Iterative Methods for Polynomial Equations Based on Vieta's Theorem

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Abstract. Base Vieta's theorem, iterative methods for polynomial equations are proposed and all roots of polynomial equation can be found simultaneously. The convergence of methods is preliminarily discussed. Examples are given.

Introduction

In mathematics, a univariate polynomial is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ ($n \in Z^+, a_n \neq 0, a_j \in R, i = 0,1,\cdots,n$). The natural number n is known as the degree of the polynomial. A polynomial equation of degree n has exactly n roots. In elementary algebra, methods such as the quadratic formula are given for solving all first degree and second degree polynomial equations in one variable. There are also formulas for the cubic and quadratic equations. For higher degrees, the Abel–Ruffini theorem asserts that there cannot exist a general formula in radicals. However, root-finding algorithms may be used to find numerical approximations of the roots of a polynomial expression of any degree[1-4]. In this paper, the iterative method is used to solve the polynomial equation of any degree based on the Vieta's theorem. A usual iterative method for the polynomial equations can be found one root[5-8], but the method of this paper can be found simultaneously all the roots of the polynomial equation.

Vieta's Theorem

Vieta's Theorem for polynomial equations says that if a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ ($a_n \neq 0$) has n different roots $x_1, x_2, \cdots, x_n$, then

$$x_1 + x_2 + \cdots + x_n = -\frac{a_{n-1}}{a_n} \tag{1}$$

$$x_1x_2 + x_1x_3 + \cdots x_1x_n + x_2x_3 + x_2x_4 + \ldots + x_{n-1}x_n = \frac{a_{n-2}}{a_n} \tag{2}$$

$$x_1x_2x_3 + x_1x_2x_4 + \cdots x_1x_2x_n + x_1x_3x_4 + x_1x_3x_5 + \ldots + x_{n-2}x_{n-1}x_n = -\frac{a_{n-3}}{a_n} \tag{3}$$

$$x_1x_2x_3 \cdots x_{n-1}x_n = (-1)^n \frac{a_0}{a_n} \tag{4}$$

Iterative Methods for Quadratic Formulas

If A quadratic equation has 2 different roots x_1, x_2, Vieta's root forumula is: $x_1 + x_2 = -\frac{a_1}{a_2}$, $x_1 x_2 = \frac{a_0}{a_2}$.

Then $x_1 = -\frac{a_1}{a_2} - x_2$, $x_2 = \frac{a_0}{a_2 x_1}$. Therefore iterative methods for quadratic formulas are

$$\begin{cases} x_1^{(k+1)} = -\dfrac{a_1}{a_2} - x_2^{(k)} \\ x_2^{(k+1)} = \dfrac{a_0}{a_2 x_1^{(k)}} \end{cases} \tag{5}$$

Start with $x_1^{(0)}$, $x_2^{(0)} = \frac{a_0}{a_2 x_1^{(0)}}$. The calculation can be carried out by using the iteration formula.

It seems reasonable that $x_1^{(k+1)}$ could be used in place of $x_1^{(k)}$ in the computation of $x_2^{(k+1)}$. Therefore, the formula (5) is improved:

$$\begin{cases} x_1^{(k+1)} = -\dfrac{a_1}{a_2} - x_2^{(k)} \\ x_2^{(k+1)} = \dfrac{a_0}{a_2 x_1^{(k+1)}} \end{cases} \tag{6}$$

Uses $\left|x_i^{(k+1)} - x_i^{(k)}\right| \le \varepsilon$ ($i = 1,2$) as the condition of the end of algorithm. This method only needs add, subtract, multiply and divide, without square root operation.

Examples 1: Solve the following polynomial equation: $x^2 - 5x + 6 = 0$. Start with $x_1^{(0)} = 4$ and $\varepsilon = 0.00005$, after 43 steps, formula (5) meet the accuracy. Here $x_1^{(43)} = 3.00010025293625$, $x_2^{(43)} = 1.99993316694257$. After 23 steps, formula (6) meet the accuracy. Here $x_1^{(23)} = 3.00006683305743$, $x_2^{(23)} = 1.99995544562095$.

Iterative Methods for Polynomial Equation

To begin, solve the formula (1) for x_1, the formula (1) for x_2 and so on to obtain the rewritten equations:

$$x_1 = -\frac{a_{n-1}}{a_n} - x_2 - x_3 - \cdots - x_n$$

$$x_2 = \frac{{a_{n-2}}/{a_n} - x_1 x_3 - x_1 x_4 - \cdots - x_{n-1} x_n}{x_1 + x_3 + x_4 + \cdots x_n}$$

$$\cdots$$

$$x_n = \frac{(-1)^n a_0}{a_n x_1 x_2 \cdots x_{n-1}}$$

The iterative method is

$$\begin{cases} x_1^{(k+1)} = -\dfrac{a_{n-1}}{a_n} - x_2^{(k)} - x_3^{(k)} - \cdots - x_n^{(k)} \\ x_2^{(k+1)} = \dfrac{a_{n-2}/a_n - x_1^{(k)}x_3^{(k)} - x_1^{(k)}x_4^{(k)} - \cdots - x_{n-1}^{(k)}x_n^{(k)}}{x_1^{(k)} + x_3^{(k)} + x_4^{(k)} + \cdots + x_n^{(k)}} \\ \vdots \\ x_n^{(k+1)} = \dfrac{(-1)^n a_0}{a_n x_1^{(k)} x_2^{(k)} \cdots x_{n-1}^{(k)}} \end{cases} \quad (7)$$

In particular, when $n=3$, there are $x_1 + x_2 + x_3 = -\dfrac{a_2}{a_3}$, $x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{a_1}{a_3}$ and $x_1x_2x_3 = -\dfrac{a_0}{a_3}$.

The iterative method for the cubic equation is

$$\begin{cases} x_1^{(k+1)} = -\dfrac{a_2}{a_3} - x_2^{(k)} - x_3^{(k)} \\ x_2^{(k+1)} = \dfrac{a_1/a_3 - x_1^{(k)}x_3^{(k)}}{x_1^{(k)} + x_3^{(k)}} \\ x_3^{(k+1)} = \dfrac{-a_0}{a_3 x_1^{(k)} x_2^{(k)}} \end{cases} \quad (8)$$

Examples 2: Solve the following polynomial equation: $x^3 - 2x^2 - x + 2 = 0$. Start with $x_1^{(0)} = -0.5$, $x_2^{(0)} = 3$ and $\varepsilon = 0.00005$. After 27 steps, formula (8) meet the accuracy. Here $x_1^{(27)} = 1.99998447786764$, $x_2^{(27)} = -1.00000337062216$, $x_3^{(27)} = 0.99999270665868$.

Convergence of the Iteration Methods

We can use Jacobi method or Gauss-Seidel method to solve the linear systems. If some convergence conditions are required, the Jacobi method or Gauss-Seidel method converge to the unique solution[1-2]. For the iterative method of this paper, it is obvious that the iterative equation is nonlinear, and its convergence is more complicated. Here we only discuss the convergence of quadratic equation.

Firstly, the convergence of the improved method is discussed. From formula (6):

$$x_1^{(k+1)} = -\frac{a_1}{a_2} - x_2^{(k)} = -\frac{a_1}{a_2} - \frac{a_0}{a_2 x_1^{(k)}}$$

$$x_2^{(k+1)} = \frac{a_0}{a_2 x_1^{(k+1)}} = \frac{a_0}{a_2(-\dfrac{a_1}{a_2} - x_2^{(k)})} = \frac{a_0}{-a_1 - a_2 x_2^{(k)}}$$

From the iteration theory, the iteration function of x_1 , x_2 are $\varphi_1(x)=-\frac{a_1}{a_2}-\frac{a_0}{a_2 x}$, $\varphi_2(x)=\frac{a_0}{-a_1-a_2 x}$ respectively.

By contraction mapping theorem, we therefore obtain that if $|\varphi'_1(x)|<1$ and $|\varphi'_2(x)|<1$, formula (6) converges.

We have $\varphi_1^{'}(x)=\frac{a_0}{a_2 x_2^2}$ and $\varphi_2^{'}(x)=-\frac{a_0 a_2}{(a_1+a_2 x)^2}$.

Therefore condition of formula (6) is

$$\begin{cases} \left|\frac{a_0}{a_2 x^2}\right|<1 \\ \left|-\frac{a_0 a_2}{(a_1+a_2 x)^2}\right|<1 \end{cases}$$

So, when $\frac{a_1}{a_2}\geq 0$, condition of formula (6) is $x>\sqrt{\left|\frac{a_0}{a_2}\right|}$, or $x<-\frac{a_1}{a_2}-\sqrt{\left|\frac{a_0}{a_2}\right|}$; when $\frac{a_1}{a_2}<0$, condition of formula (6) is $x>-\frac{a_1}{a_2}+\sqrt{\left|\frac{a_0}{a_2}\right|}$, or $x<-\sqrt{\left|\frac{a_0}{a_2}\right|}$.

From formula (5):

$$x_1^{(k+1)}=-\frac{a_1}{a_2}-x_2^{(k)}=-\frac{a_1}{a_2}-\frac{a_0}{a_2 x_1^{(k-1)}}$$

$$x_2^{(k+1)}=\frac{a_0}{a_2 x_1^{(k)}}=\frac{a_0}{a_2(-\frac{a_1}{a_2}-x_2^{(k-1)})}=\frac{a_0}{-a_1-a_2 x_2^{(k-1)}}$$

We set $\varphi_1(x)=-\frac{a_1}{a_2}-\frac{a_0}{a_2 x}$, $\varphi_2(x)=\frac{a_0}{-a_1-a_2 x}$. Suppose x_1^*, x_2^* are roots of quadratic equation. Therefore $x_1^*=\varphi_1(x_1^*)$, $x_2^*=\varphi_1(x_2^*)$, $x_1^{(k+1)}=\varphi_1(x_1^{(k-1)})$, and $x_2^{(k+1)}=\varphi_1(x_2^{(k-1)})$.

By Lagrange's mean value theorem

$$x^*-x_1^{(k+1)}=\varphi_1(x_1^*)-\varphi_1(x_1^{(k-1)})=\varphi'_1(\xi)(x_1^*-x_1^{(k-1)})$$

Where ξ is interior point between x_1^* and $x_1^{(k-1)}$.

Then, $\varphi'_1(\xi)=\frac{x^*-x_1^{(k+1)}}{x_1^*-x_1^{(k-1)}}=\frac{x^*-x_1^{(k+1)}}{x_1^*-x_1^{(k)}}\cdot\frac{x^*-x_1^{(k)}}{x_1^*-x_1^{(k-1)}}$

When $k\to\infty$, $\left[\frac{x^*-x_1^{(k+1)}}{x_1^*-x_1^{(k)}}\right]^2\to|\varphi'_1(x^*)|$

For the x_2 case, the derivation process is similar. Therefore the condition of formula (5) is $\sqrt{|\varphi'_1(x)|}<1$ and $\sqrt{|\varphi'_2(x)|}<1$. That is $|\varphi'_1(x)|<1$ and $|\varphi'_2(x)|<1$,formula (5) converges.

Conclusion

For the roots of the algebraic equations of general iterative method, each iteration only to find a root. In this paper, the method can get n roots. and the method is simple, easy to program. In this paper, we discuss the convergence of the iterative method for the quadratic equation, and the convergence of the other conditions is further studied.

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