Weighted Wavelet Estimate in Semiparametric Models with Heteroscedastic Errors

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***Abstract*—Using wavelet smoothing method, we consider the semiparametric regression model with independent heteroscedastic errors. We investigate the asymptotic normality and weak consistence rates of wavelet estimators.**

Keywords-Semiparametric regression model, Wavelet estimate, Asymptotic normality, Weak consistence rates, Heteroscedastic error

I. INTRODUCTION

Consider the following semiparametric regression model

$$y_i = x_i\beta + g(t_i) + \sigma_i e_i,\ i = 1,2,\cdots,n. \tag{1}$$

where $\{y_i\}$ are scalar response variables, $\beta \in R^1$ is an unknown parameter, $\sigma_i^2 = h(u_i)$, $g(\cdot)$ and $h(\cdot)$ are unknown functions on $[0,1]$, $\{t_i\}$ and $\{u_i\}$ are two nonrandom sequences on $[0,1]$, $\{x_i\}$ are nonrandom design points, and $\{e_i\}$ are independent identically distribution random variables with $Ee_1 = 0$ and $Ee_1^2 = 1$.

The model (1) has been extensively studied, and there exist many important results. Gao et al.[1] investigated the asymptotic normality of weighted least squares estimator of the parameter. When $t_i = u_i$,You and Chen[2] and Ran and Zhu[3] discussed the test of heteroscedastic errors in model (1). When (x_i, t_i, u_i) are random design points, Sun and Zhao[4] discussed the model (1) by the near neighbor method, and obtained asymptotic normality and weak consistence rates of estimators. When the errors are negatively associated (NA) random variables, Ren and Chen[5] investigated strong convergence of these estimators in [1], and discussed asymptotic normality and weak consistence rates of these estimators. In the paper, we give wavelet weighted estimators of parameter β, nonparameter $g(t)$ and variance function $h(u)$ by the wavelet smooth method and least squares method, and investigate some asymptotic properties. The organization of this paper is as follows: the weighted wavelet estimators and main results are given in section 2. The proofs of main results are presented in section 3.

II. THE MAIN RESULTS

Suppose that there exists a scaling function $\phi(x)$ in the Schwartz space S_l and a multiresolution analysis V_m in the concomitant Hilbert space $L^2(R)$, with its reproducing kernel $E_m(t,s)$ given by

$$E_m(t,s) = 2^m E_0(2^m t, 2^m s) = 2^m \sum_{k\in Z} \phi(2^m t - k)\phi(2^m s - k),$$

where Z denotes the collection of integers.

Let $A_i = [s_{i-1}, s_i]$ be intervals that partition [0, 1] with $t_i \in A_i$ and $1 \le i \le n$, and $\tilde{x}_i = x_i - \sum_{j=1}^{n}\int_{A_j} E_m(t_i,s)ds x_j$, $\tilde{y}_i = y_i - \sum_{j=1}^{n}\int_{A_j} E_m(t_i,s)ds y_j$, $T_n^2 = \sum_{i=1}^{n} h^{-1}(u_i)\tilde{x}_i^2$. By the wavelet estimate method (see [1]), we obtain three wavelet estimators given by

$$\hat{\beta}_n = T_n^{-2}\sum_{i=1}^{n} h^{-1}(u_i)\tilde{x}_i\tilde{y}_i \tag{2}$$

$$\hat{g}_n(t) = \hat{g}_0(t,\hat{\beta}_n) = \sum_{i=1}^{n}(y_i - x_i\hat{\beta}_n)\int_{A_i} E_m(t,s)ds \tag{3}$$

and

$$\hat{h}_n(u) = \sum_{i=1}^{n}(\tilde{y}_i - \tilde{x}_i\hat{\beta}_n)^2 \int_{A_i} E_m(u,s)ds \tag{4}$$

To obtain our results, the following conditions are sufficient.

(A1) There exists a bounded function $f(t), t\in[0,1]$ such that $x_i = f(t_i) + v_i$, $i = 1,2,\cdots,n$, where v_i satisfy $\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} v_i^2 = \Sigma_0\ (0 < \Sigma_0 < \infty)$, $\lim_{n\to\infty} n^{-1/2}\sum_{i=1}^{n} v_i = \Sigma_1$ $(|\Sigma_1| < \infty)$ and $\max_{1\le i\le n}|v_i| = O(n^{-1/2})$.

(A2) $0 < m_0 \le \min_{1\le i\le n} h(u_i) \le \max_{1\le i\le n} h(u_i) \le M_0 < \infty$.

(A3) $g(\cdot)$, $f(\cdot)$ and $h(\cdot)$ belong to the Sobolev space with order $\alpha > 1/2$.

(A4) $g(\cdot)$, $f(\cdot)$ and $h(\cdot)$ satisfy the Lipschitz condition with order $\gamma > 0$.

(A5) $\phi(\cdot)$ is in the Schwartz space with order $l \geq \alpha$, which satisfies the Lipschitz condition with order l and has a compact support. Furthermore, $\hat{\phi}(\xi) - 1 = O(\xi)$ as $\xi \to 0$, where $\hat{\phi}$ is the Fourier transform of ϕ.

(A6) $\max_{1\le i\le n}(s_i - s_{i-1}) = O(n^{-1})$ and $2^m = O(n^{1/3})$.

Remark 1. The conditions (A1) and (A2) are same as that ones in [1] and [6]. The condition (A4) is weaker than that one in [1]. The conditions (A3)-(A6) are basic conditions of wavelet estimation method (see [7-10]).

Theorem 1. Suppose that conditions (A1)-(A6) hold. Then for $\alpha > 3/2$ and $\gamma \ge 1/3$,

$$n^{1/2}\left(\hat{\beta}_n - \beta\right) \xrightarrow{D} N\left(0, \Sigma_0^{-1}\right), \quad n \to \infty.$$

Theorem 2. Suppose that conditions (A1)-(A6) hold. If $Ee_1^6 < \infty$, then for $\alpha > 3/2$ and $\gamma \ge 1/3$,

$$\sup_{0\le u\le 1}\left|\hat{h}_n(u) - h(u)\right| = O_p\left(n^{-1/3}\log n\right), \quad n \to \infty.$$

Theorem 3. Suppose that conditions (A1)-(A6) hold. Then for $\alpha > 3/2$ and $\gamma \ge 1/3$,

$$\sup_{0\le t\le 1}\left|\hat{g}_n(t) - g(t)\right| = O_p\left(n^{-1/3}\log n\right), \quad n \to \infty.$$

Theorem 4. Suppose that conditions (A1)-(A6) hold. Then for $\alpha > 3/2$ and $\gamma \ge 1/3$,

$$\frac{\hat{g}_n(t) - \bar{g}(t)}{\left(Var\left(\hat{g}_n(t)\right)\right)^{1/2}} \xrightarrow{D} N(0,1), \forall t \in [0,1], n \to \infty,$$

where

$$\bar{g}(t) = \sum_{i=1}^n \int_{A_i} E_m(t,s)\,ds\, g(t_i), Var\left(\hat{g}_n(t)\right) = \sum_{i=1}^n h(u_i)\left(\int_{A_i} E_m(t,s)\,ds\right)^2.$$

III. Proofs of Theorems

Throughout this paper, let C denote a generic positive constant which could take different value at each occurrence. To prove the main results, we first introduce some lemmas.

Lemma 1[9]. Let $\tau_m = 2^{-m(\alpha-1/2)}$ when $1/2 < \alpha < 3/2$, $\tau_m = \sqrt{m}2^{-m}$ when $\alpha = 3/2$, $\tau_m = 2^{-m}$ when $\alpha > 3/2$. If conditions (A2)-(A5) hold, then

$$\sup_t \left| h(t) - \sum_{k=1}^n \left(\int_{A_k} E_m(t,s)\,ds\right) h(t_k)\right| = O\left(n^{-\gamma}\right) + O(\tau_m)$$

and

$$\sup_t \left| g(t) - \sum_{k=1}^n \left(\int_{A_k} E_m(t,s)\,ds\right) g(t_k)\right| = O\left(n^{-\gamma}\right) + O(\tau_m).$$

Lemma 2[8]. If condition (A4) holds, then

(1) $\left|E_0(t,s)\right| \le C_k/(1+|t-s|)^k$ and $\left|E_m(t,s)\right| \le 2^m C_k/(1+2^m|t-s|)^k$ for $k \in N$, where C_k is a real constant depending only on k;

(2) $\sup_{0\le S\le 1}\left|E_m(t,s)\right| = O\left(2^m\right)$;

(3) $\sup_t \int_0^1 \left|E_m(t,s)\right| ds \le C$;

(4) $\int_0^1 E_m(t,s)\,ds \to 1, n \to \infty$.

Lemma 3. If conditions (A1)-(A6) hold, then $\lim_{n\to\infty} n^{-1}T_n^2 = \Sigma_2$, $n \to \infty$.

Proof. Let

$$\tilde{f}(t_i) = f(t_i) - \sum_{j=1}^n \int_{A_j} E_m(t_i,s)\,ds\, f(t_j),$$

$$\tilde{v}_i = v_i - \sum_{j=1}^n \int_{A_j} E_m(t_i,s)\,ds\, v_j.$$

Note that

$$n^{-1}T_n^2 = n^{-1}\sum_{i=1}^n h^{-1}(u_i)\tilde{f}^2(t_i) + 2n^{-1}\sum_{i=1}^n h^{-1}(u_i)\tilde{f}(t_i)\tilde{v}_i + n^{-1}\sum_{i=1}^n h^{-1}(u_i)\tilde{v}_i^2.$$

By Lemma 1, Lemma 2 and Abel inequality, the Lemma 3 follows from the condition (A1).

Lemma 4[11]. Let $\{\xi_i, i = 1,2,\cdots,n\}$ be a independent random sequence with $E(\xi_i) = 0$ and $Var(\xi_i) < \infty$. If $\sup_{1\le i\le n} E|\xi_i|^r \le C < \infty$ for some $r \ge 2$, $\sup_{1\le i,j\le n}|a_{ij}| = O\left(n^{-p_1}\right)(0 < p_1 < 1)$ and $\sum_{i=1}^n a_{ij} = O\left(n^{p_2}\right)\left(p_2 \ge \max\{0, 2/r - p_1\}\right)$, then

$$\max_{1\le j\le n}\left|\sum_{i=1}^n a_{ij}\xi_i\right| = O\left(n^{-s}\log n\right), \quad a.s. \quad s = (p_1 - p_2)/2.$$

Let $r = 3, p_1 = 2/3, p_2 = 0$, $a_{ij} = W_i(t_j)$. Then by Lemma 4, we have

$$\max_{1\le j\le n}\left|\sum_{i=1}^n W_i(t_j)\xi_i\right| = O\left(n^{-1/3}\log n\right), \quad a.s.. \tag{5}$$

Lemma 5. Let $\{\xi_i, i = 1,2,\cdots,n\}$ be a independent random sequence with $E(\xi_i) = 0$ and $Var(\xi_i) < \infty$. If $\sup_{1\le i\le n} E|\xi_i|^r \le C < \infty$ for some $r \ge 2$, and conditions (A2)-(A6) hold, then

$$\sup_{0\le t\le 1}\left|\sum_{i=1}^n \varepsilon_i \int_{A_i} E_m(t,s)\,ds\right| = O_p(n^{-1/3}\log n).$$

Proof. Write

$$\sup_{0\le t\le 1}\left|\sum_{i=1}^n \varepsilon_i \int_{A_i} E_m(t,s)\,ds\right|$$

$$\le \sup_{0\le t\le 1}\left|\sum_{i=1}^n \varepsilon_i \int_{A_i} \sum_{j=1}^n \left(E_m(t,s) - E_m(s_j,s)\right) I\left(s_{j-1} < t \le s_j\right) ds\right|$$

$$+\sup_{0\le t\le 1}\left|\sum_{i=1}^{n}\varepsilon_i\int_{A_i}\sum_{j=1}^{n}E_m\left(s_j,s\right)I\left(s_{j-1}<t\le s_j\right)ds\right|. \quad (6)$$

The result follows from the (5) (6) and the (2.10) in [8].

Proof of Theorem 1. Let $\varepsilon_i=\sigma_i e_i,\overline{\varepsilon}_i=\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m\left(t_i,s\right)ds$. By (2), we have

$$n^{1/2}\left(\hat{\beta}_n-\beta\right)=nT_n^{-2}\left(n^{-1/2}\sum_{i=1}^{n}h^{-1}\left(u_i\right)\tilde{x}_i\tilde{g}\left(t_i\right)+n^{-1/2}\sum_{i=1}^{n}h^{-1}\left(u_i\right)\tilde{x}_i\varepsilon_i-n^{-1/2}\sum_{i=1}^{n}h^{-1}\left(u_i\right)\tilde{x}_i\overline{\varepsilon}_i\right)$$

$$=nT_n^{-2}\left(I_1+I_2-I_3\right). \quad (7)$$

By conditions (A1) (A2), Lemma 1, Lemma 2 and Abel inequality, we obtain

$$\left|I_1\right|\le n^{1/2}\max_{1\le i\le n}\left|h^{-1}\left(u_i\right)\right|\max_{1\le i\le n}\left|\tilde{f}\left(t_i\right)\right|\max_{1\le i\le n}\left|\tilde{g}\left(t_i\right)\right|$$
$$+C\left(n^{-\gamma}+\tau_m\right)\left(C+n^{1/2}\max_{1\le i\le n}\int_0^1\left|E_m\left(t_i,s\right)\right|ds\max_{1\le j\le n}\left|v_j\right|\right)$$
$$=O\left(n^{-2\gamma+1/2}\right)+O\left(n^{-1/6}\right)+O\left(n^{-\gamma}+\tau_m\right)\to 0 \quad (8)$$

By Lemma 1 and Lemma 2, we obtain

$$E\left(I_3\right)^2=n^{-1}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}h^{-1}\left(u_i\right)\tilde{x}_i\int_{A_j}E_m\left(t_i,s\right)ds\right)^2E\varepsilon_j^2$$
$$\le Cn^{1/3}\left(n^{-\gamma}+\tau_m+n^{-1/2}+n^{-1/2}\right)=O\left(n^{-1/3}\right)\to 0 \quad (9)$$

Thus, by Chebyshev inequality, we have

$$I_3=O_p\left(n^{-1/6}\right)\to 0 \quad (10)$$

By Lemma 3 and $\max_{1\le i\le n}\left|\tilde{x}_i\right|<\infty$, we easily obtain

$$\sum_{i=1}^{n}E\left|a_ie_i\right|^{2+\delta}\Big/\left(\sum_{i=1}^{n}E\left(a_ie_i\right)^2\right)^{(2+\delta)/2}\to 0,\ n\to\infty \quad (11)$$

for some $\delta>0$, where $a_i=n^{-1/2}\Sigma_2^{-1/2}h^{-1}\left(u_i\right)\tilde{x}_i\sigma_i$. Thus, by central limit theorem, we obtain

$$\Sigma_2^{-1/2}I_2\to_D N\left(0,1\right),\ n\to\infty. \quad (12)$$

Hence the desired conclusion follows from the (7) (8) (10) and (12).

Proof of Theorem 2. By (4) and $\tilde{y}_i=\tilde{x}_i\beta+\tilde{g}\left(t_i\right)+\tilde{\varepsilon}_i$, we have

$$\hat{h}_n\left(u\right)=$$
$$n\left(\beta-\hat{\beta}_n\right)^2\cdot n^{-1}\sum_{i=1}^{n}\tilde{x}_i^2\int_{A_i}E_m\left(u,s\right)ds+\sum_{i=1}^{n}\tilde{g}^2\left(t_i\right)\int_{A_i}E_m\left(u,s\right)ds+$$
$$\sum_{i=1}^{n}\tilde{\varepsilon}_i^2\int_{A_i}E_m\left(u,s\right)ds+2n^{1/2}\left(\beta-\hat{\beta}_n\right)\cdot n^{-1/2}\sum_{i=1}^{n}\tilde{g}\left(t_i\right)\int_{A_i}E_m\left(u,s\right)ds+$$
$$2n^{1/2}\left(\beta-\hat{\beta}_n\right)\cdot n^{-1/2}+2\sum_{i=1}^{n}\tilde{g}\left(t_i\right)\tilde{\varepsilon}_i\int_{A_i}E_m\left(u,s\right)ds$$
$$\hat{=}n\left(\beta-\hat{\beta}_n\right)^2I_1+I_2+I_3+2n^{1/2}\left(\beta-\hat{\beta}_n\right)I_4+2n^{1/2}\left(\beta-\hat{\beta}_n\right)I_5+2I_6. \quad (13)$$

Note that $\lim_{n\to\infty}n^{-1}\sum_{i=1}^{n}\tilde{x}_i^2=\Sigma_0$, and by Lemma 2 and Lemma 3, we easily obtain

$$\sup_{0\le u\le 1}\left|I_i\right|=O\left(n^{-2/3}\right)\to 0(i=1,2),$$
$$\sup_{0\le u\le 1}\left|I_4\right|=O\left(n^{-1/2-\gamma}\right)+O\left(n^{-1/2}\tau_m\right)\to 0. \quad (14)$$

In the following, we shall prove

$$\sup_{0\le u\le 1}\left|I_3-h\left(u\right)\right|=O_p\left(n^{-1/3}\log n\right). \quad (15)$$

In fact, by Cauchy inequality and Lemma 2, we obtain that

$$\sup_{0\le u\le 1}\left|I_3-h\left(u\right)\right|$$
$$\le\sup_{0\le u\le 1}\left|\sum_{i=1}^{n}h\left(u_i\right)\left(e_i^2-1\right)\int_{A_i}E_m\left(u,s\right)ds\right|+\sup_{0\le u\le 1}\left|\sum_{i=1}^{n}h\left(u_i\right)\int_{A_i}E_m\left(u,s\right)ds-h\left(u\right)\right|$$
$$\max_{1\le i\le n}\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m\left(t_i,s\right)ds\right)^2+2\sup_{0\le u\le 1}\left|\left\{\sum_{i=1}^{n}h\left(u_i\right)\left(e_i^2-1\right)\int_{A_i}E_m\left(u,s\right)ds+\sum_{i=1}^{n}h\left(u_i\right)\int_{A_i}E_m\left(u,s\right)ds\right\}^{1/2}\right|$$
$$\cdot\sup_{0\le u\le 1}\left|\left\{\sum_{i=1}^{n}\int_{A_i}E_m\left(u,s\right)ds\left(\sum_{j=1}^{n}\varepsilon_j\int_{A_j}E_m\left(t_i,s\right)ds\right)^2\right\}^{1/2}\right|$$
$$\hat{=}\sup_{0\le u\le 1}\left|I_{31}\right|+\sup_{0\le u\le 1}\left|I_{32}\right|+\max_{1\le i\le n}\left|I_{33}\right|+2\sup_{0\le u\le 1}\left\{I_{31}+I_{34}\right\}^{1/2}\sup_{0\le u\le 1}\left\{\sum_{i=1}^{n}\int_{A_i}E_m\left(u,s\right)ds\left|I_{33}\right|\right\}^{1/2} \quad (16)$$

Let $\eta_i=h\left(u_i\right)\left(e_i^2-1\right)$, then by (5), we have

$$\sup_{0\le u\le 1}\left|I_{31}\right|=O_p\left(n^{-1/3}\log n\right). \quad (17)$$

By Lemma 1, Lemma 2 and Lemma 4, we obtain

$$\sup_{0\le u\le 1}\left|I_{32}\right|=O_p\left(n^{-1/3}\right),$$
$$\max_{1\le i\le n}\left|I_{33}\right|=O_p\left(n^{-2/3}\log^2 n\right),\ \left|I_{34}\right|\le C. \quad (18)$$

Hence, the (15) follows from (16)-(18). By Cauchy inequality and (14) (15), we obtain

$$\sup_{0\le u\le 1}\left|I_6\right|\le\sup_{0\le u\le 1}\left|I_2^{1/2}I_3^{1/2}\right|=O\left(n^{-1/3}\right)\to 0,$$
$$\sup_{0\le u\le 1}\left|I_5\right|\le\sup_{0\le u\le 1}\left|I_1^{1/2}I_2^{1/2}\right|=O\left(n^{-2/3}\right)\to 0 \quad (19)$$

Therefore, the desired conclusion follows from (13)-(15)(19) and Theorem 1.

Proof of Theorem 3. Note that

$$\sup_{0\le t\le 1}\left|\hat{g}_n\left(t\right)-g\left(t\right)\right|\le\sup_{0\le t\le 1}\left|\left(\beta-\hat{\beta}_n\right)\sum_{i=1}^{n}\tilde{x}_i\int_{A_i}E_m\left(t,s\right)ds\right|$$
$$+\sup_{0\le t\le 1}\left|\sum_{i=1}^{n}g\left(t_i\right)\int_{A_i}E_m\left(t,s\right)ds-g\left(t\right)\right|$$
$$+\sup_{0\le t\le 1}\left|\sum_{i=1}^{n}\varepsilon_i\int_{A_i}E_m\left(t,s\right)ds\right|\hat{=}n^{1/2}\left|\beta-\hat{\beta}_n\right|I_1+I_2+I_3. \quad (20)$$

By Lemma 2 and condition (A1), we have

$$I_1\le Cn^{-1/2}\sup_{0\le t\le 1}\left(\left|\sum_{i=1}^{n}f\left(t_i\right)\int_{A_i}E_m\left(t,s\right)ds\right|+\left|\sum_{i=1}^{n}v_i\int_{A_i}E_m\left(t,s\right)ds\right|\right)$$
$$\le Cn^{-1/2}\max_{1\le i\le n}\left|f\left(t_i\right)\right|\sup_{0\le t\le 1}\int_0^1\left|E_m\left(t,s\right)\right|ds+Cn^{-1/2}\max_{1\le i\le n}\left|v_i\right|\sup_{0\le t\le 1}\int_0^1\left|E_m\left(t,s\right)\right|ds$$
$$=O\left(n^{-1/2}\right). \quad (21)$$

By Theorem 1, (20) (21), Lemma 1 and Lemma 5, we complete the proof of Theorem 3.

Proof of Theorem 4. It is easily shown that $\left\{\sigma_i e_i \int_{A_i} E_m(t,s)\,ds, i=1,2,\cdots,n\right\}$ satisfy the Lindeberg condition. Therefore, the desired conclusion follows from the central limit theorem and Theorem 3.

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