Practical Stability of Dynamic Systems with Time Delays in Terms of Two Measurements

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***Abstract*—The paper discusses practical stability and strict practical stability of differential equation on ring neighborhood by means of Lyapunov function. The paper develops $(h_0,h)-$ practical stability by employing two auxiliary functions and using comparative method. Some criteria and results which are used to guarantee practical stability for differential equation are given by the method of Lyapunov function, which enables us to obtain results under weaker assumptions.**

Keywords-practical stability, strict practical stability, delay differential equations, $(h_0,h)-$ practical stability, ring neighborhood, comparative theorem

I. INTRODUCTION

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Theory of stable in the sense of Lyapunov has been researched enough and can be widely used in concrete problems of the real world. But the stable domain or the domain of attractive is not large enough to allow the desired deviation to cancel out, thus a notion of stability is desired from practical considerations. S. Leela [1] first puts forward the definition of practical stable, and he gives a systematic study of the theory of practical stability and gives some sufficient conditions which guarantee the strict stability of the considered system. Mohapatra [2] advanced the concept of strict stable for different system .V.L.akshmilantham [3] develop the concepts of strict practical stable to delay differential equations by using Lyapunov function. Kobayashi [4] gives boundedness in differential equation with finite delay by using auxiliary function. In the paper we develop strict practical stable on ring neighborhood D_k. Significance of the paper is that dV/dt is variable number outside of D_k. We research $(h_0,h)-$practical stability of differential equation.

II. MAIN RESULTS

We consider the functional differential equation with finite delay

$$x' = f(t,x_t) \qquad (1)$$

$f: R_+ \times C_H \to R^n$ is completely continuous, C_H is a subset of the space of continuous functions $\varphi:[-h,0]\to R^n$,$\|\varphi\|=\sup\limits_{-h\le s\le 0}|\varphi(s)|$, $\varphi\in C_H$,and $0<H\le+\infty$, Let $x(t)=x(t,t_0,\varphi)$ denotes a solution of (1)with initial condition $x_{t_0}=\varphi$. We always assume the existence of solution $x(t,t_0,\varphi)$ or x_t of (1) for $t\ge t_0$, and $f(t,0)=0$.

Let U,V : $R_+\times C_H\to R_+$, $\gamma: R_+\to R_+$, and $g_\lambda: R\to R,\ 0<\lambda<H$ are continuous. Suppose there exists a $r_0>0$ such that $g_\lambda(r)>0$ for $0<r<r_0$, $g_\lambda(r)\ge\alpha_\lambda r$ for $r\ge r_0$, where $\alpha_\lambda>0$.

Theorem1. Suppose that

(1) $0\le V(t,\varphi)\le W_1(\|\varphi\|)$, there are β,λ,A, such that

$W_2(|\varphi(0)|)\le U(t,\varphi)\le W_3(|\varphi(0)|)$, $0\le\beta<\lambda<A$,

$V_{(1)}'(t,x_t)\le-\gamma_\beta(t)$, if $\beta\le\inf\limits_{t-h\le s\le t}|x(s)|<H$

$V_{(1)}'(t,x_t)\le-g_\lambda\left(U_{(1)}'(t,x_t)\right)$, if $\lambda\le|x(t)|<H$

where $U_{(1)}'(t,x_t)$ is finite and locally integrable or $U(t,x_t)$ is absolutely continuous for any solution of (1).

(2) There exists a constant $S>0$, such that $\int_t^{t+s}\gamma_\beta(s)\,ds>W_1(A)$ uniformly in $t\in R_+$,

(3)Let $L=\frac{1}{2}\min\left[\alpha_\lambda,\inf\limits_{\delta_1\le r\le r_0} g_\lambda(r)/r_0\right]$, such that

$W_1(\lambda)<L\left(W_2(A)-W_3(\lambda)\right)$

where $\delta_1=W_2(A)-W_3(\lambda)/2(h+S)$

Then the zero solution of (1) is uniformly practical stability with respect (λ, A).

Proof. For some $t_0 \in R$ and $\varphi \in C_H$ with $\|\varphi\| < \lambda$, let $x(t) = x(t, t_0, \varphi)$ be a solution of (1) on $(t_0, +\infty)$. We claim that $\|\varphi\| < \lambda$ implies $\|x(t)\| < A$ .If it is not true, there exists a solution $x_t(t_0, \varphi)$ with $\|\varphi\| < \lambda$ and t_1, t_2, such that $|x(t_1)| = A$, $x(t_2) = \lambda$, $|x(t)| < A$ on $[t_0, t_1)$. Since in view of (2), we have $t_1 - t_2 < h + S$, for otherwise

$$0 \le v(t_2+h+S) \le v(t_2+h) - \int_{t_2+h}^{t_2+h+S} \gamma_\lambda(s)\,ds < W_1(A) - W_1(A) = 0$$

This is impossible.

Put $Q_1 = \{t \in [t_2, t_1]; u'(t) \ge r_0\}$, $Q_2 = \{t \in [t_2, t_1]; \delta_1 \le u'(t) < r_0\}$ and $Q_3 = [t_2, t_1] \setminus (Q_1 \cup Q_1)$. Then

$$W_2(A) - W_3(\lambda) = W_2(|x(t_1)|) - W_3(|x(t_2)|) \le u(t_1) - u(t_2)$$

$$\le \int_{t_2}^{t_1} u'(t)dt = \int_{Q_1} u'(t)dt + \int_{Q_2} u'(t)dt + \int_{Q_3} u'(t)dt$$

$$\le \int_{Q_1} u'(t)dt + r_0 u(Q_2) + \delta_1(t_1 - t_2)$$

That is

$$\int_{Q_1} u'(t)dt + r_0 u(Q_2) \ge \frac{1}{2}(W_2(A) - W_3(\lambda))$$

where $u(Q_2)$ denotes measure of Q_2. Since $v'(t) \le 0$, it follows that

$$v(t_1) - v(t_2) \le \int_{t_1}^{t_2} v'(t)dt \le \int_{Q_1} v'(t)dt + \int_{Q_2} v'(t)dt$$

$$\le -\int_{Q_1} g_\lambda(u'(t))dt - \int_{Q_2} g_\lambda(u'(t))dt$$

$$\le -\alpha_\lambda \int_{Q_1} u'(t)dt - \inf_{\delta_1 \le r \le r_0} g_\lambda(r) u(Q_2)$$

Since $L = \frac{1}{2}\min\left[\alpha_\lambda, \inf_{\delta_1 \le r \le r_0} g_\lambda(r)/r_0\right]$, thus $v(t_1) - v(t_2) \le -L(W_2(A) - W_3(\lambda))$.

In view of (3), then we have

$$0 \le v(t_1) \le v(t_2) - L(W_2(A) - W_3(\lambda)) < v(t_2) - W_1(\lambda) < 0$$

That is a contradiction. Hence $|x(t)| < A$ on $[t_0, +\infty)$, This proves conclusion.

Let $V(t,\varphi), U(t,\varphi) \in C[R_+ \times C_H, R_+]$, $V(t,\varphi), U(t,\varphi)$ is locally lipschitz in φ, $\lambda_i(r):\ R_+ \to R_+, i = 1,2$,and $g_\gamma(r): R \to R$ are continuous. Suppose there exist constants α、$r_0 > 0$ such that $g_\gamma(r) > 0$ for $r \le r_0$, $g_\gamma(r) \ge \alpha r$ for $r \ge r_0$.

Let $w_i(s) \in K\ (i = 1,2,3)$.

Theorem 2 Assume that

(1)

$0 \le V(t,\varphi) \le w_1(\|\varphi\|)$,

$w_2(|\varphi(0)|) \le U(t,\varphi) \le w_3(|\varphi(0)|)$;

(2) $0 < \lambda < A$ is given, the functions $V(t, x(t)), \lambda_i(t) \quad (i = 1,2)$ are such that

$D^+V(t, u(t), \xi)\big|_{(1)} \le -\lambda_1(t)$ if $\lambda \le \inf_{t-h \le s < t} |x(t)| < A$

$D^+V(t, u(t), \xi)\big|_{(1)} \le -g_\lambda(U'_{(1)}(t, x_t, \xi)) + \lambda_2(t)$

If $\lambda \le |x(t)| < A$

Where $U'_{(1)}(t, x_t)$ is finite and locally integrable or $U(t, x_t)$ is absolutely continuous for any solution of equation (1).

(3) there exists a constant c such that

$\int_t^{t+h} \lambda_2(\xi)d\xi \le c$ uniformly $t \in R_+$.

(4) there exists a constant $S > 0$ such that

$\int_t^{t+Sh} \lambda_1(\xi)d\xi > c + 1 + w_1(A)$ and

$\alpha[w_2(A) - w_3(\lambda) - r_0(S+1)h] > (S+2)c + w_1(A)$

Then the solutions $x(t, t_0, \varphi)$ of system (1) is uniformly practical stability with respect to (λ, A, σ).

Proof. $V(t) = V(t, x(t+\xi, \tau_0, \phi) - x(t, t_0, \varphi))$ and $U(t) = U(t, y(t) - x(t))$ for any $\varphi \in C_H$, $x(t) = x(t, t_0, \varphi)$,and $\|\phi - \varphi\| < \lambda, |\tau_0 - t_0| < \sigma$. Choose a constant $M: 0 < M < A$ such that

$$\alpha[w_2(M) - w_3(\lambda) - r_0(S+1)h] > (S+2)c + w_1(A)$$

First, we shall show $V(t) < w_1(M)\ on\ [t_0, +\infty)$ for any $\|\phi - \varphi\| < \lambda, |\tau_0 - t_0| < \sigma$ .If this is not true, because of condition (1) of theorem 2, we have $V(t_0) \le w_1(\|\phi - \varphi\|) < w_1(\lambda) < w_1(M)$. So there is a $t_1 \ge t_0$ such that $w_1(M) = V(t_1) \le w_1(\|x_{t_1} - y_{t_1}\|)$ and

$V(t) < w_1(M)$ on $[t_0, t_1)$ .It follows $\|x_{t_1} - y_{t_1}\| \geq M$ .Which implies $|x(t_2) - y(t_2)| \geq M$ for $t_2 \in [t_1 - h, t_1]$ .From continuous of solutions of equation (1), there is a constant $t_3 \in [t_0, t_2)$ such that $|x(t_3) - y(t_3)| = \lambda$ and $\lambda < |x(t) - y(t)| < M \quad on\, [t_3, t_2]$ .Either (i) $t_2 > t_3 + (S+1)h$ or (ii) $t_2 \leq t_3 + (S+1)h$.

If (i) holds, then

$$w_1(M) = V(t_1) \leq V(t_2) + \int_{t_2}^{t_1} \lambda_2(t)dt \leq V(t_2) + c$$

$$\leq V(t_3 + h) + \int_{t_3+h}^{t_2} V'(t)dt + c$$

$$\leq V(t_3 + h) - \int_{t_3+h}^{t_2} \lambda_1(\xi)d\xi + c$$

$$\leq w_1(M) - \int_{t_3+h}^{t_2} \lambda_1(\xi)d\xi + c \leq w_1(M) - c - 1 + c < w_1(M)$$

It leads to a contradiction.

If (ii) holds, then

$$w_1(M) = V(t_1) = V(t_2) + \int_{t_2}^{t_1} V'(t)dt \leq V(t_2) + c$$

$$\leq V(t_3) + \int_{t_3}^{t_2} V'(t)dt + c$$

$$\leq V(t_3) - \int_{t_3}^{t_2} g(u'(t))dt + \int_{t_3}^{t_2} \lambda_2(\xi)d\xi + c$$

$$\leq w_1(M) - \int_{t_3}^{t_2} g(u'(t))dt + (S+2)c$$

Put $Q = \{t \,|U'(t) > r_0, t \in [t_3, t_2]\}$ .Then

$$w_2(M) - w_3(\lambda) \leq u(t_2) - u(t_3) = \int_{t_2}^{t_3} U'(t)dt$$

$$\leq \int_Q U'(t)dt + r_0(t_2 - t_3) \leq \int_Q U'(t)dt + r_0(S+1)h$$

or

$$\int_Q U'(t)dt \geq w_2(M) - w_3(\lambda) - r_0(S+1)h$$

From condition（4）of theorem 2 we obtain

$$w_1(M) = V(t_1) \leq w_1(M) - \alpha\int_{t_3}^{t_2} U'(t)dt + (S+2)c$$

$$\leq w_1(M) - \alpha[w_2(M) - w_3(\lambda) - r_0(S+1)h] + (S+2)c$$

$$< w_1(M) - (S+2)c + (S+2)c = w_1(M)$$

It leads to a contradiction.

It follows $V(t) < w_1(M)\; on\, [t_0, +\infty)$.

We claim that $\|\phi - \varphi\| < \lambda$,$|\xi| < \sigma$

Implies $|x(t) - y(t)| < A$, $t \geq t_0$ .If the claim is not true, there exists $t_4 \geq t_0$ such that $|x(t_4) - y(t_4)| = A$ and $|x(t) - y(t)| < A$ on $[t_0, t_4)$ .Then due to $\|\phi - \varphi\| < \lambda$, we can find $t_5 \in [t_0, t_4)$, such that $|x(t_5) - y(t_5)| = \lambda$ and $\lambda < |x(t) - y(t)| < A$ on (t_5, t_4) .Then

$$V(t_4) \leq V(t_5) + \int_{t_5}^{t_4 - h} V'(t)dt + \int_{t_4 - h}^{t_4} V'(t)dt$$

$$\leq V(t_5) + \int_{t_5}^{t_4 - h} V'(t)dt + c$$

If $t_4 > t_5 + (S+1)h$, we have

$$V(t_4) \leq w_1(M) - \int_{t_5}^{t_5 + Sh} \lambda_1(\xi)dt + c$$

$$< w_1(M) - [c + 1 + w_1(A)] + c < 0$$

This contradicts to condition (1) of theorem 2.

If $t_4 \leq t_5 + (S+1)h$, we have

$$V(t_4) = V(t_5) + \int_{t_5}^{t_4} V'(t)dt$$

$$\leq V(t_5) - \int_{t_5}^{t_4} g(u'(t))dt + \int_{t_5}^{t_5+(S+1)h} \lambda_2(\xi)d\xi$$

$$\leq w_1(M) - \alpha[w_2(A) - w_3(\lambda) - r_0(S+1)h] + (S+1)c$$

$$\leq w_1(M) - (S+2)c + (S+1)c - w_1(A) < 0$$

This contradicts to condition (1) of theorem 2. The contradiction obtained proves that solutions of system（1）are uniformly practical stability with respect to (λ, A) .The proof is complete.

We consider the differential system

$$x' = f(t, x), x(t_0) = x_0, t_0 \in R_+ \qquad (2)$$

where $f \in C\left[R_+ \times R^n, R^n\right]$.

The scalar differential equation

$$u' = g(t, u), u(t_0) = u_0 \geq 0 \qquad (3)$$

Where $g \in C\left[{R_+}^2, R_+\right]$ and $g(t, u)$ is quasimonotone nondecreasing in u .

Theorem3. Assume that

(i) $0 < \lambda < A$;

(ii) $h_0, h \in \Gamma$ and h_0 is finer than h $i.e. h(t,x) \leq \Phi(h_0(t,x)), \Phi \in K$

Whenever $h_0(t,x) \leq \lambda$;

$\Gamma = \{h \in C\left[R_+ \times R^n, R_+\right] : \inf_{(t,x)} h(t,x) = 0\}$

(iii) $V \in C\left[R_+ \times R^n, R_+\right]$, $V(t,x)$ is locally Lipschitzian in x and satisfies

① $b(h(t,x)) \le V(t,x) \le a(h(t,x))$ if $\lambda < h(t,x) < A$

② $D^+V(t,x) \le g(t,V(t,x)), (t,x) \in S(h,A)$, where $g \in C\left[R_+^2, R\right]$ and $S(h,A) = \{(t,x) \in R_+ \times R^n : \lambda_1 < h(t,x) < A\}$, $\lambda_1 < \lambda$.

(iv) $\Phi(\lambda) < A$ and $a(\lambda) < b(A)$ hold.

Then the practically stable properties of (3) imply the corresponding $(h_0, h)-$practically stable properties of the system (2).

Proof. Suppose that the equation (3) is practically stable with respect to $(a(\lambda), b(A))$,so that we have $u_0 < a(\lambda)$ implies $u(t,t_0,u_0) < b(A)$ $t \ge t_0$ .Then we claim that the system (2) is $(h_0, h)-$practically stable with respect to (λ, A). If this is not true, then there would exist $t_2 > t_1 > t_0$ and a solution $x(t) = x(t,t_0,x_0)$ such that

$$h_0(t_0,x_0) < \lambda, h(t_1,x(t_1)) = \lambda_1, h(t_2,x(t_2)) = A \quad , \quad t_1 < t < t_2 .$$

Since in view of conditions (ii) and (iv), we have $h(t_0,x_0) \le \Phi(h_0(t_0,x_0)) < \Phi(\lambda) < A$

Hence, In view of (iv) and ②, we get

$$V(t,x(t)) \le r(t,t_1,V(t_1,x(t_1))),\ t_1 \le t \le t_2 .$$

where $r(t,t_1,u_0)$ is the maximal solution of (3) , $r(t_1,t_1,u_0) = u_0 = V(t_1,x(t_1))$.

$$b(A) = \lim_{t \to t_2} b(h(t,x(t))) = b(h(t_2,x(t_2))) \le V(t_2,x(t_2))$$

$$\le r(t_2,t_1,V(t_1,x(t_1))) \le r(t_2,t_1,a(\lambda_1)) < b(A).$$

That is a contradiction.This proves $(h_0,h)-$ practical stability properties of the system (2).

Theorem4. Suppose that

(1) There exists sequence of positive number $\{r_k\}$ and constants λ, A,

$$0 < \lambda < A \quad ,$$

$$l = \inf_{\|x\|=A} V(t,x) > 0 ,\ \lim_{k\to\infty} r_k = l ,\ p = \sup_{\|x\|=\lambda} V(t,x) .$$

$$b(|x(t)|) \le V(t,x(t)),\quad b(r) \in K .$$

(2) Let $D_k = \{x \mid 0 \le r_k - \eta_k \le V(t,x(t)) \le r_k\}$ $(k = 1,2,...)$ such that

$$D^+V \le 0,\ |x(t)| \ge \lambda\ ,\ x \in D_k$$

(3) For any $0 < \lambda_1 < \lambda$, there exists sequence of positive number $\{\xi_k\}$ and constant A_2 , such that $A_2 < \lambda_1$, $\lim_{k\to\infty} \xi_k = m$, $m = \sup_{\|x\|=A_2} V(t,x) > 0$.

$$b_1(|x(t)|) \le V_1(t,x(t)) \tag{4}$$

Let $D_k' = \{x \mid \xi_k \le V_1(t,x(t)) \le \xi_k + l_k\}$ $(k = 1,2,...)$ such that

$$D\_V_1 \ge 0, |x(t)| < \lambda_1\ ,\ \ x \in D_k'$$

Then the system (1) is strict practical stable.

Proof. Since in view of (1) $\lim_{k\to\infty} r_k = l$, there exists t_1, t_2 , $|x(t_1)| = \lambda$, $|x(t_2)| = A$. Since $P \le r_k \le l$, thus there exists K and $t_3, t_4 \in [t_1, t_2]$, such that

$$V(t_3,x(t_3)) = r_K - \eta_K ,V(t_4,x(t_4)) = r_K$$

So

$$V(t_3,x(t_3)) < V(t_4,x(t_4)) \tag{5}$$

On the other hand, $r_K - \eta_K \le V(t,x) \le r_K$, because of condition (2), So $V(t,x)$ is nonincrease, $V(t_4,x(t_4)) \le V(t_3,x(t_3))$. This is contradicting to (5). Thus the solution is practical stability.

In view of (3), $\lim_{k\to\infty} \xi_k = m$ there exists t_5, t_6 , $|x(t_5)| = \lambda_1, |x(t_6)| = A_2$.

There exists K' , $t_7, t_8 \in [t_5, t_6]$ such that, $t_7 < t_8$, $V_1(t_7,x(t_7)) = \xi_{K'} + l_{K'}$,

$V_1(t_8,x(t_8)) = \xi_{K'}$, So

$$V_1(t_8,x(t_8)) < V_1(t_7,x(t_7)) \tag{6}$$

On the other hand, since in view of (3), we have $V_1(t_7,x(t_7)) \le V_1(t_8,x(t_8))$, this is contradicted to (6), so the solution is strict practical stability.

REFERENCES

[1] S Leela and A. A. Martynyuk Practical Stable of Nonlinear systems [M], World Scientific Publishing Co.Pet.Ltd, 1990.

[2] V.Lakshmikantham ; R. N. Mohapatra, Strict stable of differential equations, Nonlinear Analysis [M], 1989.

[3] V. Lakshmikantham, Strict practical stable of delay differential equation, Applied Mathematics and computation [J], 2001.

[4] KatsumasaKobayashi, Stable and Boundedness in functional differential equations with finite delay [J], 2002.

[5] LIAO Xiao-xin. Mathematical theory and applications of stability [M], HuaZhong Normal University Press, 2001.

[6] ZHENG Da-zhong.. Linear System Theory [M], Beijing: Tsinghua University Press, 1992.