Vector Equilibrium Problem in Topological Vector Spaces

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Abstract: In this paper, the vector equilibrium problem in topological vecror spaces is discussed. By using the well-known Ky Fan section theorem, its existence theorems of solution are proved and the main results of reference [3][4] are generalized.

Equilibrium problem is 1994 year by mathematician Blum and Oetlli [6] the was explicitly presented for the first time. Because it contains variational inequalities, phase complement problem, fixed point problem, best approximation problem and the Nash equilibrium problem and optimization problem in economics are the special cases, so much attention is paid by many mathematics workers, the equilibrium problem of vector-valued function is a research problem which is more concerned at present. (See Research anthology [5])

1. Preliminary Knowledge

Definition 1.1 ([1]) SetX$\neq \phi$, τ is X the set family of subsets in if the following properties are met:

(I)X, $\phi \in \tau$;

(II) τ the collection of any number of subsets in the τ,namely: if $\forall \alpha \in I, G_\alpha \in \tau$,则 $\bigcup_{\alpha \in I} G_\alpha \in \tau$;

(III) τ the intersection of any finite subset of the τ , namely: if $G_1, G_2 \ldots, G_m \in \tau$, $\bigcap_{i=1}^{m} G_i \in \tau$;

is said τ is X a topology , which is called (X, τ) is a topological space.

Definition 1.2 ([1]) X is a number field R The linear space on the, τ is X a topology on the. if (X, τ) satisfies the following conditions:

(1)X The addition operation in the is continuous;

(2)X the number multiplication operation in the is continuous;

The(X, τ)is a topological linear space.

Definition 1.3 X is linear space, C$\subset$ X, said C for a cone ,if$\forall\ x \in$C, $\forall\ \lambda \geq 0$, Yes $\lambda x \in$C;

If C is a cone and is a convex set, it is said C is a convex cone; if C$\bigcap$(–C)={0} , Cone C the is the point cone.

Set P solid topological linear space Z a closed convex cone in, which intP$\neq \phi$; K solid topological linear space X Non-empty collection in; $f: K \times K \to Z$ the so-called vector equalization problem (denoted to VEP) is: Find $x \in K$,Meet (VEP) $f(x, y) \notin$–intP, $\forall y \in K$.

Definition 1.4 Set X is a topological linear space, $f: X \to R$,Said f at the point $x \in K$, on the upper half continuous, if $\overline{\lim_{y \to x}} f(y) \leq f(x)$; f in X top half Continuous, if f in X each point of the is half continuous; f at the point $x \in K$ for the next half continuous, if f at each point is the upper half continuous; f in X on the bottom half of the row, if f in X each point of the is the lower half continuous.

Special case: when $T: K \to L(X, Z)$, $\theta: K \times K \to X$ and $\eta: K \times K \to Z$ to a known mapping, take $f(x,y)=<Tx,\theta(y,x)>+\eta(x,y)$, which $L(X,Z)$ represents X to Z the space formed by the continuity operator of the, $<l,x>$expression operator $l\in L(X,Z)$ at the point $x\in X$ the value of the , the vector equalization problem above becomes a generalized variational inequality , that is: ask $x\in X$ meet (GVVI) $<Tx,\theta(y,x)>+\eta(x,y)\notin$ –int P, $\forall y\in K$.

Definition 1.5 A convex subset of set K to X , $f:K\to Z$ .If the $\forall\ x_1, x_2\in$ K,and $\lambda\in(0,1)$, $\lambda f(x_1)$ +(1– λ $f(x_2)\in f(\lambda x_1+(1-\lambda)x_2)$+P, called f function it's convex.

When Z=R, P=[0, +∞], f is the usual convex function.

Theorem 1.1 (Ky Fan intercept theorem) ([2]) Set K to Hausdorff Topological linear space X a non-empty compact convex set on the, and set L as K×K subset of the following properties:

(a) $\forall x\in K$, $(x,x)\in$ L;

(b) $\forall y\in K$, Collection L(y)={ $x\in K$: $(x,y)\in$ L} in K the is a closed set;

(c) $\forall x\in K$, Collection M(x)={ $y\in K$: $(x,y)\notin$ L} convex;

The $x_0\in$K, makes { x_0 } ×K⊂L.

2. Primary Results

Theorem 2.1 Set K to Hausdorff a non-empty compact subset on a topological linear space, P to Hausdorff topological linear space Z a closed convex cone in the, and int $P\neq\phi$. Set $f:K\times K\to Z$ meet the conditions:

(I) $\forall x\in K$, $f(x,x)\in P$;

(II) $\forall x\in K$, Mapping $y\to f(x,y)$ is convex;

(III) $\forall y\in K$, Collection $\{x\in K: f(x,y)\notin -\operatorname{int}P\}$ in K the is closed.

The $\exists\ x_0\in$K makes $f(x_0,y)\notin -\operatorname{int}P$, $\forall y\in K$.

Proof: Order $L=\{(x,y)\in K\times K: f(x,y)\notin -\operatorname{int}P\}$,by criteria (a), $\forall x\in K$, $f(x,x)\notin$ $-\operatorname{int}P$, $\because P\cap(-\operatorname{int}P)=\phi$, $\therefore f(x,x)\notin -\operatorname{int}P$.

So $(x,x)\in L$, L is an empty set.

Also by the conditions (c) , $\forall\ y\in K$, Collection $\{x\in K: f(x,y)\notin -\operatorname{int}P\}$ in K is closed,

The collection $L(y)=\{x\in K:(x,y)\in L\}$

$$=\{y\in K: f(x,y)\notin -\operatorname{int}P\}\ (\forall y\in K)$$ is a closed set.

$\forall x\in K$, the collection $M(x)=\{x\in K:(x,y)\notin L\}$

$$=\{y\in K: f(x,y)\in -\operatorname{int}P\}.$$

under the M(x) is a convex subset.

In fact, y_1, $y_2\in$M(x), $t\in(0,1)$ and $z=ty_1+(1-t)y_2$ by Criteria (b) , and note P for closed convex cone, available $f(x,z)\in tf(x,y_1)+(1-t)f(x,y_2)-P\in -\operatorname{int}P-P=-\operatorname{int}P,$

That $z\in$M(x), M(x)is convex. by Ky Fan intercept theorem, available: $\exists\ x_0\in$K makes{ x_0 } ×K⊂L,

That: $f(x_0,y)\notin$ –intP, $\forall y\in K$.

Theorem 2.2 Set K to Hausdorff a non-empty compact subset on a topological linear space, P to Hausdorff topological linear space Z a closed convex cone in the, and int $P\neq\phi$ .set $f:K\times K\to Z$ meet the conditions:

(a) $\forall x \in K$, $f(x,x) \in P$;

(b) $\forall x \in K$, mapping $y \to f(x,y)$ is convex;

$(c)'$: $\forall\ y \in K$, $f(x,y)$ about x continuous,

The $\exists x_0 \in K$ makes $f(x,y) \notin$ –intP , $\forall y \in K$.

Proof: by theorem 2.1 , we know that at this point, just by proving the criteria $(c)'$ available conditions (c), get the theorem 2.2 set up.

Conditions of proof below (c) established, $\forall y \in K$, collection in { $x \in K$: $f(x,y) \notin$ –int P} K is closed.

In fact, $\forall y \in K$, if the net{ x_α }⊂{ $x \in K$: $f(x,y) \notin -\operatorname{int} P$ }, and $x_\alpha \to x_0$, to permit $x_0 \in$ { $x \in$K: $f(x,y) \notin$ –intP }, cause $f(x,y)$ about x continuous , there are $f(x_{\alpha,}y) \to f(x_0,y)$.

∵ $f(x_{\alpha,}y) \in$ Z \ {–intP },, and Z \ {–intP } the is a closed set.

∴ $f(x_{\alpha,}y) \to f(x_0,y) \in$ Z \ {–intP }, that□ $f(x_0,y) \notin$ –int P . so $x_0 \in$ { $x \in$K: $f(x,y) \notin$ –intP }.

Inference 2.3 Set K to Hausdorff topological linear space X the last non-empty tight convex subset, P to Hausdorff topological linear space Z a closed convex cone in the, and int P $\neq \phi$. set map, $T: K \to L(X,Z)$, $\theta: K \times K \to X$, $\eta: K \times K \to Z$ meet the conditions:

(1) $\forall x \in K$,<Tx, $\theta(x,x)$>+$\eta(x,x)$ $\in$P;

(2) $\forall x \in K$,mapping y<Tx, $\theta(y,x)$>+$\eta(x,y)$ about $y \in K$ is convex.

(3) $\forall x \in K$,collection { $x \in K$:<Tx, $\theta(y,x)$>+$\eta(x,y) \notin$ –int P} in K is closed.

Then (GVVI) has a solution, namely: $\exists$ $x_0 \in$K makes <T x_0, $\theta(y,x_0)$ >+ $\eta(x_{0,}y) \notin$ –intP, $\forall y \in K$.

Proof: in theorem 2.1 , take $f(x,y)$=<Tx, $\theta(y,x)$>+$\eta(x,y)$, $\forall x,y \in K$. by the theorem 2.1 The is informed that the conclusion is correct.

Inference 2.4 Set K , X , P , Z , T , θ, η as inference 2.3 , and Conditions (a), (b) is established. also set the following conditions set up: $(c)'$ $\forall y \in K$, mapping x$\mapsto \langle Tx, \theta(y,x) \rangle$ + $\eta(x,y)$ continuous.

Then (GVVI) has a solution.

Proof: in the theorem 2.2 , fetching $f(x,y)$=<Tx, $\theta(y,x)$>+$\eta(x,y)$, $\forall x,y \in K$ .by theorem 2.2 the concludes.

3. Summary

Well known, equilibrium problem is a meaningful generalization of variational inequalities and complementary problems, the is an important research topic in nonlinear analysis. Due to many problems such as mechanics, cybernetics, differential equations, mathematical economy and optimization theory, it can be summed up as equilibrium problem, so the research of this paper has a wide application background.

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