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Two New Lifetime Distributions of $X-$Weibull Family: Theories and Applications

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Abstract

In this paper, two new distributions, weibull-rayleigh and weibull-exponential of $X-$weibull family are introduced. The various properties of theses distributions, for example, the density functions, distribution functions, hazard rate functions, moment functions and Shannon entropy are investigated and capabilities of distributions compared to other distributions. Also by using simulation from these two distributions with various parameters some of the sample statistical indexes are estimated. Two real data sets are used to illustrate the applicability of these distributions.

Key words and phrases: Exponential distribution, Simulation method, $T-X$ distribution family, Weibull distribution

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1. Introduction

Statistical distributions are important for parametric inference and applications to fit real world phenomena. Many methods have been developed to generate statistical distributions in the literature. Sometimes fit a distribution to a data set faces poor results. To solve this problem statisticians have always tried to fit a better distribution to data with production or extension the distributions. In this regard, many generalized classes of distributions to describe the different phenomena are obtained. Many types of generalized exponential distributions obtained by Khan and Jain (1978). Gupta and Kundu (1999) introduced the three-parameter "generalized exponential distribution" (location, scale and shape), to study the theoretical properties of this family and compared them with respect to the well study properties of the gamma distribution and weibull distribution. Gupta et al. (1998) proposed to model failure time data by $F^{*}(f) = [F(t)]\,\theta$ where $F(t)$ is the baseline distribution function and θ is a positive real number. The monotonicity of the failure rates were studied, in general, and some order relations are examined. Some examples including exponentiated weibull, exponential, gamma and Pareto distributions were investigated. Gupta and Kundu (2001) studied some properties of a new family of distributions, namely Exponentiated Exponential distribution, discussed in Gupta et al. (1998). They They presented two real life data sets, where it is observed that in one data set exponentiated exponential distribution has a better fit compared to weibull or gamma distribution and in the other data set weibull has a better fit than exponentiated exponential or gamma distribution. Some numerical experiments are performed to see how the maximum likelihood estimators and their asymptotic results work for finite sample sizes. Various extensions on the Gamma distribution are given by Stacy (1962). Agarwal and Al-Saleh (2001) defined a new generalized gamma distribution. The trend of this density function and statistical properties are studied. A real life data set used to fit the distribution. A new weibull type distribution was also considered and the hazard rate function of this distribution also has both the properties namely bathtub and monotonicity. Mudholkar et al.onotone hazard rates and can be adopted for testing goodness of fit of weibull as a submodel. The usefulness and flexibility of the family is illustrated by reanalyzing five classical data sets on bus-motor failures from Davis that are typical of data in repair–reuse situations and Efron's datapertainingtoahead-and-neck-cancerclinical trial.

Eugene et al. (2002) introduced a general class of distributions generated from the logit of the beta random variable. The shape properties of the beta-normal distribution discussed. Also the estimation of parameters of the beta-normal distribution by the maximum likelihood method discussed. Nadarajah and Kotz (2004) introduced a generalization referred to as the beta Gumbel distribution generated from the logit of a beta random variable. They provided a comprehensive treatment of the mathematical properties of this new distribution. Also they derived the analytical shapes of the corresponding probability density function and the hazard rate function and provided graphical illustrations. Akinsete et al. (2008) defined and studied a four-parameter beta-Pareto distribution. They discussed various properties of the distribution. The distribution found to be unimodal and has either a unimodal or a decreasing hazard rate. The expressions for the mean, mean deviation, variance, skewness, kurtosis and entropies obtained. The relationship between these moments and the parameters provided. Also they proposed the method of maximum likelihood to estimate the parameters of the distribution. The distribution applied to two flood data sets. Alzaatreh, et al. (2012a) defined and studied a new distribution, the Gamma-Pareto, and various properties of the distribution are obtained. Results for moments, limiting behavior and entropies are provided. Also the method of maximum likelihood is proposed for estimating the parameters and the distribution is applied to fit three real data sets. Alzaatreh, et al. (2012b) proposed a new method for generating discrete distributions. A special class of the distributions, namely, the T-geometric family contains the discrete analogues of continuous distributions. Some general properties of the T-geometric family of distributions obtained. Alzaatreh, et al. (2013a) a new distribution, namely, weibull-Pareto distribution defined and studied. Various properties of the weibull-Pareto distribution obtained. Results for moments, limiting behavior, and Shannon entropy provided. The method of modified maximum likelihood estimation is proposed for estimating the model parameters. Several real data sets used to illustrate the applications of weibull-Pareto distribution. Alzaatreh, et al. (2013b) proposed the $T-X$ families of distributions. These families of distributions were used to generate a new class of distributions which offer more flexibility in modeling a variety of data sets. Alzaatreh, et al. (2014) discussed some properties of gamma-X family and a member of the family, the gamma-normal distribution, is studied in detail. The limiting behaviors, moments, mean deviations, dispersion, and Shannon entropy for the gamma-normal distribution are provided.

Suppose that X is a random variable with density function $f_X(x)$ and distribution function $F_X(x)$. Also let T be a continuous random variable with density function $r(t)$ and distribution function $R(t)$defined on the interval $[a, b]$.

The new family of distributions cumulative distribution function is as follows

$$G(x) = \int_a^{W(F(x))} r(t)dt = R(W(F(x))), \tag{1}$$

where $W(F(x)$ applies to the following conditions:

a) $W(F(x)) \in [a, b]$.

b) $W(F(x)$ is non-decreasing and differentiable.

c) $\lim\limits_{x\to-\infty} W(F(x) = a$ and $\lim\limits_{x\to\infty} W(F(x) = b$.

In this family, the random variable T is a transformation variable and random variable X is a transformer variable.

By differentiating $G(x)$, the density function of $g(x)$ given by

$$g(x) = \frac{dW(F(x))}{dx} r(W(F(x))). \tag{2}$$

Since various distributions is introduced and investigated with this method. For example, Alzaatreh and Knight (2013) proposed and studied the new distribution, Gamma-half normal distribution. Various structural properties of the gamma-half normal distribution derived. Results for moments, limit behavior, mean deviations and Shannon entropy provided. To estimate the model parameters, the method of maximum likelihood estimation proposed. Three real-life data sets used to illustrate the applicability of the gamma-half normal distribution. Aljarrah et al. (2014) proposed the use of quantile functions to define the W function and some general properties of this $T-X$ system of distributions studied. Also they derived three new distributions of the $T-X$ family, namely, the normal-weibull based on the quantile of Cauchy distribution, normal-weibull based on the quantile of logistic distribution, and weibull-uniform based on the quantile of log-logistic distribution. Two real data sets applied to illustrate the flexibility of the distributions.

This paper is organized as follows. In Section 2, the weibull-rayleigh distribution is introduced. Results for hazard rate function, moment generating function and Shannon entropy is obtained. Using simulation method from this distribution with various parameters some of the sample statistical indexes are estimated. In Section 3, the weibull-exponential distribution is introduced. Results for hazard rate function, moment generating function and Shannon entropy is obtained. Using simulation method from this distribution with various parameters some of the sample statistical indexes are estimated. In Section 4, two real data sets are used to illustrate the applicability of these distributions. Concluding remarks are given in Section 5.

2. Weibull-rayleigh distribution

Let $r(x)$ be a density function of a random variable with weibull distribution (c, γ) and $f_X(x)$ and $F_X(x)$ be the density function and distribution function with rayleigh distribution with parameter b respectively, and $W(F(x)) = -\log(1-F(x))$. Then (1) is a new distribution as weibull-rayleigh distribution. The density function and distribution function of weibull-rayleigh distribution is as follows:

$$G_{WRD}(x) = 1 - \exp\left\{-\left[\frac{-\log(1-F(x))}{\gamma}\right]^c\right\} = 1 - \exp\left\{-\left(\frac{x^2}{2b^2\gamma}\right)^c\right\}, \tag{3}$$

and

$$g_{WRD}(x) = \frac{cx}{b^2\gamma}\left(\frac{x^2}{2b^2\gamma}\right)^{c-1}\exp\left\{-\left(\frac{x^2}{2b^2\gamma}\right)^c\right\}, \qquad x \geq 0, \;\; b,\, c,\, \gamma > 0. \tag{4}$$

The following lemmas give the relationship between the weibull-rayleigh distribution and weibull distribution.

Lemma 1. Suppose that X is a random variable with weibull-rayleigh distribution with parameters (c, γ, b), then the random variable $Y = \frac{X^2}{2b^2}$ is distributed as weibull distribution with parameter (c, γ).

Lemma 2. Suppose that X is a random variable with weibull distribution with parameters (c, γ), then the random variable $V\sqrt{2b^2X}$ is distributed as weibull-rayleigh distribution with parameters (c, γ, b).
The proof of these lemmas is easy.
The hazard rate function of weibull-rayleigh distribution can computed as follows:

$$h_{WRD}(x) = \frac{g(x)}{1-G(x)} = \frac{cx}{b^2\gamma}\left(\frac{x^2}{2b^2\gamma}\right)^{c-1} = \frac{cx^{2c-1}}{2^{c-1}(b^2\gamma)^c}. \tag{5}$$

If $c > \frac{1}{2}$ the rate function of weibull-rayleigh distribution is increasing, if $c < \frac{1}{2}$ the rate function is decreasing and if $c = \frac{1}{2}$ the rate function is constant and is equal to $\frac{c}{(2b\gamma)^c}$.

Theorem 1. Suppose that X is a random variable is distributed as weibull-rayleigh distribution with density function given in (4). Then it's moment generating function is as follows:

$$M_X(t) = c\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(2c(j+1))}{(b^2\gamma)^{c(j+1)}2^{c(j+1)-1}t^{2c(j+1)}}.$$

Proof. The moment generating function of weibull-rayleigh distribution is

$$M_X(t) = \int_0^{\infty} e^{tx}g(x)dx = \int_0^{\infty}\frac{c}{(b^2\gamma)^c2^{c-1}}x^{2c-1}\exp\left(-\frac{x^{2c}}{(2b^2\gamma)^c}+tx\right)dx. \tag{6}$$

Using the Taylor expansion, we have

$$M_X(t) = \sum_{j=0}^{\infty}(-1)^j\frac{c}{(b^2\gamma)^c2^{c-1}}\int_0^{\infty}\frac{x^{2c-1}x^{2cj}}{j!(2b^2\gamma)^{cj}}e^{tx}dx = c\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(2c(j+1))}{(b^2\gamma)^{c(j+1)}2^{c(j+1)-1}t^{2c(j+1)}}.$$

Theorem 2. Suppose that X is a random variable is distributed as weibull-rayleigh distribution with density function given in (4). Then the first central moment and variance of X are respectively as:

$$E_{WRD}(X) = \Gamma(1+\frac{1}{2c})\sqrt{2b^2\gamma}, \tag{7}$$

and

$$V_{WRD}(X) = b^2\gamma\left(\frac{c}{2^{c-1}}\Gamma(1+\frac{1}{c}) - 2\Gamma^2(1+\frac{1}{2c})\right). \tag{8}$$

Proof. By differentiating from (6) with respect to t, we have

$$\frac{\partial}{\partial t}M_X(t) = \frac{c}{(b^2\gamma)^c2^{c-1}}\int_0^{\infty}x^{2c}\exp\left(-\frac{x^{2c}}{(2b^2\gamma)^c}+tx\right)dx.$$

Putting $t=0$, then the first central moment is

$$\begin{aligned} E_{WRD}(X) &= \tfrac{c}{(b^2\gamma)^c 2^{c-1}} \int_0^\infty x^{2c} \exp\left(-\tfrac{x^{2c}}{(2b^2\gamma)^c}\right) dx \\ &= \tfrac{1}{(b^2\gamma)^c 2^c} \int_0^\infty u^{\frac{1}{2c}} \exp\left(-\tfrac{u}{(2b^2\gamma)^c}\right) du \\ &= \Gamma\left(1+\tfrac{1}{2c}\right)\sqrt{2b^2\gamma}. \end{aligned}$$

Also by differentiating from(6) with respect to t, the second central moment is

$$E_{WRD}(X^2) = \frac{cb^2\gamma}{2^{c-1}}\Gamma\left(1+\frac{1}{2c}\right).$$

Therefore the variance of weibull-rayleigh distribution is given by

$$V_{WRD}(X) = E(X^2) - E^2(X) = b^2\gamma\left(\frac{c}{2^{c-1}}\Gamma(1+\frac{1}{c}) - 2\Gamma^2(1+\frac{1}{2c})\right).$$

Theorem 3. Suppose that X is a random variable is distributed as weibull-rayleigh distribution with density function given in (4). Then the mode of distribution is given by

$$x = \sqrt[2c]{\frac{2^{c-1}(c-1)(b^2\gamma)^c}{c}}, \qquad c > \frac{1}{2}.$$

Proof. The proof of this theorem is easy. One can differentiate from (4) with respect to x and equal it to zero.

Theorem 4. Suppose that X is a random variable is distributed as weibull-rayleigh distribution with density function given in (4). Then the Shannon entropy is given by

$$\eta_X = -E(\log\frac{X}{b^2}) + w(1-\frac{1}{c}) + \ln(\frac{\gamma}{c}) + 1,$$

where $w = \int_1^\infty \left(\frac{1}{[x]} - \frac{1}{x}\right) dx$ is the Euler-Mascheroni constant.

Proof. Let X be a random variable distributed as weibull-rayleigh distribution. By the definition of Shannon entropy we have

$$\eta_X = E(-\log g(X)) = E\left(-\log\left(\frac{X}{b^2}r\left(\frac{X^2}{2b^2}\right)\right)\right) = E\left(-\log\frac{X}{b^2}\right) + E\left(-\log r\left(\frac{X^2}{2b^2}\right)\right),$$

where r is the density function of rayleigh distribution with parameter (c, γ). Since the random variable $\frac{X^2}{2b^2}$ by lemma 1 has weibull distribution, therefore $E\left(-\log(r\left(\frac{X^2}{2b^2}\right)\right)$ is the

Shannon entropy of weibull distribution and is equal to $w(1-\frac{1}{c})+\ln(\frac{\gamma}{c})+1$. This completes the proof.
To simulate from weibull-rayleigh distribution, first we simulate from rayleigh distribution with parameter (c, γ) for production the samples Y, then by lemma 2 and using the conversion $X=\sqrt{2b^2Y}$ the simulation from weibull-rayleigh distribution is obtained. The mean, median, variance, skewness and kurtosis of the samples simulation are given in Table 1. From this table we can see that all the samples simulation increases as b increases. Also the mean, median and variance increases as γ increases. It is worth mentioning that the formula in(7) and (8) are conformity with the data given in Table 1.

Table 1. Sample statistical from weibull-rayleigh distribution with various parameters

Kurtosis	Skewness	Variance	Median	Mean	b	γ	c
8.685	1.976	0.251	0.347	0.500	0.5	0.5	0.5
8.233	1.913	0.991	0.697	1	1	0.5	0.5
8.517	1.942	3.947	1.386	1.992	2	0.5	0.5
8.167	1.909	8.967	2.089	3	3	0.5	0.5
8.480	1.941	0.493	0.492	0.708	0.5	0.5	0.5
8.643	1.975	2.022	0.981	1.416	1	0.5	0.5
8.713	1.979	4.056	1.151	1.788	2	0.5	0.5
8.928	1.982	7.955	1.960	2.82	3	0.5	0.5
8.452	1.953	0.992	0.691	0.996	0.5	0.5	0.5
8.774	1.981	3.981	1.378	1.990	1	0.5	0.5
8.387	1.932	15.894	2.779	4	2	0.5	0.5
8.465	1.937	35.716	4.147	5.980	3	0.5	0.5
8.201	1.922	1.498	0.855	1.228	0.5	0.5	0.5
8.534	1.962	6.092	1.718	2.471	1	0.5	0.5
8.171	1.911	23.987	3.407	4.905	2	0.5	0.5
8.481	1.945	54.143	5.117	7.347	3	0.5	0.5
3.189	0.604	0.053	0.418	0.443	0.5	0.5	0.5
3.221	0.617	0.215	0.836	0.887	1	0.5	0.5
3.165	0.608	0.859	1.665	1.770	2	0.5	0.5
3.190	0.612	1.925	2.500	2.660	3	0.5	0.5
3.232	0.632	0.107	0.588	0.625	0.5	0.5	0.5
3.242	0.635	0.432	1.176	1.252	1	0.5	0.5
3.243	0.634	1.710	2.350	2.503	2	0.5	0.5
3.242	3.876	3.876	3.540	3.767	3	0.5	0.5
3.191	0.614	0.214	0.836	0.887	0.5	0.5	0.5
3.239	0.626	0.858	1.671	1.777	1	0.5	0.5
3.249	0.629	3.440	3.325	3.544	2	0.5	0.5
3.239	0.631	7.775	4.997	5.331	3	0.5	0.5
3.126	0.598	0.323	1.022	1.087	0.5	0.5	0.5
3.217	0.620	1.297	2.041	2.170	1	0.5	0.5
3.198	0.618	5.159	4.089	1.349	2	0.5	0.5
3.172	0.610	11.637	6.124	6.520	3	0.5	0.5
2.753	-0.079	0.016	0.456	0.453	0.5	0.5	0.5

Let $x_1, x_2, \ldots, x_n$ be a random sample from weibull-rayleigh distribution with density function (4). Then the likelihood function is given by

$$\log L(c, \gamma, b) = \sum_{i=1}^{n} \left(\log c + \log x_i - \log \gamma - 2 \log b + (c-1) \log \left(\frac{x_i^2}{2b^2\gamma} \right) - \left(\frac{x_i^2}{2b^2\gamma} \right) c \right). \tag{9}$$

By differentiating from(9) with respect to each parameter, we have

$$\frac{\partial}{\partial c} \log L(c, \gamma, b) = \frac{n}{c} - \sum_{i=1}^{n} \log \left(\frac{x_i^2}{2b^2\gamma} \right) + \sum_{i=1}^{n} \left(\frac{x_i^2}{2b^2\gamma} \right)^{c} \log \left(\frac{x_i^2}{2b^2\gamma} \right) = 0, \tag{10}$$

$$\frac{\partial}{\partial \gamma} \log L(c, \gamma, b) = -\frac{nc}{\gamma} + \frac{2b^2 c}{(2b^2\gamma)^{c+1}} \sum_{i=1}^{n} (x_i)^{2c} = 0 \tag{11}$$

and

$$\frac{\partial}{\partial b} \log L(c, \gamma, b) = -\frac{2nc}{\gamma} + \frac{4\gamma bc}{(2b^2\gamma)(c+1)} \sum_{i=1}^{n} (x_i)^{2c} = 0 \tag{12}$$

By solving the equations (10), (11) and (12) the maximum likelihood estimators is given by $\hat{c}$, $\hat{\gamma}$ and $\hat{b}$.

3. Weibull-exponential distribution

Let in formula (1) $r(t)$ and $R(t)$ be the density function and distribution function of a random variable respectively, which is distributed as weibull distribution with parameters (λ, k). Therefore

$$r(t) = \left(\frac{k}{\lambda} \right) \left(\frac{t}{\lambda} \right)^{k-1} e^{-\left(\frac{t}{\lambda} \right)}, \quad \lambda, k > 0, \quad t > 0$$

and

$$R(t) = 1 - e^{-\left(\frac{t}{\lambda} \right)}, \quad \lambda, k > 0, \quad t > 0.$$

The $X-$weibull family is as

$$G(x) = \int_{0}^{W(F(x))} \left(\frac{k}{\lambda} \right) \left(\frac{t}{\lambda} \right)^{k-1} e^{-\left(\frac{t}{\lambda} \right)} dt = 1 - e^{-\left(\frac{W(F(x))}{\lambda} \right)^{k}}.$$

Let $f_X(x)$ and $F_X(x)$ be the density function and distribution function of a random variable respectively, which is distributed as exponential distribution with parameter μ. Also, since

$X > 0$, define $W(F(x)) = \frac{F(x)}{1-F(x)}$. Then the density function and distribution function of weibull-exponential distribution is

$$G_{WRD}(x) = 1 - \exp\left(-\frac{e^{\mu x}-1}{\lambda}\right)^{k}, \quad \lambda,\, k,\, \mu > 0, \quad x > 0 \tag{13}$$

and

$$g_{WRD}(x) = \frac{k\mu}{\lambda} e^{\mu x}\left(-\frac{e^{\mu x}-1}{\lambda}\right)^{k-1} \exp\left(-\frac{e^{\mu x}-1}{\lambda}\right)^{k}, \quad \lambda,\, k,\, \mu > 0, \quad x > 0. \tag{14}$$

The following lemmas give the relationship between the weibull-exponential distribution with weibull distribution and weibull-rayleigh distribution.

Lemma 3. Suppose that X is a random variable with weibull-exponential distribution with parameters (λ, k, μ), then the random variable $Y = e^{\mu X} - 1$ is distributed as weibull distribution with parameter (λ, k). Note that if $k = 1$, the random variable Y has exponential distribution with parameter λ^{-1}.

Lemma 4. Suppose that X is a random variable with weibull distribution with parameters (λ, k), then the random variable $Y = \frac{1}{\mu}\log(X+1)$ is distributed as weibull-exponential distribution with parameters (λ, k, μ).
The proof of these lemmas is easy.

Lemma 5. Suppose that X is a random variable with weibull-exponential distribution with parameters (c, γ, μ), then the random variable $Y = \sqrt{2(e^{\mu X} - 1)}$ is distributed as weibull-rayleigh distribution with parameter (c, γ, μ).

Asymptotic behavior of density function of weibull-exponential is

$$\lim_{x\to+\infty} g_{WRD}(x) = 0,$$

and

$$\lim_{x\to 0} g_{WRD}(x) = \begin{cases} \infty & 0 < k < 1 \\ \frac{\mu}{\lambda} & k = 1 \\ 0 & k > 1. \end{cases}$$

The hazard rate function of weibull-exponential distribution can computed as follows:

$$h_{WRD}(x) = \frac{g(x)}{1-G(x)} = \frac{k\mu}{\lambda}\, e^{\mu x}\left(\frac{e^{\mu x}-1}{\lambda}\right)^{k-1}, \quad \lambda,\, k,\, \mu > 0, \quad x > 0. \tag{15}$$

The limit of hazard rate function of weibull-exponential distribution for various values of k is

$$\lim_{x\to 0} h_{WRD}(x) = \begin{cases} \infty & 0 < k < 1 \\ \frac{\mu}{\lambda} & k = 1 \\ 0 & k \geq 1, \end{cases}$$

and

$$\lim_{x\to\infty} h_{WRD}(x) = \infty.$$

Theorem 5. Suppose that X is a random variable with weibull-exponential distribution as (14). The quantile function of X^2 given by

$$Q(p) = \frac{1}{\mu}\log(\lambda\sqrt[k]{-\log(1-p)} + 1), \qquad 0 < p < 1, \;\; \lambda,\, k,\, \mu > 0. \tag{16}$$

Proof. The proof is easy by solving $G(Q(p)) = p$.
If $p = 0.5$, then the median of distribution given by

$$Q(0.5) = \frac{1}{\mu}\log(\lambda\sqrt[k]{0.69} + 1), \qquad \lambda,\, k,\, \mu > 0. \tag{17}$$

Theorem 6. Suppose that X is a random variable is distributed as weibull-exponential distribution with density function given in (14). Then the mode of distribution is given by

$$x = \frac{1}{\mu}\log(z+1),$$

where z the positive roots of the following polynomial

$$kz^{k+1} + kz^k - k\lambda^{k+1}z + (\lambda^k z - \lambda k^k) = 0.$$

Proof. By differentiating of logarithm from (14), we have that

$$\frac{\partial}{\partial x}\log g_{WED}(x) = \mu + \mu(k-1)\frac{e^{\mu x}}{e^{\mu x}-1} - \frac{k\mu}{\lambda^k}(e^{\mu x}-1)^{k-1}e^{\mu x}, \tag{18}$$

With some manipulations we have that

$$k\lambda^k e^{\mu x} - ke^{\mu x}(e^{\mu x}-1)^k - \lambda^k = 0.$$

Let $z = e^{\mu x} - 1$, then

$$kz^{k+1} + kz^k - k\lambda^{k+1}z + (\lambda^k z - \lambda k^k) = 0,$$

and this completes the proof.
Theorem 7. Suppose that X is a random variable is distributed as weibull-exponential distribution with density function given in (14). Then a bound for mathematical expectation X given by

$$E(X) \leq \frac{1}{\mu}\log\left(1 + \lambda\Gamma(1+\frac{1}{k})\right).$$

Proof. The moment generating function of weibull- exponential distribution is

$$\begin{aligned} M_X(t) &= \tfrac{k\mu}{\lambda}\int_0^\infty e^{(\mu+t)x}\left(\tfrac{e^{\mu x}-1}{\lambda}\right)^{k-1} e^{-\left(\frac{e^{\mu x}-1}{\lambda}\right)^k} dx \\ &= \tfrac{k}{\lambda}\int_0^\infty (y+1)^{\frac{t}{\mu}}\left(\tfrac{y}{\lambda}\right)^{k-1} e^{-\left(\frac{y}{\lambda}\right)^k} dy = E(Y+1)^{\frac{t}{\mu}}, \end{aligned} \tag{19}$$

where Y is distributed as weibull distribution with parameters $(\lambda,\, k)$. By differentiating from (19), we have that

$$\frac{\partial}{\partial t}M_X(t) = \frac{\partial}{\partial t}\left(E(Y+1)^{\frac{t}{\mu}}\right) = E\left(\left(\frac{1}{\mu}(Y+1)^{\frac{t}{\mu}}\right)\log(Y+1)\right).$$

Putting $t = 0$ and using Jensen's inequality, a bound for $E(X)$ given by

$$E(X) = E\left(\tfrac{1}{\mu}\log(Y+1)\right) \leq \tfrac{1}{\mu}\log\left(1 + E(Y)\right) = \tfrac{1}{\mu}\log\left(1 + \lambda\Gamma(1+\tfrac{1}{k})\right).$$

Theorem 8. Suppose that X is a random variable is distributed as weibull-exponential distribution with density function given in (14). Then a bound for it's Shannon entropy given by

$$\eta_{WE} \geq -\log\,\mu - \log\left(1 + \Gamma(1+\frac{1}{k})\right) + \eta_W, \tag{20}$$

where η_W the Shannon entropy of weibull distribution with parameters $(\lambda,\, k)$.
Proof. Shannon entropy of a random variable is defined by $E_X(-\log(g(x)))$. According to (2) we have that

$$\begin{aligned} E_X(-\log(g(x))) &= E\left(-\log\left(\tfrac{d}{dx}(W(F(x)))r(W(F(x)))\right)\right) \\ &= E\left(-\log\left(\mu e^{\mu X}\right) + \log(r(e^{\mu X} - 1))\right) \\ &= -\log\,\mu - \mu E(X) + \eta_W \\ &\geq -\log\,\mu - \log\left(1 + \Gamma(1+\tfrac{1}{k})\right) + \eta_W. \end{aligned}$$

To simulate from weibull-exponential distribution, first we simulate the samples Y using lemma 4 from weibull distribution, then we use the conversion $X = \frac{1}{\mu}\log(Y+1)$. The mean, median, variance, skewness and kurtosis of the samples simulation are given in Table 2. From this table we can see that increasing the value of μ causes the decreasing of mean, median and variance.
Increasing the value of k causes the increasing of mean, median and variance. Also increasing the value of λ causes the decreasing of mean and variance and decreasing of median. It is worth mentioning that the formula in (16) is conformity with the data given in Table 2.

Table 2. Sample statistical indexes from weibull-exponential with various parameters

Kurtosis	Skewness	Variance	Median	Mean	b	γ	c
5.618	1.787	1.367	0.391	0.982	0.5	0.5	0.5
6.265	1.887	0.385	0.237	0.453	1	0.5	0.5
8.746	1.493	0.0866	0.108	0.225	2	0.5	0.5
5.431	1.941	0.037	0.072	0.149	3	0.5	0.5
3.867	1.501	2.227	0.706	1.143	0.5	0.5	0.5
4.583	1.476	0.570	0.371	0.700	1	0.5	0.5
4.016	1.421	0.142	0.129	0.347	2	0.5	0.5
4.788	1.347	0.075	0.122	0.237	3	0.5	0.5
3.515	1.021	3.5656	1.356	2.025	0.5	0.5	0.5
3.360	0.951	1.064	0.708	0.952	1	0.5	0.5
3.804	1.092	0.259	0.340	0.478	2	0.5	0.5
4.201	1.045	0.106	0.259	0.343	3	0.5	0.5
3.170	1.011	4.886	1.675	2.479	0.5	0.5	0.5
3.104	0.889	1.219	0.703	1.213	1	0.5	0.5
2.710	0.846	0.311	0.524	0.556	2	0.5	0.5
4.202	1.045	0.106	0.258	0.340	3	0.5	0.5
4.167	1.112	0.359	0.552	0.745	0.5	0.5	0.5
3.233	1.091	0.084	0.300	0.187	1	0.5	0.5
3.590	0.911	0.021	0.153	0.187	2	0.5	0.5
4.025	1.053	0.008	0.084	0.119	3	0.5	0.5
2.662	0.813	0.668	1.014	1.193	0.5	0.5	0.5
2.988	0.748	0.188	0.519	0.608	1	0.5	0.5
3.126	0.709	0.048	0.257	0.297	2	0.5	0.5
3.261	0.779	0.018	0.185	0.197	3	0.5	0.5
2.557	0.256	1.355	1.713	1.838	0.5	0.5	0.5
2.756	0.351	0.319	0.881	0.929	1	0.5	0.5
2.769	0.403	0.081	0.451	0.476	2	0.5	0.5

Let $x_1, x_2, \ldots, x_n$ be a random sample from weibull-exponential distribution with density function (14). Then the likelihood function is given by

$$\begin{aligned}\log\, L(\lambda,\, k,\, \mu) = n \log\, k + n \log\, \mu - n \log\, \lambda + \mu \textstyle\sum_{i=0}^{n} x_i \\ +(k-1) \textstyle\sum_{i=1}^{n} \log\, \left(\frac{e^{\mu x_i - 1}}{\lambda}\right) - \sum_{i=1}^{n} \left(\frac{e^{\mu x_i - 1}}{\lambda}\right)^k.\end{aligned} \tag{21}$$

By differentiating from (21) with respect to each parameter, we have

$$\frac{\partial}{\partial c}\log L(\lambda,\, k,\, \mu) = -\frac{n}{\lambda} - (k-1)\frac{n}{\lambda} + \frac{k}{\lambda^{k+1}}\sum_{i=1}^{n}\left(\frac{e^{\mu x_i}-1}{\lambda}\right)^{k} = 0, \tag{22}$$

$$\frac{\partial}{\partial k}\log L(\lambda,\, k,\, \mu) = \frac{n}{k} + \sum_{i=1}^{n}\log\left(\frac{e^{\mu x_i}-1}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{e^{\mu x_i}-1}{\lambda}\right)^{k}\left(\frac{e^{\mu x_i}-1}{\lambda}\right) = 0, \tag{23}$$

and

$$\frac{\partial}{\partial \mu}\log L(\lambda,\, k,\, \mu) = \frac{n}{k} + \sum_{i=1}^{n} x_i + (k-1)\sum_{i=1}^{n}\frac{x_i e^{\mu x_i}}{e^{\mu x_i}-1} - \frac{k}{\lambda^{k}}\sum_{i=1}^{n} x_i\ e^{\mu x_i}(e^{\mu x_i}-1)^{k-1} = 0, \tag{24}$$

By solving the equations (22), (23) and (24) the maximum likelihood estimators is given by $\hat{\lambda}$, $\hat{k}$ and $\hat{\mu}$.

4. Application of distributions

In this section, two real data sets are used to illustrate the applicability of these distributions. The first data set we are considering comes from Smith and Naylor (1987). This data shows the power of glass fibers with length 1.5 mm which are measured in the England international physics laboratory. Alzaatreh, et al. (2014) fitted two gamma- normal distributions (four parametric, GND4p and two parametric, GND2p) to this data set and showed that this distribution Works better than Birnbaum–Saunders distribution (BSD) and Beta Birnbaum–Saunders distribution (BBSD). The maximum likelihood estimators along with comparison between the weibull-rayleigh distribution and gamma- normal distribution and other distributions in terms of Akaike information criterion are computed and given in Table 3.

Table 3. Estimation of parameters for the first data set

Distribution	GND4p	GND2p	BBSD	BSD	WRD
Estimated parameters	$\hat{\alpha} = 0.587$	$\hat{\alpha} = 18.084$	$\hat{\alpha} = 0.363$	$\hat{\alpha} = 0.269$	$\hat{\alpha} = 0.989$
	$\hat{\beta} = 0.187$	$\hat{\beta} = 0.146$	$\hat{\beta} = 7857.56$	$\hat{b} = 1.391$	$\hat{\gamma} = 2.617$
	$\hat{\mu} = 0.2089$		$\hat{a} = 1.050$		$\hat{b} = 1.112$
	$\hat{\sigma} = 0.391$		$\hat{b} = 30.478$		
Akaike information criterion	34.429	34.110	37.552	48.378	33.486

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From Table 3, we can see that weibull-rayleigh distribution has the minimum value of Akaike information criterion, so that it is the best fit.
The second data set we are considering comes from Cordeiro and Bager (2015). This data shows the production level of milk for 107 Cattles at the beginning of birth. Cordeiro and Bager (2015) fitted the beta power distribution (BPD) to this data and show that the beta power distribution is better than power distribution of this data set. We fit weibull-exponential distribution (WED), weibull distribution (WED), weibull-Logistics distribution, exponential-weibull distribution and generalized-exponential distribution to this data set. The Akaike information criterion and Bayes information criterion along with the maximum likelihood estimators are computed and given in Table 4. It shows that weibull-exponential distribution has the minimum value of Akaike information criterion and Bayes information criterion, so that it is the best fit than the other distributions.

Table 4. Estimation of parameters for the second data set

Distribution	WED	BPD	WLD	WD	GE
Estimated parameters	$\hat{k} = 1.017$	$\hat{a} = 0.270$	$\hat{c} = 3.771$	$\hat{k} = 2.601$	$\hat{a} = 3.714$
	$\hat{\lambda} = 15.288$	$\hat{b} = 42.022$	$\hat{b} = 12.126$	$\hat{\theta} = 0.523$	$\hat{b} = 0.238$
	$\hat{\mu} = 5.057$	$\hat{a} = 6.640$	$\hat{m} = 0.272$		
		$\hat{\beta} = 0.775$			
Akaike information criterion	-52.254	-47.502	-47.154	-38.695	-6.077
Bayes information criterion	-44.235	-39.525	-39.136	-33.349	-0.731

5. Concluding remarks

In this study, we introduced weibull-rayleigh distribution and weibull-exponential distribution of $X-$weibull family. The density functions, distribution functions, hazard rate functions, moment functions and Shannon entropy investigated and capabilities of distributions compared to other distributions. Also by using simulation from these two distributions with various parameters some of the sample statistical indexes estimated. Two real data sets used to illustrate the applicability of these distributions.

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References

Akinsete, A., Famoy, F. and Lee, C. (2008), The beta-pareto distribution, *Statistics*, 42, 547-563.

Agarwarl, S. K. and Al-saleh, J. A. (2001), Generalized gamma type distribution and its hazard rate function, *Communications in Statistics-Theory and Methods*, 30(2), 309-318.

Alzaatreh, A. Lee, c. and Famoy, F. (2012a), Gamma-pareto distribution and its applications, *Journal of Modern Applied Statistical Methods*, 11(1), 78-94.

Alzaatreh, A. Lee, C. and Famoy, F. (2012b), On the discrete analogues of continuous distributions, *Statistical Methodology*, 9, 589-603.

Alzaatreh, A., Famoy, F. and Lee, C. (2013a), Weibull-pareto distribution and its applications,*Communications in statistics-Teory and methods*, 42(9), 1673-1691.

Alzaatreh, A. Lee, C. and Famoy, F. (2013b), A new method for generating families of continuous distributions, *Metron*, 71(1), 63-79.

Alzaatreh, A. and Knight, K. (2014), On the Gamma-half normal distribution and its applications, *Journal of Modern Applied Statistical Methods* 12(1), 103-119.

Aljarrah, M., Lee, C. and Famoy, F. (2014), On generating T-X family of distributions using quantile function, *Journal of Statistical Distributions and Application*, 1:2.

Cordeiro, G. M. and Bager, R. S. (2015), The beta power distribution,Brazilian *Journal of Probability and Statistics*, 26(1), 88-112.

Eugene, N., Lee, C. and Famoye, F. (2002), The-normal distribution and its applications, *Communications in Statistics-Theory and Methods*, 31(4), 497-512.

Gupta, R. D. and Kundu, D. (1999), Generalized exponential distributions *Australian New Zealand Journal of Statistics*, 41(2), 173-188.

Gupta, R. D. and Kundu, D. (2001), Exponentiated-exponential family: an alternative to gamma and weibull distributions, *Biometrical Journal*, 43, 117-130.

Gupta, R., Gupta, P. L. and Gupta, R. D. (1998), Modeling failure time data by lehman alternatives, *Communications in Statistics-Theory and Methods*, 27(4), 887-904

Khan, M. S. H. and Jain, G. C. (1978), A class of distributions in the first emptiness of a semi-infinite reservoir, *Biometrical Journal*, 20, 243-252.

Mudholkar, G. S., Srivastava, D. K. and Fremer, M. (1995), The exponentiated weibull family: A reanalysis of the bus-motor-failure data, *Thechnometrics*, 37(4), 436-445.

Nadarajah, S. and Kotz, S. (2004), The data Gumbel distribution, *Reliability Engineering and System*, 91(6), 698-697.

Smith, R. L. and Naylor, J. C. (1987), A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *Journal of Royal Statistical Society Series*, 36, 358-369.

Stacy, E. W. (1962), A generalization of the gamma distribution, *Annals of Mathematical Statistics*, 33, 1187-1192.