ρ_z	0.5487

The results of *MSE*'s and *PRE*'s of all the estimators considered in this paper are summarized in Table 3.

Table 3: MSE's and PRE's of all the estimators

t_i	MSE	PRE's	t_i	MSE	PRE's
t_0	1344.07	100.00	t_{G1}	4408.34	33.01
t_1	373.32	389.62	t_{G2}	46.27	3144.94
t_2	768.06	189.45	t_{G3}	2249.14	64.70
t_3	43.74	3362.67	t_{G4}	1131.00	128.66
t_4	820.09	177.43	t_{G5}	93.11	1562.88
t_5	1044.42	139.32	t_{G6}	52.99	2746.27
t_6	187.08	777.79	t_{G7}	7321.20	19.88
t_7	23.67	6158.08	t_{78}	5100.04	28.53
t_{GE}	**21.62**	**6729.21**	t_{G9}	753.10	193.22
			t_{G10}	1816.00	80.13

The results presented in Table 3 indicate that the proposed generalized semi-exponential type estimator works considerably better than all other exponential and non-exponential estimators considered in this paper. The proposed estimator has the least mean square error as compare to other estimators.

6.   Conclusion

A generalized semi-exponential type estimator is proposed in this paper for the estimation of finite

population mean with two auxiliary variables under the framework of systematic sampling. The expressions of approximate bias and mean square error of the proposed estimator are derived. The algebraic expressions for the mean square error of the proposed estimator are compared with other existing estimators both theoretically and empirically. The efficiency comparisons are also carried out using the data taken from Tailor *et al.* (2013). The results illustrated in Table 3 show that the proposed generalized semi-exponential type estimator is more efficient than the other estimators considered in this paper in term of higher percent relative efficiency.

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APPENDIX A

Table 1: Some Special Cases of Proposed Estimator

Estimators	Mean Square Errors	v_1	α	v_2
$t_0 = \bar{y}_{sys}$	$MSE(t_0) = \bar{Y}^2 C_0^2$	0	0	0
$t_{G1} = \bar{y}_{sys} \exp\left(\dfrac{\bar{Z} - \bar{z}_{sys}}{\bar{Z}}\right)$	$MSE(t_{G1}) = \bar{Y}^2\left(C_0^2 + C_2^2\left(1 - 2H_{yz}\sqrt{\rho_{yz}^*}\right)\right)$	0	1	1
$t_4 = \bar{y}_{sys} \exp\left(\dfrac{\bar{Z} - \bar{z}_{sys}}{\bar{Z} + \bar{z}_{sys}}\right)$	$MSE(t_4) = \bar{Y}^2\left(C_0^2 + 0.25C_2^2\left(1 - 4H_{yz}\sqrt{\rho_{yz}^*}\right)\right)$	0	1	2
$t_{G2} = \bar{y}_{sys} \exp\left(\dfrac{\bar{z}_{sys} - \bar{Z}}{\bar{Z}}\right)$	$MSE(t_{G2}) = \bar{Y}^2\left(C_0^2 + C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*}\right)\right)$	0	-1	1
$t_5 = \bar{y}_{sys} \exp\left(\dfrac{\bar{z}_{sys} - \bar{Z}}{\bar{Z} + \bar{z}_{sys}}\right)$	$MSE(t_5) = \bar{Y}^2\left(C_0^2 + 0.25C_2^2\left(1 + 4H_{yz}\sqrt{\rho_{yz}^*}\right)\right)$	0	-1	2
$t_1 = \bar{y}_{sys}\left(\dfrac{\bar{X}}{\bar{x}_{sys}}\right)$	$MSE(t_1) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right)\right)$	1	0	0
$t_{G3} = \bar{y}_{sys}\left(\dfrac{\bar{X}}{\bar{x}_{sys}}\right)\exp\left(\dfrac{\bar{Z} - \bar{z}_{sys}}{\bar{Z}}\right)$	$MSE(t_{G3}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 - 2H_{yz}\sqrt{\rho_{yz}^*} + 2H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	1	1	1
$t_{G4} = \bar{y}_{sys}\left(\dfrac{\bar{X}}{\bar{x}_{sys}}\right)\exp\left(\dfrac{\bar{Z} - \bar{z}_{sys}}{\bar{Z} + \bar{z}_{sys}}\right)$	$MSE(t_{G4}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + 0.25C_2^2\left(1 - 4H_{yz}\sqrt{\rho_{yz}^*} + 4H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	1	1	2
$t_{G5} = \bar{y}_{sys}\left(\dfrac{\bar{X}}{\bar{x}_{sys}}\right)\exp\left(\dfrac{\bar{z}_{sys} - \bar{Z}}{\bar{Z}}\right)$	$MSE(t_{G5}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*} - 2H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	1	-1	1
$t_{G6} = \bar{y}_{sys}\left(\dfrac{\bar{X}}{\bar{x}_{sys}}\right)\exp\left(\dfrac{\bar{z}_{sys} - \bar{Z}}{\bar{Z} + \bar{z}_{sys}}\right)$	$MSE(t_{G6}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 - 2H_{yx}\sqrt{\rho_{yx}^*}\right) + 0.25C_2^2\left(1 + 4H_{yz}\sqrt{\rho_{yz}^*} - 4H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	1	-1	2
$t_2 = \bar{y}_{sys}\left(\dfrac{\bar{x}_{sys}}{\bar{X}}\right)$	$MSE(t_1) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 + 2H_{yx}\sqrt{\rho_{yx}^*}\right)\right)$	-1	0	0
$t_{G7} = \bar{y}_{sys}\left(\dfrac{\bar{x}_{sys}}{\bar{X}}\right)\exp\left(\dfrac{\bar{Z} - \bar{z}_{sys}}{\bar{Z}}\right)$	$MSE(t_{G7}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 + 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 - 2H_{yz}\sqrt{\rho_{yz}^*} - 2H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	-1	1	1
$t_{G8} = \bar{y}_{sys}\left(\dfrac{\bar{x}_{sys}}{\bar{X}}\right)\exp\left(\dfrac{\bar{Z} - \bar{z}_{sys}}{\bar{Z} + \bar{z}_{sys}}\right)$	$MSE(t_{G8}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 + 2H_{yx}\sqrt{\rho_{yx}^*}\right) + 0.25C_2^2\left(1 - 4H_{yz}\sqrt{\rho_{yz}^*} - 4H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	-1	1	2
$t_{G9} = \bar{y}_{sys}\left(\dfrac{\bar{x}_{sys}}{\bar{X}}\right)\exp\left(\dfrac{\bar{z}_{sys} - \bar{Z}}{\bar{Z}}\right)$	$MSE(t_{G9}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 + 2H_{yx}\sqrt{\rho_{yx}^*}\right) + C_2^2\left(1 + 2H_{yz}\sqrt{\rho_{yz}^*} + 2H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	-1	-1	1
$t_{G10} = \bar{y}_{sys}\left(\dfrac{\bar{x}_{sys}}{\bar{X}}\right)\exp\left(\dfrac{\bar{z}_{sys} - \bar{Z}}{\bar{Z} + \bar{z}_{sys}}\right)$	$MSE(t_{G10}) = \bar{Y}^2\left(C_0^2 + C_1^2\left(1 + 2H_{yx}\sqrt{\rho_{yx}^*}\right) + 0.25C_2^2\left(1 + 4H_{yz}\sqrt{\rho_{yz}^*} + 4H_{xz}\sqrt{\rho_{xz}^*}\right)\right)$	-1	-1	2