Stability of Strong Tracking Filter with Markovian Packet Losses

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Abstract-**We consider the stability of strong tracking filter (STF) in the networked control system. First, we study the stability of strong tracking filter for systems with colored observation noises using augmentation and a theorem of bounded stochastic functions. Then by analyzing the error covariance of STF in different network environment, we prove the boundedness of the peak error covariance.**

Keywords -Strong tracking filter, Nonlinear system, Colored noises, Networked control system, Peak error covariance.

I. INTRODUCTION

In the recent years, networked control systems have gained more and more attention from both the communication filed and control filed. It has very broad application from navigation to environment monitoring. The reason for this phenomenon is due to several advantages of the system. It can increase the speed and agility of the system, reduce the system wiring and monitor the system easily. However, it also has some disadvantages such as the packet delayed and dropped which affect the accuracy of the transmitted information. This makes it necessary to calculate the peak error covariance.

Many researchers study the error covariance of the linear time invariant system in the networked control system. [1] studied error covariance from the expectation angel and concluded the bound of the packet arrival rate, below which, the expectation of error covariance would go infinity. [2] considered the issue in a probabilistic perspective Pr[Pk<M] and extended the result of [1]. [3] concluded a necessary and sufficient condition for the stability of discrete-time networked control system using pole placement.

Most studies are about the linear networked control system using Kalman filtering. However, in reality, many systems are nonlinear. Moreover, strong tracking filter is superior to Kalman filter in many aspects, for example it can adapt the abrupt changes and track the system better, it has self-accommodation to the inaccuracy of the model as well as parameters. The contribution of this paper is that we put forward nonlinear issue in the networked control system using strong tracking filtering.

II. PROBLEM FORMULATION

We consider the following nonlinear system with colored noises

$$\begin{cases} x_{k+1} = f(k,u_k,x_k) + \Gamma_k v_k \\ y_{k+1} = h(k+1,x_{k+1}) + B_{k+1}e_{k+1} + D_{k+1}\xi_{k+1} \end{cases}$$

where v_k, ξ_k are Gaussian white noises, and e_k is colored observation noise

$$\begin{cases} E(v_k) = 0, \mathrm{cov}(v_k,v_j) = R_k\delta_{k,j} \\ E(\xi_k) = 0, \mathrm{cov}(\xi_k,\xi_j) = S_k\delta_{k,j} \\ \mathrm{cov}(v_k,\xi_j) = 0 \end{cases}.$$

We decompose the nonlinear function $f(k,u_k,x_k), h(k+1,x_{k+1})$ by Taylor expansion.

$$f(k,u_k,x_k) = f(k,u_k,\hat{x}_k) + F_k(x_k - \hat{x}_k) + \varphi(x_k,\hat{x}_k,k)$$

$$h(k,x_k) = h(k,\hat{x}_k) + H_k(x_k - \hat{x}_k) + \chi(x_k,\hat{x}_k,k)$$

The nonlinear system can be simplified as linear time-variant system

$$\begin{cases} x_{k+1} = F_k x_k + t_k + \Gamma_k v_k \\ y_k = H_k x_k + B_k e_k + D_k \xi_k + m_k \end{cases}$$

$$t_k = f(k,u_k,x_k) - F_k \hat{x}_k + \varphi(x_k,\hat{x}_k,k), m_k = h(k,x_k) - H_k \hat{x}_k + \chi(x_k,\hat{x}_k,k)$$

To remove the colored noise, we use augmentation to simplify the system. Here we define

$$\begin{cases} \alpha_{k+1} = \bar{A}\alpha_k + \bar{C}\varepsilon_k \\ e_k = \bar{B}\alpha_k \end{cases}, E(\varepsilon_k) = 0$$

$$Z_k = [x_k^T, \alpha_k^T]^T, w_k = [v_k^T, \varepsilon_k^T], T_k = \begin{bmatrix} t_k \\ 0 \end{bmatrix}, A_k = \begin{bmatrix} F_k & 0 \\ 0 & \bar{A} \end{bmatrix}, L_k = \begin{bmatrix} \Gamma_k & 0 \\ 0 & \bar{C} \end{bmatrix}, E_k = [H_k, B_k\bar{B}]$$

Then we get the augmented state and observation equations:

$$\begin{cases} Z_{k+1} = A_k Z_k + L_k w_k + T_k \\ y_k = E_k Z_k + D_k \xi_k + m_k \end{cases}$$

According to Zhou D.H in [4], the algorithm of strong tracking filter is as follows:

$$\begin{cases} P_k = [I - K_k E_k]P_k^- \\ P_{k+1}^- = \lambda_{k+1} A_K P_k A_k^T + Q_k \\ K_k = P_k^- E_k [E_k P_k^- E_k^T + S_k]^{-1} \end{cases}$$

where $\lambda_{k+1} = \begin{cases} \lambda_0, \lambda_0 \geq 1 \\ 1, \lambda_0 < 1 \end{cases}$, and

$$\begin{cases} \lambda_0 = tr[N_{k+1}/M_{k+1}] \\ N_{k+1} = \bar{V}_{k+1} - E_{k+1}Q_k E_{k+1}^T - S_{k+1}, M_{k+1} = E_{k+1}A_k P_k A_k^T E_k^T \\ \bar{V}_{k+1} = E_{k+1}P_{k+1}^- E_{k+1}^T + S_{k+1} \end{cases}.$$

Then we can get the following recursive algorithm of error covariance:

$$P_{k+1} = \lambda_{k+1}A_k P_k A_k^T - \lambda_{k+1}A_k P_k E_k^T(E_k P_k E_k^T + S_k)E_k P_k A_k^T + Q_k$$

III. MAIN RESULTS

When the packet arrival rate $\gamma_t = 1$, we have the bound of the error covariance matrix of strong tracking filter for the nonlinear system with colored observation noises.

Condition A1:

$$\|A_k\| \le \overline{a}, \|L_k\| \le \overline{L}, \|E_k\| \le \overline{e}, \|D_k\| \le \overline{d}, \|Q_k\| \le \overline{q}, \underline{p}I \le P_k \le \overline{p}I, S_k \ge \underline{s}I$$

Theorem 1

Under condition A1, $P_k \le d_k^{(1)}P_0 + d_k^{(2)}$, as well as $P_{k+1} < P_k$, that is to say the error covariance matrix is decreasing.

To prove the theorem, we introduce 2 lemmas.

Lemma 1: Assume there is a stochastic process $V_n(\varsigma_n)$ as well as real numbers $\overline{v}$, $\underline{v}$, $\mu > 0$ and $0 < \alpha \le 1$ such that

$$\underline{v}\|\varsigma_n\|^2 \le V_n(\varsigma_n) \le \overline{v}\|\varsigma_n\|^2 \quad , \quad \text{and}$$
$$E\{V_{n+1}(\varsigma_{n+1}) \mid \varsigma_n\} - V_n(\varsigma_n) \le \mu - \alpha V_n(\varsigma_n)$$

Then the stochastic process is exponentially bounded in mean square. We have

$$E\left\{\|\varsigma_n\|^2\right\} \le \frac{\overline{v}}{\underline{v}}E\left\{\|\varsigma_0\|^2\right\}(1-\alpha)^n + \frac{\mu}{\underline{v}}\sum_{i=1}^{n-1}(1-\alpha)^i$$

Moreover, the stochastic process is bounded with probability one. [5]

Lemma 2: There exits a real number $0 < \alpha \le 1$, such that $\Pi_n = P_n^{-1}$ satisfies the inequality

$$\left(A_k - K_k E_k\right)^T \Pi_{n+1}\left(A_k - K_k E_k\right) \le (1-\alpha)\Pi_n$$

The proof is similar as that in [5]

$$P_{k+1} = \lambda_{k+1}(A_k - K_k E_k)P_k(A_k - K_k E_k)^T + Q_k$$
$$+ \lambda_{k+1}K_k E_k P_k(A_k - K_k E_k)^T$$

As

$$\lambda_{k+1}K_k E_k P_k$$
$$= \lambda_{k+1}A_k[A_k^- K_k E_k][A_k^-(A_k - K_k E_k)P_k]^T A_k^T$$

We can study the two parts $A_k^- K_k E_k$,

$A_k^-(A_k - K_k E_k)P_k$ respectively.

$$[A_k^-(A_k - K_k E_k)P_k]$$
$$= P_k - P_k E_k^T(E_k P_k E_k^T + R_k)^{-1}E_k P_k$$

From the matrix inversion lemma,

$$[A_k^-(A_k - K_k E_k)P_k]$$
$$= (P_k^- + E_k^T R_k^- E_k)^{-1} > 0$$
$$A_k^- K_k E_k = P_k E_k^T(E_k P_k E_k^T + R_k)^{-1}E_k > 0$$

Meanwhile, from Zhou., we conclude

$$\lambda_{k+1} = trace(N_{k+1} / M_{k+1}) > 0$$

So,

$$\lambda_{k+1}K_k E_k P_k$$
$$= \lambda_{k+1}A_k[A_k^- K_k E_k][A_k^-(A_k - K_k E_k)P_k]^T A_k^T > 0$$

Next, we come to the equation:

$$P_{k+1} = \lambda_{k+1}(A_k - K_k E_k)P_k(A_k - K_k E_k)^T + Q_k$$
$$+ \lambda_{k+1}K_k E_k P_k(A_k - K_k E_k)^T$$
$$\ge \lambda_{k+1}(A_k - K_k E_k)P_k(A_k - K_k E_k)^T + Q_k$$
$$= \lambda_{k+1}(A_k - K_k E_k)*$$
$$[P_k + \frac{1}{\lambda_{k+1}}(A_k - K_k E_k)^{-1}Q_k(A_k - K_k E_k)^{-T}](A_k - K_k E_k)^T$$

then inversing the two side of the upper equation, we can get the following relationship:

$$(A_k - K_k E_k)^T \Pi_{k+1}(A_k - K_k E_k)$$
$$\le \frac{\Pi_k}{\lambda_{k+1}}[1 + \frac{1}{\lambda_{k+1}}(A_k - K_k E_k)^{-1}Q_k(A_k - K_k E_k)^{-T}\Pi_k]^{-1}$$

Which completes the proof

$$1 - \alpha = \frac{1}{\lambda_{k+1}}\left[1 + \frac{1}{\lambda_{k+1}}\frac{\overline{q}}{\underline{p}\left(\overline{a} + \frac{\overline{ape}^2}{\underline{s}}\right)^2}\right]^{-1}$$

Now we come to the state and observation equation as follows:

$$\begin{cases} Z_{k+1} = A_k Z_k + L_k w_k + T_k \\ y_k = E_k Z_k + D_k \xi_k + m_k \end{cases}$$

Based on the Kalman filtering, we can get

$$\hat{Z}_{k+1} = A_k \hat{Z}_k + T_k + K_k(y_k - E_k \hat{Z}_k - m_k)$$

As $\Delta_k = z_k - \hat{z}_k$, we can yield

$$\Delta_{k+1} = (A_k - K_k E_k)\Delta_k + L_k w_k - K_k D_k \xi_k$$

We define $V_n(\varsigma_n) = \varsigma_n^T \Pi_n \varsigma_n$

$$V_{n+1}(\varsigma_{n+1}) = \Delta_k^T(A_k - K_k E_k)^T \Pi_{n+1}(A_k - K_k E_k)\Delta_k + w_k^T L_k^T \Pi_{n+1}[2(A_k - K_k E_k)\Delta_k]$$
$$-2\xi_k^T D_k^T K_k^T \Pi_{n+1}L_k w_k - \xi_k^T D_k^T K_k^T \Pi_{n+1}[2(A_k - K_k E_k)\Delta_k] + w_k^T L_k^T \Pi_{n+1}L_k w_k + \xi_k^T D_k^T K_k^T \Pi_{n+1}K_k D_k \xi_k$$

As ξ_k is independent of Π_{n+1}, Δ_k, so the expectation of $\xi_k^T D_k^T K_k^T \Pi_{n+1}L_k w_k$ and

$\xi_k^T D_k^T K_k^T \Pi_{n+1}\left[2(A_k - K_k E_k)\Delta_k\right]$ is zero.

We assume $E(w_k w_k^T) = wI, tr(L_k L_k^T) \le \Upsilon$,

Then $E(w_k^T L_k^T \Pi_{n+1}L_k w_k) \le \dfrac{w\Upsilon}{\underline{p}}$.

Similarly, we assume $E\left(\xi_k \xi_k^T\right) = \xi I, tr\left(K_k D_k (K_k D_k)^T\right) \le m$,

then $E\left(\xi_k^T D_k^T K_k^T \Pi_{n+1} K_k D_k \xi_k\right) \le \dfrac{\xi m}{\underline{p}}$.

According to lemma 1, we have the following conclusion

$$P_k \le \frac{\overline{p}}{\underline{p}}\|P_0\|(1-\alpha)^k + \overline{p}\left(\frac{w\Upsilon}{\underline{p}} + \frac{\xi m}{\underline{p}}\right)\sum_{i=1}^{k-1}(1-\alpha)^i .$$

We can simplify the inequality as

$$P_k \le A(1-\alpha)^k + B\sum_{i=1}^{k-1}(1-\alpha)^i \ (A>0, B>0) .$$

Accordingly,

$$P_{k+1} - P_k \le -A\alpha(1-\alpha)^k - B\alpha\sum_{i=1}^{k-1}(1-\alpha)^i , P_{k+1} \le P_k + C_0 .$$

which completes the proof of theorem 1.

In the networked control system, we consider the packet arrival rate $\gamma_t = 1$ if we get the packet successfully and $\gamma_t = 0$ if the packet is lost.

$\gamma_t = 1, P_{k+1} = \lambda_{k+1}A_k P_k A_k^T - \lambda_{k+1}A_k P_k E_k^T(E_k P_k E_k^T + S_k)E_k P_k A_k^T + Q_k$

$\gamma_t = 0, P_{k+1} = A_k P_k A_k^T + Q_k$

According to [6], the author defines a_k as the kth packet loss after successive packet arrivals and b_k as the kth packet arrival after successive packet losses.

$$a_i^* = a_i - b_{i-1}, b_i^* = b_i - a_i$$
$$P(a_i^* - 1 = k) = (1-p)^k p, P(b_i^* - 1 = k) = (1-q)^k q$$

([6], lemma2)

Now we calculate the bound of the peak error covariance matrix

$$P_{b_k}\left(P_{b_k} = P_{b_k|b_{k-1}}\right) .$$

Theorem 2

We assume condition

$A2: \left|\overline{a}^2(1-q)\right| < 1$, under which we can draw the conclusion that the peak error covariance matrix is bounded.

Proof:

At first, we define

$F_K(P_K) = P_{k+1} = \lambda_{k+1}A_k P_k A_k^T - \lambda_{k+1}A_k P_k E_k^T(E_k P_k E_k^T + S_k)E_k P_k A_k^T + Q_k$

$\phi(i,j) = F_i\left[...F_{j+1}\left[F_j(P)\right]\right], G(i,j) = A_i...A_j$

Similar in [6], we first prove

$$E\left[\left\|P_{b_{k+1}+1}\right\|\,\Big|\,P_{b_k+1} = P\right]$$

$$= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left\|F_{b_k+i+j}\left[\begin{array}{l}G(b_k+i+j-1,b_k+i)\phi(b_k+i-1,b_k+1)G^T(b_k+i+j-1,b_k+i)\\+...A_{b_k+i+j-1}Q_{b_k+i+j-2}A_{b_k+i+j-1}^T+Q_{b_k+i+j-1}\end{array}\right]\right\|$$

$$\times(1-p)^{i-1}p(1-q)^{j-1}q = \Lambda(P)$$

According to theorem 1, $P_{k+1} \le P_k + C_0$, we can conclude

$$\Lambda(P) \le \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left\|\begin{array}{l}G(b_k+i+j-1,b_k+i)\phi(b_k+i-1,b_k+1)G^T(b_k+i+j-1,b_k+i)\\+...A_{b_k+i+j-1}Q_{b_k+i+j-2}A_{b_k+i+j-1}^T+Q_{b_k+i+j-1}\end{array}\right\|$$

$$\times(1-p)^{i-1}p(1-q)^{j-1}q + C$$

$$\le\left(\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left\|\left[G(b_k+i+j-1,b_k+i)\phi(b_k+i-1,b_k+1)G^T(b_k+i+j-1,b_k+i)\right]\right\|\right.$$

$$\left.+\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left\|\begin{array}{l}G(b_k+i+j-1,b_k+i+1)Q_{b_k+i}G^T(b_k+i+j-1,b_k+i+1)+...\\A_{b_k+i+j-1}Q_{b_k+i+j-2}A_{b_k+i+j-1}^T+Q_{b_k+i+j-1}\end{array}\right\|\right)$$

$$\times(1-p)^{i-1}p(1-q)^{j-1}q + C = \Lambda_1 + \Lambda_2 + C .$$

(1) Under condition A1, we can infer:

$$\left\|G(b_k+i+j-1,b_k+i)\right\| \le \overline{a}^j ,$$

According to theorem 1, we can infer:

$$\phi(b_k+i-1,b_k+1) \le \alpha\|P\| + \beta$$

Therefore $\Lambda_1 \le \sum_{j=1}^{\infty}\overline{a}^{2j}\left(\alpha\|P\|+\beta\right)(1-q)^{j-1}q$

When $\left|\overline{a}^2(1-q)\right| < 1$, we have $\Lambda_1 < \infty$.

(2)

$\Lambda_2 \le \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left(\overline{a}^{2(j-1)}\left\|Q_{b_k+i}\right\|+...\overline{a}^2\left\|Q_{b_k+i+j-2}\right\|+\left\|Q_{b_k+i+j-1}\right\|\right)\times(1-p)^{i-1}p(1-q)^{j-1}q$.

As $\left\|Q_k\right\| \le \overline{q}$, we can

get $\Lambda_2 \le \sum_{j=1}^{\infty}\left(\sum_{j=0}^{\infty}\overline{a}^{2j}\right)\times(1-q)^{j-1}q\overline{q}$

When $\left|\overline{a}^2(1-q)\right| < 1$, we have $\Lambda_2 < \infty$.

From above, we infer $E\left[\left\|P_{b_{k+1}+1}\right\|\,\Big|\,P_{b_k+1} = P\right] \le \delta\|P\| + C$

Next, in the same way we can caculate

$$E\left[\left\|P_{b_{k+1}}\right\|\,\Big|\,P_{b_k+1} = P\right] \le \sum_{j=1}^{\infty}\overline{a}^{2j}\left(\alpha\|P\|+\beta\right)(1-q)^{j-1}q + O(1)$$

When $\left|\overline{a}^2(1-q)\right| < 1$, we

infer $E\left[\left\|P_{b_{k+1}}\right\|\,\Big|\,P_{b_k+1} = P\right] < \infty$

IV. CONCLUSION

In this paper, we prove the bound of peak error covariance matrix of STF in the networked control system, and give the necessary and sufficient condition which guarantees the boundedness. For future work, we will compare the kalamn filter with strong tracking filter and it is of interest to find the upper and lower bound of the error covariance matrix.

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