A Global Optimization Algorithm for Sum of Quadratic Ratios Problem with Coefficients

Ji Ying
School of Management , Academy of Fundamental and Interdisciplinary Science
Harbin Institute of Technology
Harbin,China

Li Yijun
School of Management
Harbin Institute of Technology
Harbin,China

***Abstract*—In this paper a global optimization algorithm for solving sum of quadratic ratios problem with coefficients and nonconvex quadratic function constraints (NSP) is proposed. First, the problem NSP is converted into an equivalent sum of linear ratios problem with nonconvex quadratic constraints (LSP). Using linearization technique, the linearization relaxation of LSP is obtained. The whole problem is then solvable using the branch and bound method. In the algorithm, lower bounds are derived by solving a sequence of linear lower bounding functions for the objective function and the constraint functions of the problem NSP over the feasible region. The proposed algorithm is convergent to the global minimum through the successive refinement of the solutions of a series of linear programming problems. The numerical examples demonstrate that the proposed algorithm can easily be applied to solve problem NSP.**

Keywords- Quadratic Ratios Problem, quadratic constraints problem, linearization relaxation, branch and bound, global convergence

I. INTRODUCTION

In this paper, we consider the global optimization of the sum of quadratic ratios problem with coefficients and nonconvex quadratic function constraints of the following form,

$$NSP(X):\begin{cases}\min f(x)=\sum_{j=1}^{p} f_j(x)=\sum_{j=1}^{p} v_j\dfrac{n_j(x)}{d_j(x)}\\ s.t.\quad g_m(x)\le 0,\quad m=1,\dots,M\\ \qquad X=\{x:\underline{x}_n\le x_n\le \overline{x}_n, n=1,\dots,M\}\end{cases}$$

where

$$n_j(x)=c_j+b_j^T x+\frac{1}{2}x^T A_j x$$

$$d_j(x)=r_j+e_j^T x+\frac{1}{2}x^T D_j x$$

$$g_m(x)=h_m+w_m^T x+\frac{1}{2}x^T G_m x$$

and c_j , r_j and h_m are all arbitrary real number, $b_j,e_j,w_m\in R^N$, and $A_j,D_j,G_m\in R^{N\times N}$ are symmetric not positive semidefinite matrixes, $j=1,\dots,p$, $m=1,\dots,M$. So the constraint set is nonconvex. v_j are real constant coefficients, $j=1,\dots,p$.

The purpose of this paper is to introduce a new global algorithm to solve the problem NSP . The algorithm works by solving a sum of linear ratios problem with coefficients and nonconvex constraints LSP that is equivalent to problem NSP . The main feature of this algorithm is summarized as follows. Firstly, at each iteration a linear programming problem is solved and the optimal objective value of this linear programming problem provides a lower bound on the optimal objective value of the original fractional programming problem NSP . Then the proposed linear programming method for problem NSP is more convenient in the computation than the parametric programming (or concave minimization) methods of [1], thus any effective linear programming algorithm can be used to solve this nonlinear programming problem NSP . Secondly, the given method can solve general NSP problem, but sum of linear ratios problem [2] and other methods reviewed above (see Refs. [1], for example) can only treat special cases of problem NSP . Our method is also different form Qu's method [3], since their method is based on Lagrangian relaxation. Thirdly, the main computation involves solving a sequence of linear programming problems, for which standard simplex algorithm are available. Finally, numerical computation shows that the proposed method is superior to the method in [3].

II. LINEAR RELAXATION

Assumption 1 Without loss of generality, let $v_j>0$ $(j=1,\dots,T)$, $v_j<0$ $(j=T+1,\dots,p)$. For each $j=1,\dots,p$, it holds that there exist positive scalars l_j,u_j,L_j and U_j satisfy $0<l_j\le n_j(x)\le L_j$ and $0<u_j\le d_j(x)\le U_j$ for all x belonging to the feasible region.

Introducing positive variables t_j and s_j, and setting $t_j = n_j(x)$ and $s_j = d_j(x)$, the problem NSP then leads to the following equivalent problem LSP:

$$LSP(X,\Omega):\begin{cases}\min & f(x)=\sum_{j=1}^{p} f_j(x)=\sum_{j=1}^{T} c_j\frac{t_j}{s_j}+\sum_{j=T+1}^{p} c_j\frac{t_j}{s_j}\\ s.t. & t_j-n_j(x)\le 0, j=1,\dots,T,\\ & -t_j+n_j(x)\le 0, j=T+1,\dots,p,\\ & d_j(x)-s_j\le 0, j=1,\dots,T,\\ & -d_j(x)+s_j\le 0, j=T+1,\dots,p,\\ & g_m(x)\le 0, m=1,\dots,M,\\ & X=\{x:\underline{x}_i\le x_i\le \overline{x}_i, i=1,\dots,N\}\\ & \Omega=\{(t,s)\in R^{2p}:0<l_j\le t_j\le L_j, 0<u_j\le s_j\le U_j\}\end{cases}$$

Given any $Y=[\underline{y},\overline{y}]\subset Y^0\subset R^{2p+N}$, for any convex function $\varphi(y)$ defined in Y which is at least subdifferential, we have the following property: For any $z\in Y$, the sub-differential set $\partial\varphi(z)$ of $\varphi(y)$ at z is, $\partial\varphi(z)=\{v:\varphi(y)\ge\varphi(z)+<v,y-z>\}$.

Theorem 1 For any $z\in Y$, we have $\varphi(y)\ge\varphi(z)+<v,y-z>,\forall y\in Y$, where $v\in\partial\varphi(z)$.

All the details of the linearization technique for generating relaxations will be given in the following Theorem 4. Given any $Y=[\underline{y},\overline{y}]\subset Y^0\subset R^{2p+N}$, $z=(z_n)_{(2p+N)\times 1}\in Y$ and $\forall y=(y_n)_{(2p+N)\times 1}\in Y$, the following notations are introduced:

$$Lf_0(y)=\sum_{j=1}^{p} v_j y_{N+j}\min\{\frac{1}{u_j},\frac{1}{U_j}\}=\sum_{j=1}^{T} v_j y_{N+j}\frac{1}{U_j}+\sum_{j=T+1}^{p} v_j y_{N+j}\frac{1}{u_j}$$

$$Uf_0(y)=\sum_{j=1}^{p} v_j y_{N+j}\max\{\frac{1}{u_j},\frac{1}{U_j}\}=\sum_{j=1}^{T} v_j y_{N+j}\frac{1}{u_j}+\sum_{j=T+1}^{p} v_j y_{N+j}\frac{1}{U_j}$$

$$Lf_m(y,z)=Lf_m^1(y,z)-\frac{\lambda_m}{2}Uq(y)$$

$$Uf_m(y,\overline{z})=f_m^1(y)-\frac{\lambda_m}{2}Lq(y,\overline{z})$$

$$Lf_m^1(y,z)=f_m^1(z)+<\nabla f_m^1(z),y-z>,$$

$$f_m^1(y)=\frac{\lambda_m}{2}\|y\|^2+f_m(y), m=1,\dots,2p+M,$$

$$Lq(y,\overline{z})=2<\overline{z},y>-\|z\|^2,$$

$$Uq(y)=(\underline{y}+\overline{y})^T y-\underline{y}^T\overline{y}$$

where λ_m is defined as (1) such that $\lambda_m I+H_m$ is positive definite, $z,\overline{z}\in Y$ are fixed vectors and functions $Lf_m^1(y,z)$ and $Uf_m(y,\overline{z})$ have the argument y and depend on parameters z and $\overline{z}$, respectively.

Theorem 2 Given $z,\overline{z}\in Y\subset Y^0$, consider the functions $f_0(y),Lf_0(y),Uf_0(y)$, $f_m(y),Lf_m(y,z),\ Uf_m(y,\overline{z})$ for any $y\in Y$, where $m=1,\dots,2p+M$. Then the following two statements are valid.

(i) The functions $Lf_0(y),Uf_0(y)$, and $Lf_m(y,z)$ are all linear functions about y and $Uf_m(y,\overline{z})$ is a convex function about y. Moreover the functions $f_0(y),Lf_0(y),Uf_0(y)$, $f_m(y),Lf_m(y,z),\ Uf_m(y,\overline{z})$ satisfy:

$$Lf_0(y)\le f_0(y)\le Uf_0(y) \quad (2)$$

$$Lf_m(y,z)\le f_m(y)\le\ Uf_m(y,\overline{z}), m=1,\dots,2p+M \quad (3)$$

(ii) The maximal errors of bounding $f_0(y)$ using $Lf_0(y)$ and $Uf_0(y)$, and bounding $f_m(y)$ using $Lf_m(y,z)$ and $Uf_m(y,\overline{z})$, $m=1,\dots,2p+M$, satisfy

$$\lim_{\|\overline{y}-\underline{y}\|\to 0} LE_{\max}^0=\lim_{\|\overline{y}-\underline{y}\|\to 0} UE_{\max}^0=0 \quad (4)$$

$$\lim_{\|\overline{y}-\underline{y}\|\to 0} LE_{\max}^m=\lim_{\|\overline{y}-\underline{y}\|\to 0} UE_{\max}^m=0 \quad (5)$$

where

$$LE_{\max}^0=\max_{y\in Y} f_0(y)-Lf_0(y), UE_{\max}^0=\max_{y\in Y} Uf_0(y)-f_0(y),$$

$$LE_{\max}^m=\max_{y\in Y} f_m(y)-Lf_m(y,z), UE_{\max}^m=\max_{y\in Y} Uf_m(y,\overline{z})-f_m(y).$$

Next by means of Theorem 4, we can give the linear relaxation of the problem LSP. Let $Y^k=[\underline{y}^k,\overline{y}^k]\subset Y^0$, consequently we construct the the corresponding approximation relaxation linear programming (LP) of problem LSP in Y^k as follows:

$$LP(Y^k):\begin{cases}\min & Lf_0(y)\\ s.t. & Lf_m(y,z^k)\le 0, m=1,\dots,2p+M,\\ & y\in Y^k\end{cases} \quad (6)$$

where $z^k=(z_n^k)_{(2p+N)\times 1}\in Y^k$ is one constant vector, in our computation we choose $z_n^k=\frac{\underline{y}_n^k+\overline{y}_n^k}{2}$, $n=1,\dots,2p+N$.

Theorem 3 The linear programming $LP(Y^k)$ provides the lower bound of the optimal value of the problem QFP over the rectangle Y^k.

III. THE ALGORITHM AND GLOBAL CONVERGENCE

ALGORITHM LBBF

Step 0: Initialization

0.1: Give a sufficient small positive number ε and set $k:=0$, $\Re_k=\{1\}$, $q(k)=1$, $Y^{q(k)}=Y^q=Y^0$. Set an initial upper bound $\Theta^*=\infty$.

0.2: Solving the $LP(Y^{q(k)})$, denote the optimal solution and minimum value $(y^{q(k)},LBV_{q(k)})$, then $\Theta_k = f_0(y^{q(k)})$. Set the initial lower bound $LBV(k) = LBV_{q(k)}$ and the initial feasible point $y^k = y^{q(k)}$.

0.3: If $\Theta_k - LBV(k) \le \varepsilon$, then stop with $y^{q(k)}$ as the approximal global solution to problem LSP.

Step 1: (Partitioning step) According to the branching rule stated in 3.1, we choose a branching variable y_l to partition $Y^{q(k)}$ into two subrectangles $Y^{q(k,1)}$ and $Y^{q(k,2)}$. Replace $q(k)$ by these two new node indices $q(k,1)$ and $q(k,2)$ in $\Re_k$.

Step 2: (Feasibility check) For each new node indices $q(k,s)$ where $s = 1,2$, the corresponding rectangle $Y^{q(k,s)}, (s = 1,2)$, compute the lower bounds $\bar{L}f_0 = \min_{y \in Y^{q(k,s)}} Lf_0(y)$ and $\bar{L}f_m = \min_{y \in Y^{q(k,s)}} Lf_m(y, z^k)$ of $Lf_0(y)$ and $Lf_m(y, z^k)$ over the rectangle $Y^{q(k,s)}$ respectively where $m = 1,...,2p+M, s = 1,2$. If there exists $m = 0,1,...,2p+M$ such that one of the lower bounds satisfies $\bar{L}f_0 > LBV(k)$ or $\bar{L}f_m > 0$, for some $m = 1,...,2p+M$, then the corresponding node indices $q(k,s)$ will be removed. If $q(k,s), s = 1,2$ have all been removed then return to Step 4.

Step 3: (Updating upper bound and deleting step) For the remaining subrectangle update the corresponding parameters. Solve the programming $LP(Y^{q(k,s)})$, where $s = 1$ or $s = 2$ or $s = 1,2$, and denote the optimal solutions and optimal values $(y^{q(k,s)}, LBV_{q(k,s)})$. Then if possible update the best available upper bound $\Theta_k = \min\{\Theta_k, f_0(y^{q(k,s)})\}$. If $LBV_{q(k,s)} > \Theta_k$, then remove the corresponding node.

Step 4: (Convergence test) Fathom any nonimproving nodes by setting $\Re_{k+1} = \Re_k - \{q \in \Re_k : LBV_q \ge \Theta_k - \varepsilon\}$. If $\Re_{k+1} = \varnothing$, then terminate with Θ_k as the optimal value and $y^*(\tau)$ (where $\tau \in \Omega$) as the global solution, where $\Omega = \{\tau : f_0(y^*(\tau)) = \Theta_k\}$, Otherwise, $k = k+1$.

Step 5: Set the lower bound $LBV(k) = \min\{LBV_q : q \in \Re_k\}$, then select an active node $q(k) \in \arg\min\{LBV_q : q \in \Re_k\}$, let $y^k = y^{q(k)}$ and go to step 1.

Theorem 4 (convergence of algorithm LBBF)

(i) If the above algorithm terminates finitely at iteration k, then y^k is the globaloptimal solution to problem LSP;

(ii) Otherwise the algorithm generates an infinite sequence of iteration such that along any infinite branch of the branch and bound tree, any accumulation point of the sequence $\{y^k\}$ will be the global optimal solution of the problem LSP, and Θ_k is nonincreasing while $LBV(k)$ is nondecreasing, moreover they satisfy $\lim_{k\to\infty} \Theta_k = \lim_{k\to\infty} LBV(k) = f^*$ where f^* stands for the optimal value of problem LSP.

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Email address : jiying_1981@126.com

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