# Some Improvement upon Algorithm IrrCharSer

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***Abstract*—Li presents the U-set to improve the algorithm CharSer that can compute a characteristic series from any given polynomial set. We modify the definition of U-set and present a new version of the algorithm IrrCharSer. Examples are given to show that the improvement can avoid the redundant branches of the decomposition tree.**

***Keywords -- polynomial set, U-set,zero decomposition, algorithm IrrCharSer***

## I. INTRODUCTION

The research on polynomial system is used in many modern science branches such as: algebraic geometry, automatic theorem discovery, proof and reasoning, isolation of solution of polynomial system, inequation proof, computer algebra, computer vision, automatic control system. Initial set is one of the most fundamental definition in this area. Based on initial set,Wu [4–7] presents some algorithms to decompose the polynomial set into ascending sets and other sets. Li [1,2] modifies the algorithm CharSer using U-set instead of initials and gets some wonderful improvements. We'll modify the definition of U-set and make some improvement on algorithm IrrCharSer which decompose the polynomial set into irreducible ascending sets.

In section I, we introduce some preliminaries of polynomial theory, improve the definition of U-set and give the algorithm of the modified U-set. The main results are put forward in section II, in which the algorithm IrrCharSer is optimized. Examples are given in section III to show that the advantages to the original algorithms.

Let $K$ be a field with characteristic 0, $K[x_1,\cdots,x_n]$ ( $K[X]$ for short) the ring of polynomials in the variables $(x_1,\cdots,x_n)$ ( $X$ for short) with coefficients in $K$ and $\widetilde{K}$ be the extension field of $K$. Let $T$ be a triangular set in $K[X]$ , $p\in K[X]\setminus K$ , then $\mathrm{res}(p,T)$ stands for the resultant of $p$ with respect to $T$.

For any n-dimension polynomial set $P$ in $K[X]$, and any polynomial system, $[P,Q]$, the set of all zeros in $\widetilde{K}^n$ are denoted respectively as the following:

$$Zero(P)=\{z\in\widetilde{K}^n : p(z)=0, \forall p\in P\}$$

$$Zero(P/Q)=\{z\in\widetilde{K}^n : p(z)=0, q(z)\neq 0, \forall p\in P, q\in Q\}$$

For any triangular set $T$ in $K[X]$, Li [1, 2] presents the definition of $U_T$. We modify it as the following:

Definition 1.1 The U-set for triangular set $T$ is

$$U_T=\{c:\mathrm{res}(c,T)=0,\ c\in\mathrm{ini}(T)\}\cup \{r: one\ r\in R_f,\ \mathrm{res}(\mathrm{ini}(f),T)\neq 0,\ R_f\cap K=\varnothing,\ f\in T\}$$

in which

$$R_f=\{\mathrm{res}(c,T)\neq 0 : c\in C_f\}\quad \forall f\in T$$

$C_f$ is the set of coefficients of $f$ and $\mathrm{ini}(T)$ is the set of initials of $T$.

It's easy to see that all the propositions in [1, 2] hold for the new definition of U-set. And the following algorithm is presented to compute the U-set.

**Algorithm** (CompA): $U_T\leftarrow CompA(T)$ .This algorithm computers the U-set of any given triangular set $T\in K[X]$.

**C1.Let** $U\leftarrow\varnothing, T^*\leftarrow T$;

**C2.** while $T^*\neq\varnothing$ do:

**C2.1.** Choose any one polynomial f from $T^*$, and set $T^*\leftarrow T^*\setminus\{f\}$;

**C2.2.** Let $r\leftarrow\mathrm{res}(\mathrm{ini}(f),T)$;

**C2.2.1.** if $r=0$ then $U\leftarrow U\cup\{\mathrm{ini}(f)\}$;

**C2.2.2.** if $r\neq 0$ and $C_f\cap K=\varnothing$ then let $R_f=\{res(c,T):\forall c\in C_f\}\setminus\{0\}$;

**C2.2.3.** if $R_f\cap K=\varnothing$ then choose any one element $R$ from $R_f$;

**C2.2.4.** Set $U\leftarrow U\cup\{R\}$;

**C3.** Let $U_T\leftarrow U$.

## II. MAIN RESULTS

The irreducible decomposition is one of the most classical algorithm to decompose the solution sets of polynomial set. It's presented by Wu [7]. One can find the details from some text book such as [3]. Using U-set to take the place of initials, we get the following algorithm IrrCharSerA according to the IrrCharSer algorithm in [3].

**Algorithm** (IrrCharSerA): $\Psi\leftarrow IrrCharSerA(P)$. Given a nonempty polynomial set $P\subset K[X]$ , this algorithm decomposes it into irreducible characteristic ascending sets with the same solution set.

**IA1.** Let $\Phi\leftarrow\{P\}, \Psi\leftarrow\varnothing$;

**IA2**. While $\Phi\neq\varnothing$ do:

**IA2.1.** Choose any one element $F$ from $\Phi$, and let $\Phi\leftarrow\Phi\setminus\{F\}$;

**IA2.2.** Compute $C \leftarrow \mathrm{CharSet}(F)$;

**IA2.3.** If $C$ is noncontradictory then

**IA2.3.1**. $[k, D, G] \leftarrow \mathrm{Factor}(C)$;

**IA2.3.2**. If $k = 0$ then

**IA2.3.2.1**. Compute $U_C \leftarrow \mathrm{CompA}(C)$.

**IA2.3.2.2**. Let $\Psi \leftarrow \Psi \cup \{C\}$, and

$$\Phi \leftarrow \Phi \cup \bigcup_{I \in U_C} \{F \cup C \cup \{I\}\};$$

**IA2.3.3.** If $k \neq 0$ then let

$$U_{C^{(k)}} \leftarrow \mathrm{CompA}\left(C^{(k)}\right)$$

and

$$\Phi \leftarrow \Phi \cup \bigcup_{T \in U_{C^{(k)}}} \{F \cup C \cup \{T\}\} \cup \bigcup_{g \in G} \{F \cup C \cup \{g\}\}$$

We can get that

$$Zero(P) = \bigcup_{C \in \Psi} Zero(C / U_C), \quad \Psi = \mathrm{IrrCharSerA}(P)$$

Because we just use the U-set instead of initial set, the termination and correctness of this algorithm are easily checked from the definition of U-set.

Here, the function $\mathrm{Factor}(C)$ [3] determines whether the triangular set is irreducible or not and factor the reducible elementary.

**Algorithm** (Factor): $[k, D, F] \leftarrow Factor(C)$. For any fine triangular polynomial set $C \subset K[X]$, this algorithm computes an integer $k$, a polynomial $D$ and a polynomial set $F \subset K[X]$, such that $\left|0 \leq k \leq |C|\right|$, and

I. If $k = 0$, then $C$ is irreducible;

II. If $k = 1$, then $C$ is reducible, $|F| > 1$. $C_1$, the first polynomial of class $p_1$ in $C$, can be factored into $C_1 = \prod_{f \in F} f$ over field $K_0 = K\left(x_1, \cdots, x_{p_1 - 1}\right)$. And every $f \in F \subset K_0[x_{p_1}]$ is irreducible in $K_0$;

III. If $k > 1$, then $C$ is reducible and $C^{\{k-1\}}$ is irreducible, $|F| > 1$. $C_k$, the *k*th polynomial in $C$ has a factorization $DC_k \doteq \prod_{f \in F} f$ over the field $K_{k-1}$ which is the extension field of $K$ with the adjoining triangular set $C^{\{k-1\}}$. Every $f \in F \subset K_{k-1}[x_{p_k}]$ is irreducible over $K_{k-1}$.

In III., we extend the base field $K$ into the extension field $K_{k-1}$ in a specific way : $K_{k-1} = K\left(x_1, \cdots, x_{p_k - 1}\right)$. For any $1 \leq j \leq k-1$, $x_{p_j}$, the leading variable of $C_j$, is considered as an algebraic element with adjoining polynomial $C_j$ while the other variables are adjoined as transcendental ones.

## III. APPLICATIONS

In this section, we will present two examples to compare the our algorithm with IrrCharSer in maple package *epsilon*.

**Example 1.** Let $P = \{p_1, p_2, p_3\}$, and

$$\begin{aligned} p_1 &= -x_1 - x_3 + x_4^2, \\ p_2 &= x_4 + x_1 x_4 - x_2 x_4 - x_4^2, \\ p_3 &= x_1^2 - x_1 x_4 - x_2 x_4 \end{aligned}$$

with the variable ordering $x_1 \prec x_2 \prec x_3 \prec x_4$ in the polynomial ring $\mathrm{Q}[x_1, x_2, x_3, x_4]$ with coefficient field $\mathrm{Q}$.

With algorithm IrrCharSerA, We get the decomposition of its solution set as the following:

$$Zero(\mathrm{P}) = Zero(\mathrm{C}_1) \cup Zero(\mathrm{C}_2) \cup Zero(\mathrm{C}_3 / \{x_1\})$$

in which

$$\mathrm{C}_1 = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} \mathrm{C}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 - 1 \\ -1 + x_4 \end{bmatrix} \mathrm{C}_3 = \begin{bmatrix} -x_2^2 + x_2 + x_1 \\ C_{32} \\ (-x_1 - x_2) x_4 + x_1^2 \end{bmatrix}$$

$$C_{32} = (-x_1^2 - x_1 - 2x_1 x_2 - x_2) x_3 + x_1^4 - x_1^3 - x_1^2 - 2x_1^2 x_2 - x_1 x_2$$

**Remark 1** Using the Maple software package *epsilon*, we can get the following decomposition:

$$Zero(\mathrm{P}) = Zero(\mathrm{C}_1) \cup Zero(\mathrm{C}_2) \\ \cup Zero(\mathrm{C}_3 / \{(-x_1 - x_2) x_1^2, -x_1^2 - 2x_1 x_2 - x_1 - x_2\})$$

in which $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_1$ are the same as above. The difference between the two decompositions is the U-set and the initials of $\mathrm{C}_3$. Using the algorithm CompA, one can compute the U-set of $C_3$

$$\mathrm{U}_{\mathrm{C}_3} = \{x_1\}$$

and

$$ini(\mathrm{C}_3) = \{(-x_1 - x_2) x_1^2, -x_1^2 - 2x_1 x_2 - x_1 - x_2\}.$$

Cause $Zero(\mathrm{C}_3 \cup \{x_1 + x_2\}) = \{x_1 = 0, x_2 = 0\}$, we have

$$Zero(\mathrm{C}_3 / \{x_1\}) \\ = Zero(\mathrm{C}_3 / \{(-x_1 - x_2) x_1^2, -x_1^2 - 2x_1 x_2 - x_1 - x_2\}).$$

So, we can find that the decomposition using U-set is more concise than the one using the initials.

**Example 2.** Given a polynomial set:

$$F = \{f_1, f_2, f_3\} \subset Q[x_1, x_2, x_3, x_4]$$

in which

$$\begin{aligned} f_1 &= x_1 - x_2 + x_3 + x_2^2, \\ f_2 &= -x_2 + x_1^2 + x_2^2 - x_3 x_4, \\ f_3 &= x_1 + x_1^2 + x_1 x_4 + x_4^2 \end{aligned}$$

The IrrCharSerA returns the following decomposition of solution set with the variables ordering $x_1 \prec x_2 \prec x_3 \prec x_4$:

$$Zero(\mathrm{F}) = Zero(\mathrm{C}_1) \cup Zero(\mathrm{C}_2) \\ \cup Zero(\mathrm{C}_3) \cup Zero(\mathrm{C}_4 / \{x_1, x_1 + 1, x_1 - 1\})$$

in which

$$C_1=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} C_2=\begin{bmatrix}x_1\\x_2-1\\x_3\\x_4\end{bmatrix} C_3=\begin{bmatrix}x_1-1\\x_2^2-x_2+1\\x_3\\x_4^2+x_4+2\end{bmatrix} C_4=\begin{bmatrix}C_{41}\\x_3+x_2^2-x_2+x_1\\(x_1-x_2+x_2^2)x_4-x_2+x_1^2+x_2^2\end{bmatrix}$$

$$C_{41}=(x_1^2+1)(x_2^4-2x_2^3)+(4x_1^2+x_1^3+1)x_2^2+(-3x_1^2-x_1^3)x_2+x_1^3+x_1^4$$

**Remark 2** Using the ics function in *epsilon* package, we get that

$$Zero(F)=Zero(C_1)\bigcup Zero(C_2)\bigcup Zero(C_3)$$
$$\bigcup Zero(C_4/\{x_1-x_2+x_2^2,x_1^2+1\})\bigcup Zero(C_5/\{x_1+3,x_1\})$$

in which $C_1,C_2,C_3,C_4$ are the same as above while

$$C_5=\begin{bmatrix}x_1^2+1,\\(x_1+3)x_2^2+(-3-x_1)x_2+x_1-1,\\(x_1+3)x_3+2x_1,\\x_1x_4-x_1-1\end{bmatrix}$$

It's easy to find that:

$$Zero(C_5/\{x_1+3,x_1\})\subset Zero(C_4/\{x_1,x_1+1,x_1-1\})\cdot$$

So we can see that U-set is useful to avoid the redundant branches of the decomposition tree.

**Example 3.** Let polynomial set $H=\{h_1,h_2,h_3\}$ be in the polynomial ring $Q[x_1,x_2,x_3,x_4]$ with coefficient field $Q$ and the variable ordering $x_1\prec x_2\prec x_3\prec x_4$.

$$h_1=-(x_1+1)x_4+x_3^2-x_1x_3-x_1+1,$$
$$h_2=-x_3x_1+x_2^2-x_2x_1+x_1^2-x_1+1,$$
$$h_3=(x_3-x_1)x_4-x_3x_2-x_2^2$$

Using the algorithm IrrCharSerA, one can decompose the polynomial set $H$ into the series of irreducible ascending sets $\{C_1,C_2,C_3,C_4\}$ in which

$$C_1=\begin{bmatrix}c_{11}\\-x_3x_1+x_2^2-x_2x_1+x_1^2-x_1+1\\c_{13}\end{bmatrix}$$

$$C_2=\begin{bmatrix}x_1+1\\x_2^4+2x_2^3+6x_2^2+5x_2+8\\x_3+x_2^2+x_2+3\\(x_2^2+x_2+2)x_4-x_2^3-3x_2\end{bmatrix}$$

$$C_3=\begin{bmatrix}x_1\\x_2^2+1\\x_3^2-x_3x_2-x_2\\x_4-x_3x_2-x_2-1\end{bmatrix} C_4=\begin{bmatrix}x_1\\x_2^2+1\\x_3+x_2\\x_4\end{bmatrix}$$

$$c_{11}=-x_2^6+3x_1x_2^5+(-4x_1^2+3x_1-3)x_2^4+(4x_1^3-5x_1^2$$
$$+6x_1)x_2^3+(-x_1^4+6x_1^3-9x_1^2+6x_1-3)x_2^2+(x_1^5$$
$$-3x_1^4+6x_1^3-5x_1^2+3x_1)x_2-2x_1^4+5x_1^3-5x_1^2$$
$$+3x_1-1$$

$$c_{13}=(x_1^3+x_1^2)x_4-x_2^4+2x_2^3x_1+(-2x_1^2+2x_1-2)x_2^2$$
$$(x_1^2-2x_1+2)x_2x_1+2x_1^3-3x_1^2+2x_1-1$$

And the zero set relationship is

$$Zero(H)=Zero(C_1/\{x_1+1,x_1\})$$
$$\bigcup Zero(C_2)\bigcup Zero(C_3)\bigcup Zero(C_4)$$

**Remark 3** Using the function *ics* in *epsilon* package, we can get that

$$Zero(H)=Zero(C_1/\{x_1+1,x_1\})\bigcup Zero(C_2/\{x_2^2+x_2+2\})$$
$$Zero(C_3)\bigcup Zero(C_4)$$

in which $C_1,C_2,C_3,C_4$ are the same as above. It's easy to see that

$$Zero(C_1)\subseteq Zero(H) \text{ and } Zero(C_2)\subseteq Zero(H)$$

**Example 4.** Given a polynomial set $P=\{p_1,p_2,p_3\}$ in the polynomial ring $Q[x_1,x_2,x_3,x_4]$, and

$$p_1=-2x_4x_1-x_3^2-2x_3x_1,$$
$$p_2=2x_4^2+2x_4+x_1,$$
$$p_3=2(1+x_2)x_3+x_2$$

one can decompose it into

$$Zero(P)=Zero(C_1/\{x_1\})\bigcup Zero(C_2)\bigcup Zero(C_3)$$

using the algorithm IrrCharSerA, in which

$$C_1=\begin{bmatrix}c_{11}\\2(1+x_2)x_3+x_2\\c_{13}\end{bmatrix} C_2=\begin{bmatrix}x_1\\x_2\\x_3\\1+x_4\end{bmatrix} C_3=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$c_{11}=(32x_1^3+48x_1^2-16x_1+1)x_2^4+8(16x_1^2+16x_1-3)x_2^3x_1$$
$$8(24x_1^2+14x_1-1)x_2^2x_1+32x_1^2(4x_1+1)x_2+32x_1^3$$
$$c_{13}=8(1+x_2)^2x_4x_1-x_2(x_2(4x_1-1)+4x_1)$$

**Remark 4.**Using function *ics*, polynomial set $P$ in *Example 4* can be decomposed into

$$Zero(P)=Zero(C_1/I_1)\bigcup Zero(C_2)\bigcup Zero(C_3)$$
$$\bigcup Zero(C_4/I_4)\bigcup Zero(C_5/I_5)$$

in which $C_1,C_2,C_3$ are as above and

$$I_1=\{x_2+1,32x_1^3+48x_1^2-16x_1+1,x_1\}$$
$$I_4=\{x_2+1,8x_1-1,x_1\},\ I_5=\{2x_1-1,x_1\}$$

$$C_4 = \begin{bmatrix} 32x_1^3 + 48x_1^2 - 16x_1 + 1 \\ (16x_1 - 2)x_2^2 + 2(14x_1 - 1)x_2 + 12x_1 - 1 \\ 2(x_2 + 1)x_3 + x_2 \\ 8x_4x_1 + 12x_1 - 1 \end{bmatrix}$$

$$C_5 = \begin{bmatrix} 32x_1^3 + 48x_1^2 - 16x_1 + 1 \\ 2(2x_1 - 1)x_2 + 4x_1 - 1 \\ 2x_3 + 4x_1 - 1 \\ 8x_1x_4 - 4x_1 + 1 \end{bmatrix}$$

In fact, there exists

$$Zero\ (C_1/\{x_1\}) = Zero\ (C_1/I_1) \cup Zero\ (C_4/I_4) \cup Zero\ (C_5/I_5)$$

The two examples above show that the decomposition using U-set is more concise than the one using the initials.

## IV. Conclusion

In this paper, we modify the definition of U-set and improve the algorithm IrrCharSer. The decomposition results of Example1,3 and 4 show that the modified algorithm can return the more concise result. The result of Example 2 show that it also can avoid the redundant branches of the decomposition tree. The U-set works more efficiently than initials. We'll modify the other algorithms with U-set in future.

## V. Acknowledgement

This work is partially supported by National Natural Science Foundation of China (61074189). The author would like to thank his colleagues for their advices and encouragement.

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