

Bifurcation analysis of a modified cytotoxic T-lymphocyte response model with two delays

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Keywords: cytotoxic T-lymphocyte, stability, periodic solution, delay, Hopf bifurcation.

Abstract: In this paper, a modified cytotoxic T-lymphocyte response model with two delays is considered. By regarding the delays as the bifurcation parameter, the local asymptotic stability of the positive equilibrium is studied. We found that the system undergoes a Hopf bifurcation of a nonconstant periodic solution at positive equilibrium when the delays across through a sequence of bifurcated periodic solutions. And, we investigate the direction and the stability of bifurcated periodic solution.

1. Introduction

Recently, population models of immune systems have been studied extensively. The cytotoxic T-lymphocyte has plays an important role in medicine science, and it has been widely studied by many researchers (see, e.g. [1,2] and the references therein). In [3,4] the following differential equations system was considered

$$\begin{cases} x_1' = \gamma - dx_1 - \beta x_1 x_2, \\ x_2' = \beta x_1 x_2 - ax_2 - px_2 x_3, \\ x_3' = cx_2 x_3 - bx_3. \end{cases} \quad (1.1)$$

It is of necessary to incorporate time delays into the models for reflecting the dynamical behaviours. So in this paper, we investigate the CTL response system with two delays, as follows

$$\begin{cases} x_1' = \gamma - dx_1 - \beta x_1 x_2(t - \tau_1), \\ x_2' = \beta x_1(t - \tau_2) x_2 - ax_2 - px_2 x_3, \\ x_3' = cx_2 x_3 - bx_3. \end{cases} \quad (1.2)$$

where $\gamma, \beta, d, a, p, c, b$ are positive constants, $x_1, x_2, x_3, \tau_1, \tau_2$, are functions respect to t . It is well known that periodic solutions can arise through the Hopf bifurcation [5-7] in delay differential equations. Therefore, in this paper, we take bifurcation phenomenon and the properties of periodic solutions of system (1.2) into consideration.

The organization of this paper is as follows. In section 2, we discuss the stability of the positive equilibrium and the existence of Hopf bifurcation. In section 3, the direction of Hopf furcation and the stability of bifurcated periodic solutions are determined.

2. Stability of positive equilibrium and Hopf bifurcation

It is easy to see that system (1.2) has three equilibrium. Following are the equilibrium states:

$$\begin{aligned} A: E_1(m_1, 0, 0), \quad m_1 = \frac{\gamma}{d}; \quad B: E_2(m_2, m_3, 0), \quad m_2 = \frac{a}{\beta}; m_3 = \frac{(\gamma\beta - ad)}{a\beta}; \\ C: E^*(x_1^*, x_2^*, x_3^*), \quad x_1^* = \frac{c\gamma}{cd + b\beta}, \quad x_2^* = \frac{b}{c}, \quad x_3^* = \frac{[c\gamma\beta - a(cd + b\beta)]}{p(cd + b\beta)}. \end{aligned}$$

For convenient, we take the following notes.

$$(H_1): c\gamma\beta > a(cd + b\beta), (H_2): (d + \beta x_2^*)^2 > 2cp x_2^* x_3^*, (H_3): (cp x_2^* x_3^*)^2 > 2cp(d + \beta x_2^*) x_2^* x_3^* + (\beta^2 x_1^* x_2^*)^2.$$

If system (1.2) satisfies (H_1) , it has a unique positive equilibrium and two boundary equilibria. Now, we do investigate the stability of the positive equilibrium.

Let $u_1(t) = x_1(t - \tau_2), u_2(t) = x_2(t), u_3(t) = x_3(t), \tau = \tau_1 + \tau_2$ then system (1.2) becomes

$$\begin{cases} u_2'(t) = \gamma - du_1(t) - \beta u_1(t)u_2(t - \tau), \\ u_1'(t) = \beta u_1(t)u_2(t) - au_2(t) - pu_2(t)u_3(t), \\ u_3'(t) = cu_2(t)u_3(t) - bu_3(t). \end{cases} \quad (2.1)$$

The characteristic of (2.1) at E^* is

$$\lambda^3 + \lambda^2(d + \beta x_2^*) + \lambda c p x_2^* x_3^* + \lambda \beta^2 x_1^* x_2^* e^{-\lambda \tau} + c p (d + \beta x_2^*) x_2^* x_3^* = 0. \quad (2.2)$$

Let $M_1 = d + \beta x_2^*, M_2 = c p x_2^* x_3^*, M_3 = \beta^2 x_1^* x_2^*$. Then, we arrive at

$$\lambda^3 + M_1 \lambda^2 + M_2 \lambda + M_3 e^{-\lambda \tau} \lambda + M_1 M_2 = 0. \quad (2.3)$$

In the following, we regard the time delay as the parameter to consider the local stability of the positive equilibrium and the Hopf bifurcation of the system (2.1).

Lemma 2.1 Assume that (H_1) is hold. Then, the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ of system is locally asymptotically stable in the absence of delay.

Proof When $\tau = 0$, (2.3) reduces to

$$\lambda^3 + M_1 \lambda^2 + (M_2 + M_3) \lambda + M_1 M_2 = 0. \quad (2.4)$$

Clearly, by Routh-Hurwitz criterion, we know that all the roots of (2.4) have negative real parts. Then, the conclusion is obtained.

Lemma 2.2 Suppose that $(H_1), (H_2)$ and (H_3) is satisfied. Then, at

$$\tau_j = \frac{1}{\omega_0} \arccos \left[\frac{M_1 \omega_0^2 - M_1 M_2}{\omega_0^3 - M_2 \omega_0} \right] + \frac{j\pi}{\omega_0}, j = 0, 1, 2, \dots \quad (2.5)$$

(2.3) has a simple pair of conjugate purely imaginary roots $\pm i\omega_0$, where $\omega_0 = d$.

Furthermore, we have the following results

- (i) If $\tau \in [0, \tau_0)$, then all the roots of (2.3) have negative real parts;
- (ii) If $\tau = \tau_0, (H_2)$ and (H_3) is satisfied then (2.3) has a simple pair of conjugate purely imaginary roots $\pm i\omega_0$, and all other roots have negative real parts.

Proof Now for $\tau > 0$, let $\lambda = i\omega$ is a root of (2.3), then

$$(i\omega)^3 + M_1 (i\omega)^2 + M_2 (i\omega) + M_3 e^{-i\omega \tau} + M_1 M_2 = 0,$$

separating the real and imaginary parts, we obtain

$$-M_1 \omega^2 + M_3 \omega \sin(\omega \tau) + M_1 M_2 = 0, \quad (2.6)$$

$$-\omega^3 + M_3 \omega \cos(\omega \tau) + M_2 \omega = 0. \quad (2.7)$$

It is easy to see from (2.6) and (2.7) that

$$\omega^6 + (M_1^2 - 2M_2)\omega^4 + (M_2^2 - 2M_1^2 M_2 - M_3^2)\omega^2 + M_1^2 M_2^2 = 0. \quad (2.8)$$

Let $\omega^2 = q$, then (2.8) reduce to

$$q^3 + (M_1^2 - 2M_2)q^2 + (M_2^2 - 2M_1^2 M_2 - M_3^2)q + M_1^2 M_2^2 = 0. \quad (2.9)$$

Now, by Descartes' rule of sign, for an equation with real coefficients, non-existence of a positive root of (2.9) is that the coefficients are positive, namely, $M_1^2 > 2M_2$ and $M_2^2 > 2M_1^2 M_2 + M_3^2$. So $i\omega$ is not a root of (2.3). Then, the proof is complete.

Lemma 2.3 Assume that $(H_1), (H_2)$ and (H_3) hold.

- (1) If $\tau \in [0, \tau_0)$, then the positive equilibrium E^* of system (1.2) is asymptotically stable;
- (2) If $\tau > \tau_0$, then the positive equilibrium E^* of system (1.2) is unstable;
- (3) System (1.2) can undergo a Hopf bifurcation at the positive equilibrium E^* when $\tau = \tau_j (j = 0, 1, 2, \dots)$, where τ_j is defined by (2.5).

Proof Through (2.6) and (2.7), we can get (2.5) obtained. So, $j = 0$, (2.5) gives

$$\tau_0 = \frac{1}{\omega_0} \arccos \left[\frac{M_1 \omega_0^2 - M_1 M_2}{\omega_0^3 - M_2 \omega_0} \right],$$

which is the smallest time delay for which the stable solution becomes unstable. Thus the

characteristic equation (2.3) has a pair of purely imaginary roots when $\omega = \omega_0$ and $\tau = \tau_0$.

Let $\lambda(\tau) = v(\tau) + i\omega(\tau)$ be the root of (2.3) such that when $\tau = \tau_j$ satisfying $v(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. Differentiating two sides of (2.3) with respect to τ , we have

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\lambda=i\omega_0} = \frac{3\omega_0^4 + 2(M_2^2 M_2)\omega_0^2 + M_2^2 - 2M_1^2 M_2 - M_3^2}{M_3^2 \omega_0^2}. \quad (2.10)$$

In (2.9) denoting the left side by Ω , and differentiation it at $q_2 = \omega_0^2$. Combing with (2.10), we get

$$\left[\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right]_{\lambda=i\omega_0} = \frac{1}{M_3^2 \omega_0^2} \frac{d\Omega}{dq} \Big|_{q=\omega_0^2}. \quad (2.11)$$

The sign of (2.11) is nonzero, so the characteristic (2.3) cannot have a multiple imaginary roots. Therefore, transversality condition holds. According to the bifurcation theorem for functional differential equation. One of the main results of this paper is as follows.

Theorem 2.1 Assume that $(H_1), (H_3), (H_3)$ hold.

- (a) Then the positive equilibrium E^* of system (1.2) is asymptotically stable for all $\tau \geq 0$;
- (b) If the above is hold, then the positive equilibrium E^* of system (1.2) is asymptotically stable for $\tau \in [0, \tau_0)$;
- (c) If H_1 and H_4 hold, then model system (2.1) is unstable for $\tau \geq \tau_0$. Hopf bifurcation occurs at $\tau = \tau_0$.

3. Stability of bifurcated periodic solutions

Now, we investigate the direction and stability of the periodic solutions bifurcated from the positive equilibrium E^* . Without loss of generality, we denote critical values $\tau = \tau_j, (j = 0, 1, 2, \dots)$ by τ when the characteristic equation (2.3) has a pair of purely imaginary roots $\pm i\omega_0$, we denote τ as $\tau = \tilde{\tau} + \mu, \mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation value of system (2.1).

Let $y_1(t) = u_1(t) - x_1^*, y_2(t) = u_2(t) - x_2^*, y_3(t) = u_3(t) - x_3^*$, then (2.1) reduces to

$$\begin{cases} y_1'(t) = u_1'(t) = -dy_1(t) - \beta y_1(t)y_2(t-\tau) - \beta y_1(t)x_2^*y_2(t-\tau), \\ y_2'(t) = u_2'(t) = (y_2(t) + x_2^*)(\beta y_1(t) - py_3(t)), \\ y_3'(t) = u_3'(t) = cy_2(t)(x_3^* + y_3(t)). \end{cases} \quad (3.1)$$

Let $W_1(t) = y_1(\tau t), W_2(t) = y_2(\tau t)$ and $W_3(t) = y_3(\tau t)$, then we have

$$\begin{cases} W_1'(t) = y_1'(t)\tau = \tau[-(d + \beta x_2^*)W_1(\tau t) - \beta W_2(t-1)x_1^* - \beta W_1(t)W_2(t-1)], \\ W_2'(t) = y_2'(t)\tau = \tau(W_2(t) + x_2^*)(\beta W_1(t) - pW_3(t)), \\ W_3'(t) = y_3'(t)\tau = \tau cW_2(t)(W_3(t) + x_3^*). \end{cases} \quad (3.2)$$

Then, system (2.1) can be rewritten as the following functional differential in $C([1-\infty), \mathbb{R}^3)$ as

$$W'(t) = L_\mu(W(t)) + F(\mu, W(t)), \quad (3.3)$$

where $W(t) = (W_1(t), W_2(t), W_3(t))^T \in \mathbb{R}^3$, $L_\mu : C \rightarrow \mathbb{R}$ and $F : \mathbb{R} \times C \rightarrow \mathbb{R}$ are defined respectively as follows.

$$L_\mu(\phi) = (\tilde{\tau} + \mu) \begin{pmatrix} -d - \beta x_2^* & 0 & 0 \\ \beta x_2^* & 0 & -px_2^* \\ 0 & cx_3^* & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + (\tilde{\tau} + \mu) \begin{pmatrix} 0 & -\beta x_2^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix}, \quad (3.4)$$

and

$$F(\mu, W(t)) = (\tilde{\tau} + \mu) \begin{pmatrix} -\beta \phi_1(0)\phi_2(-1) \\ \beta \phi_1(0)\phi_2(0) - p\phi_2(0)\phi_3(0) \\ c\phi_2(0)\phi_3(0) \end{pmatrix}, \quad (3.5)$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C([-1, 0], \mathbb{R}^3)$. By Riesz representation theorem, there exist a

matrix whose components are bounded variation function $\eta(\theta, \mu)$ for $\theta \in C([-1, 0], R^3)$, such that

$$L_\mu \phi = \int_{-1}^0 \phi(\theta) d\eta(\theta, \mu), \text{ for } \theta \in C([-1, 0], R^3). \quad (3.6)$$

For this representation, we can choose $\eta(\theta, \mu)$ as follows.

$$\eta(\theta, \mu) = (\tilde{\tau} + \mu) \begin{pmatrix} -d - \beta x_2^* & 0 & 0 \\ \beta x_2^* & 0 & -p x_2^* \\ 0 & c x_3^* & 0 \end{pmatrix} \delta(\theta) + (\tilde{\tau} + \mu) \begin{pmatrix} 0 & -\beta x_2^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1), \quad (3.7)$$

where δ is the Dirac delta function i.e. $\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ -1, & \theta = 0. \end{cases}$

For $\phi \in C([-1, 0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi}{d\theta}, & -1 \leq \theta < 0, \\ \int_{-1}^0 \phi(\theta) d\eta(\theta, \mu), & \theta = 0. \end{cases} \quad (3.8)$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \leq \theta < 0, \\ F(\theta, \mu), & \theta = 0. \end{cases} \quad (3.9)$$

From (3.8) and (3.9), the system (3.3) is equivalent to

$$W_t' = A(\mu)W_t + R(\mu)W_t, \quad (3.10)$$

where $W_t(\theta) = W(t + \theta)$, for $-1 \leq \theta < 0$. For $\psi \in C([-1, 0], R^3)$, the adjoint operator A^* of A is defined by

$$A^* = \begin{cases} -\frac{d\psi(s)}{ds}, & 0 < s \leq 1, \\ \int_{-1}^0 \psi(-s) d\eta(s, 0), & s = 0. \end{cases} \quad (3.11)$$

associate with a bilinear form

$$\langle \psi(s), \phi(s) \rangle = \overline{\psi(0)}\phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^\theta (\bar{\psi})^T(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \quad (3.12)$$

where $\eta(\theta) = \eta(\theta, 0)$. The normal form of eigenvector l of A belonging to the eigenvalue $i\omega_0 \tilde{\tau}$ and the eigenvector l^* of A^* belonging to the eigenvalue $-i\omega_0 \tilde{\tau}$. The eigenvector are given $A(0)l(\theta) = i\omega_0 \tilde{\tau} l(\theta)$. Then,

$$l(\theta) = (\alpha_1, \alpha_2, 1)^T e^{i\omega_0 \tilde{\tau} \theta}, \quad \alpha_1 = \frac{-\omega_0^2 + c p x_2^* x_3^*}{c \beta x_2^* x_3^*}, \quad \alpha_2 = \frac{i\omega_0}{c x_3^*} \quad (3.13)$$

If $l(s)^* = D(\alpha_1^*, \alpha_2^*, 1)^T e^{i\omega_0 \tilde{\tau} s}$, be the eigenvector of $A^*(0)$, similarly,

$$A^*(0)l(s)^* = -i\omega_0 \tilde{\tau} l^*(s), \quad \alpha_1^* = \frac{-\omega_0^2 + c p x_2^* x_3^*}{p \beta x_1^* x_2^*} e^{i\omega_0 \tilde{\tau}}, \quad \alpha_2^* = \frac{i\omega_0}{p x_2^*}. \quad (3.14)$$

Since the property of adjoint, we have $\langle l^*(s), l(\theta) \rangle = 1$, and $\langle l^*(s), \bar{l}(\theta) \rangle = 0$, we get

$$\bar{D} = \frac{1}{1 + \alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* + \beta \tilde{\tau} \alpha_2 \alpha_1^* x_1^* e^{-i\omega_0 \tilde{\tau}}}.$$

Let N_t be the solution of (3.3), when $\mu = 0$. Let's define

$$z(t) = \langle l^*, N_t \rangle; \quad G(t, \theta) = N_t(\theta) - 2 \operatorname{Re}[z(t)l(\theta)]. \quad (3.15)$$

On the center manifold C_0 , we have

$$G(t, \theta) = G(z(t), \bar{z}(t), \theta), \quad (3.16)$$

and

$$G(z, \bar{z}, \theta) = G_{20}(\theta) \frac{z^2}{2} + G_{11}(\theta) z \bar{z} + G_{02}(\theta) \frac{\bar{z}^2}{2} + G_{30}(\theta) \frac{z^3}{6} + \dots, \quad (3.17)$$

where z and \bar{z} are local coordinate for center manifold C_0 , in the direction of l^* and \bar{l}^* . Note that G is real if N_t is real. We only study the real solutions. For solution $N_t \in C_0$ of (3.10), since $\mu = 0$, then

$$z'(t) = \langle l^*, N_t' \rangle = i\omega_0 \tilde{z} z + \bar{l}^* F(0, G(z, \bar{z}, 0)) + 2\text{Re}[z(t)l(\theta)] \triangleq i\omega_0 \tilde{z} z + \bar{l}^* F_0(z, \bar{z}). \quad (3.18)$$

We rewrite the equation as

$$z'(t) = i\omega_0 \tilde{z} z(t) + g(z, \bar{z}), \quad (3.19)$$

where

$$g(z, \bar{z}) = \bar{l}^* F_0(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \dots$$

It follows from (3.15) and (3.17) that

$$\begin{aligned} N(t) &= (N_{1t}, N_{2t}, N_{3t}) \\ &= G_{20}(\theta) \frac{z^2}{2} + G_{11}(\theta) z \bar{z} + G_{02}(\theta) \frac{\bar{z}^2}{2} + (\alpha_1, \alpha_2, 1) e^{i\omega_0 \tilde{t} \theta} z + (\bar{\alpha}_1, \bar{\alpha}_2, 1) e^{-i\omega_0 \tilde{t} \theta} \bar{z} + \dots, \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} N_{1t}(0) &= G_{20}^{(1)}(0) \frac{z^2}{2} + G_{11}^{(1)}(0) z \bar{z} + G_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + \dots \\ N_{2t}(0) &= G_{20}^{(2)}(0) \frac{z^2}{2} + G_{11}^{(2)}(0) z \bar{z} + G_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \alpha_2 z + \bar{\alpha}_2 \bar{z} + \dots \\ N_{3t}(0) &= G_{20}^{(3)}(0) \frac{z^2}{2} + G_{11}^{(3)}(0) z \bar{z} + G_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + z + \bar{z} + \dots \\ N_{1t}(-1) &= G_{20}^{(1)}(-1) \frac{z^2}{2} + G_{11}^{(1)}(-1) z \bar{z} + G_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \alpha_1 z e^{-i\omega_0 \tilde{t}} + \bar{\alpha}_1 \bar{z} e^{i\omega_0 \tilde{t}} + \dots \\ N_{2t}(-1) &= G_{20}^{(2)}(-1) \frac{z^2}{2} + G_{11}^{(2)}(-1) z \bar{z} + G_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \alpha_2 z e^{-i\omega_0 \tilde{t}} + \bar{\alpha}_2 \bar{z} e^{i\omega_0 \tilde{t}} + \dots \\ N_{3t}(-1) &= G_{20}^{(3)}(-1) \frac{z^2}{2} + G_{11}^{(3)}(-1) z \bar{z} + G_{02}^{(3)}(-1) \frac{\bar{z}^2}{2} + z e^{-i\omega_0 \tilde{t}} + \bar{z} e^{i\omega_0 \tilde{t}} + \dots \end{aligned} \quad (3.21)$$

then, we get

$$\begin{aligned} g(z, \bar{z}) &= \tilde{D} \left[-\alpha_1^* \beta \left(G_{20}^{(1)}(0) \frac{z^2}{2} + G_{11}^{(1)}(0) z \bar{z} + G_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + \dots \right) \right. \\ &\quad \left(G_{20}^{(2)}(-1) \frac{z^2}{2} + G_{11}^{(2)}(-1) z \bar{z} + G_{02}^{(2)}(-1) \frac{\bar{z}^2}{2} + \alpha_2 z e^{-i\omega_0 \tilde{t}} + \bar{\alpha}_2 \bar{z} e^{i\omega_0 \tilde{t}} + \dots \right) \\ &\quad + \alpha_2^* \beta \left(G_{20}^{(1)}(0) \frac{z^2}{2} + G_{11}^{(1)}(0) z \bar{z} + G_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + \dots \right) \\ &\quad \left(G_{20}^{(2)}(0) \frac{z^2}{2} + G_{11}^{(2)}(0) z \bar{z} + G_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \alpha_2 z + \bar{\alpha}_2 \bar{z} + \dots \right) \\ &\quad \left. - p \left(G_{20}^{(2)}(0) \frac{z^2}{2} + G_{11}^{(2)}(0) z \bar{z} + G_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \alpha_2 z + \bar{\alpha}_2 \bar{z} + \dots \right) \right. \\ &\quad \left. \left(G_{20}^{(3)}(0) \frac{z^2}{2} + G_{11}^{(3)}(0) z \bar{z} + G_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + z + \bar{z} + \dots \right) \right. \\ &\quad \left. + c \left(G_{20}^{(2)}(0) \frac{z^2}{2} + G_{11}^{(2)}(0) z \bar{z} + G_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \alpha_2 z + \bar{\alpha}_2 \bar{z} + \dots \right) \right. \\ &\quad \left. \left(G_{20}^{(3)}(0) \frac{z^2}{2} + G_{11}^{(3)}(0) z \bar{z} + G_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + z + \bar{z} + \dots \right) \right]. \end{aligned} \quad (3.22)$$

Comparing the coefficients with (3.19), we have

$$\begin{aligned}
 g_{20} &= 2\tilde{\tau}\bar{D}(-\beta\alpha_1^*\alpha_2\alpha_1^*e^{-i\omega_0\tilde{\tau}} + \beta\alpha_1\alpha_2\alpha_2^* - p\alpha_2\alpha_2^* + c\alpha_2) \\
 g_{11} &= 2\tilde{\tau}\bar{D}[(-\beta\alpha_1^*)(\alpha_1\alpha_2e^{i\omega_0\tilde{\tau}} + \alpha_1\alpha_2e^{-i\omega_0\tilde{\tau}}) + \alpha_2^*(\beta(\alpha_1\alpha_2 + \beta\alpha_2\alpha_1) \\
 &\quad - p(\beta(\alpha_2 + \alpha_2)) + 2c(\beta(\alpha_2 + \alpha_2))] \\
 g_{02} &= 2\tilde{\tau}\bar{D}[-\beta\alpha_1\alpha_2\alpha_1^*e^{i\omega_0\tilde{\tau}} + \alpha_2^*(\beta\alpha_1\alpha_2 - p\alpha_2) + c\alpha_2] \\
 g_{21} &= 2\tilde{\tau}\bar{D}\left((- \beta\alpha_1^*\left(\frac{1}{2}G_{20}^{(1)}(0)\alpha_2e^{i\omega_0\tilde{\tau}} + G_{11}^{(1)}(0)\alpha_2e^{-i\omega_0\tilde{\tau}} + \alpha_1G_{11}^{(2)}(-1) + \frac{1}{2}\alpha_1G_{20}^{(2)}(-1)\right) \right. \\
 &\quad + \alpha_2^*\left[\beta\left(\frac{1}{2}G_{20}^{(1)}(0)\alpha_1 + G_{11}^{(1)}(0)\alpha_2 + G_{11}^{(2)}(0)\alpha_1 + \frac{1}{2}G_{20}^{(2)}(0)\alpha_1\right) \right. \\
 &\quad \left. \left. - p\left(\frac{1}{2}G_{20}^{(2)}(0) + G_{11}^{(2)}(0) + G_{11}^{(3)}(0)\alpha_2 + \frac{1}{2}G_{20}^{(3)}(0)\alpha_2\right) \right] \right. \\
 &\quad \left. + c\left(\frac{1}{2}G_{20}^{(2)}(0) + G_{11}^{(2)}(0) + G_{11}^{(3)}(0)\alpha_2 + \frac{1}{2}G_{20}^{(3)}(0)\alpha_2\right)\right).
 \end{aligned} \tag{3.23}$$

In order to determinate g_{21} , we focus on the computation of $G_{20}(\theta)$ and $G_{11}(\theta)$. From (3.10), (3.15) and (3.19), we can obtain

$$G' = N_t - z'l - \bar{z}\bar{l} = \begin{cases} A(0)G(t, \theta) - 2\text{Re}[\bar{l}^*(0)F_0l(\theta)], & -1 \leq \theta < 0, \\ A(0)G(t, \theta) - 2\text{Re}[\bar{l}^*(0)F_0l(\theta)] + F_0, & \theta = 0. \end{cases}$$

Let

$$G' = A(0)G(t, \theta) + Q(z, \bar{z}, \theta), \tag{3.24}$$

where

$$Q(z, \bar{z}, \theta) = Q_{20}(\theta)\frac{z^2}{2} + Q_{11}(\theta)z\bar{z} + Q_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{3.25}$$

Differentiating two sides of (3.17) with respect to t , we have

$$G' = G_z z' + G_{\bar{z}} \bar{z}'. \tag{3.26}$$

According to (3.25) and (3.26), we obtain

$$(A(0) - 2i\omega_0\tilde{\tau}I) = -Q_{20}(\theta), \quad A(0)N(\theta) = -Q_{11}(\theta) \tag{3.27}$$

From (3.24), when $-1 \leq \theta < 0$, we get

$$Q(z, \bar{z}, \theta) = -\bar{l}^*(0)F_0l(\theta) - \overline{F_0l(0)}\bar{l}(\theta) = -g(z, \bar{z})l(\theta) - \overline{g(z, \bar{z})l(\theta)}.$$

Thus,

$$Q_{20}(\theta) = -g_{20}l(\theta) - \overline{g_{02}l(\theta)}, \quad Q_{11}(\theta) = -g_{11}l(\theta) - \overline{g_{11}l(\theta)}.$$

We can obtain that

$$G'_{20} = 2i\omega_0\tilde{\tau}G_{20} + g_{20}l(\theta) + \overline{g_{20}l(\theta)},$$

then

$$G_{20}(\theta) = \frac{ig_{20}l(0)}{\omega_0\tilde{\tau}}e^{i\omega_0\tilde{\tau}\theta} + \frac{i\overline{g_{02}l(0)}}{3\omega_0\tilde{\tau}}e^{-i\omega_0\tilde{\tau}\theta} + E_1e^{2i\omega_0\tilde{\tau}\theta}.$$

Similarly,

$$G_{11}(\theta) = -\frac{ig_{11}l(0)}{\omega_0\tilde{\tau}}e^{i\omega_0\tilde{\tau}\theta} + \frac{i\overline{g_{02}l(0)}}{3\omega_0\tilde{\tau}}e^{-i\omega_0\tilde{\tau}\theta} + E_2e^{2i\omega_0\tilde{\tau}\theta},$$

Next, we focus on the computation of E_1 and E_2 . From (3.27), we achieve

$$\int_{-1}^0 d\eta(\theta)G_{20}(\theta) = 2i\omega_0\tilde{\tau}G_{20}(\theta) - Q_{20}(\theta), \quad \int_{-1}^0 d\eta(\theta)G_{11}(\theta) = -Q_{11}(\theta).$$

where $\eta(\theta) = \eta(0, \theta)$. By (3.22), we have

$$Q_{20}(0) = g_{20}l(0) - \overline{g_{02}l(0)} + 2\tilde{\tau} \begin{pmatrix} -\beta\alpha_1\alpha_2e^{-i\omega_0\tilde{\tau}} \\ \beta\alpha_1\alpha_2 - p\alpha_2 \\ c\alpha_2 \end{pmatrix},$$

and

$$Q_{11}(0) = -g_{11}l(0) - \overline{g_{11}l(0)} + 2\tilde{\tau} \begin{pmatrix} -\beta(\alpha_1\overline{\alpha_2}e^{i\omega_0\tilde{\tau}} + \overline{\alpha_1}\alpha_2e^{-i\omega_0\tilde{\tau}}) \\ \beta(\alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2) - p(\alpha_2\overline{\alpha_2}) \\ c(\alpha_2\overline{\alpha_2}) \end{pmatrix}.$$

Then

$$[i\omega_0\tilde{\tau}I - \text{int}_{-1}^0 e^{i\omega_0\tilde{\tau}\theta} d\eta(\theta)]l(\theta) = 0, [-i\omega_0\tilde{\tau}I - \text{int}_{-1}^0 de^{-i\omega_0\tilde{\tau}\theta} \eta(\theta)]l(\theta) = 0.$$

We get

$$(2i\omega_0\tilde{\tau}I - \text{int}_{-1}^0 e^{2i\omega_0\tilde{\tau}\theta} d\eta(\theta))l(\theta) = 2\tilde{\tau} \begin{pmatrix} -\beta\alpha_1\alpha_2e^{-i\omega_0\tilde{\tau}} \\ \beta\alpha_1\alpha_2 - p\alpha_2 \\ c\alpha_2 \end{pmatrix}.$$

It follows that

$$E_1^{(1)} = \frac{2}{B} \begin{vmatrix} -\beta\alpha_1\alpha_2e^{-i\omega_0\tilde{\tau}} & \beta x_1^*e^{-2i\omega_0\tilde{\tau}} & 0 \\ \beta\alpha_1\alpha_2 - p\alpha_2 & 2i\omega_0 & px_2^* \\ c\alpha_2 & -cx_3^* & 2i\omega_0 \end{vmatrix}, E_1^{(2)} = \frac{2}{B} \begin{vmatrix} 2i\omega_0 + d + \beta x_2^* & -\beta\alpha_1\alpha_2e^{-i\omega_0\tilde{\tau}} & 0 \\ -\beta\alpha_2^* & \beta\alpha_1\alpha_2 - p\alpha_2 & px_2^* \\ 0 & c\alpha_2 & 2i\omega_0 \end{vmatrix},$$

$$E_1^{(3)} = \frac{2}{B} \begin{vmatrix} 2i\omega_0 + d + \beta x_2^* & \beta x_1^*e^{-2i\omega_0\tilde{\tau}} & -\beta\alpha_1\alpha_2e^{-i\omega_0\tilde{\tau}} \\ -\beta x_2^* & 2i\omega_0 & \beta\alpha_1\alpha_2 - p\alpha_2 \\ 0 & -cx_3^* & c\alpha_2 \end{vmatrix}, B = \begin{vmatrix} 2i\omega_0 + d + \beta x_2^* & \beta x_1^*e^{-2i\omega_0\tilde{\tau}} & 0 \\ -\beta x_2^* & 2i\omega_0 & px_2^* \\ 0 & -cx_3^* & 2i\omega_0 \end{vmatrix}$$

Similarly,

$$E_2^{(1)} = \frac{2}{B'} \begin{vmatrix} -\beta(\alpha_1\overline{\alpha_2}e^{i\omega_0\tilde{\tau}} + \alpha_2\overline{\alpha_1}e^{i\omega_0\tilde{\tau}}) & \beta x_1^* & 0 \\ \beta(\alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1}) - p(\alpha_2 + \overline{\alpha_2}) & 0 & px_2^* \\ c\alpha_2 + \overline{\alpha_2} & -cx_3^* & 0 \end{vmatrix},$$

$$E_2^{(2)} = \frac{2}{B'} \begin{vmatrix} d + \beta x_2^* & -\beta(\alpha_1\overline{\alpha_2}e^{i\omega_0\tilde{\tau}} + \alpha_2\overline{\alpha_1}e^{i\omega_0\tilde{\tau}}) & 0 \\ -\beta x_2^* & \beta(\alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1}) - p(\alpha_2 + \overline{\alpha_2}) & \beta(\alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1}) - p(\alpha_2 + \overline{\alpha_2}) \\ 0 & c\alpha_2 + \overline{\alpha_2} & 0 \end{vmatrix},$$

$$E_2^{(3)} = \frac{2}{B'} \begin{vmatrix} d + \beta x_2^* & \beta x_1^* & -\beta(\alpha_1\overline{\alpha_2}e^{i\omega_0\tilde{\tau}} + \alpha_2\overline{\alpha_1}e^{i\omega_0\tilde{\tau}}) \\ -\beta x_2^* & 0 & \beta\alpha_1\alpha_2 - p\alpha_2 \\ 0 & -cx_3^* & c\alpha_2 + \overline{\alpha_2} \end{vmatrix}, B' = \begin{vmatrix} d + \beta x_2^* & \beta x_1^* & 0 \\ -\beta x_2^* & 0 & px_2^* \\ 0 & -cx_3^* & 0 \end{vmatrix}.$$

Thus, we can calculate the following values

$$C_1(0) = \frac{i}{2\omega_0\tilde{\tau}} (g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{ReC_1(0)}{Re\lambda'(\tilde{\tau})}, \beta_2 = ReC_1(0), T_2 = -\frac{ImC_1(0) + \mu_2 Im\lambda'(\tilde{\tau})}{\omega_0\tilde{\tau}},$$

which determine the qualities of furcated periodic solutions in the center manifold at the critical value $\tilde{\tau}$. According to the center manifold theorem, we have the following theorem which is the main result of this paper.

Theorem 3.1 Assume that $(H_1), (H_2), (H_3)$ and (H_4) hold.

(Y_1) μ_2 determine the direction of Hopf bifurcation. If $\mu_2 > 0 (\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical);

(Y_2) β_2 determine the stability of bifurcated solutions. If $\beta_2 > 0 (\beta_2 < 0)$, then the bifurcated solutions are stable (unstable);

(Y_3) T_2 determine the periodic of bifurcated solutions. If $T_2 > 0 (T_2 < 0)$, then the periodic increases (decreases).

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