Mean Values of Fuzzy Numbers with Evaluation Measures and the Measurement of Fuzziness

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Abstract

This paper discusses an evaluation method of fuzzy numbers as mean values and measurement of fuzziness defined by fuzzy measures, and the presented method is also applicable to fuzzy random variables and fuzzy stochastic processes in decision making modeling. We compare the measurement of fuzziness and the variance as criteria to measure uncertainty.

Keyword: Mean value, fuzzy number, fuzzy measure, measurement of fuzziness, possibility measure, necessity measure

1. Introduction

Estimation of uncertain quantities is one of the important topics in decision making. In dynamical systems such as option pricing in financial engineering (Yoshida [9, 10]), we need to pay attention to criteria of objective functions with uncertainty since the evaluation is related to the precision and reliability of decisions. In this paper we focus on evaluation of fuzzy quantities as uncertainty. The criterion should not be adhoc and it should be established from some reasonable and theoretical viewpoint. We give the valuation of fuzzy quantities by a kind of mean values, and further we discuss a criterion to measure the fuzziness in decision making problems.

In decision making problems, the most popular methods to evaluate fuzzy quantities are the defuzzification and ordering of fuzzy quantities, and many authors have examined the defuzzification method for fuzzy numbers in various applications ([8],[3],[6],[1],[4]). From the viewpoint of measure theory, Campos and Munoz [1] gave the following type evaluation of fuzzy numbers:

$$\int_0^1 h(\alpha)\,\mathrm{d}m(\alpha), \tag{1}$$

where the function $h(\alpha)$ is an estimation of the α-cut of the fuzzy numbers and m is a probability measure. López-Díaz and Gil [5] studied this type of evaluation in a general form with randomness. When we use the defuzzification methods like (1) in decision making modeling, it is needed to discuss the meaning of the measure m on $[0,1]$ and to give its reasonable construction. In decision making with fuzzy numbers, the meaning of criteria is important and we discuss it from the viewpoint of measure theory. To introduce the mean values of fuzzy numbers, we need to demonstrate the following three items:

(Q.i) How do we represent the values which fuzzy numbers $\tilde{a}$ take?

(Q.ii) How do we define the measure induced from the fuzzy numbers $\tilde{a}$?

(Q.iii) How should we give the mean value of the fuzzy numbers $\tilde{a}$ by the measure?

In this paper, we estimate fuzzy numbers by fuzzy measures, which are called *evaluation measures*, and the results are given by mean values and measurement of fuzziness. Especially we focus on the estimation methods with the possibility measure and the necessity measure for its numerical computation in modeling. This method is also applicable to fuzzy random variables and fuzzy stochastic processes. Next we compare the measurement of fuzziness and the variance as citeria to measure uncertainty.

2. Mean values of fuzzy numbers

By using fuzzy measures, we present a method to estimate fuzzy numbers. Campos and Munoz [1] studied an evaluation of fuzzy numbers in the form (1). In decision making with fuzzy numbers, we discuss the meaning of the estimation from the viewpoint of measure theory, and then fuzzy measures are used to evaluate a confidence degree that a fuzzy number takes values in an interval. Let $\mathbb{R}$ be the set of all real numbers and let $\mathcal{I}$ denote the set of all bounded closed intervals.

Definition ([7]). *A map $g : \mathcal{I} \mapsto \mathbb{R}$ is called a* mean function *on $\mathcal{I}$ if g satisfies the following (g.i), (g.ii) and (g.iii):*

(g.i) $g(I) \in I$ *for* $I \in \mathcal{I}$*;*

(g.ii) $g(I_1) \leq g(I_2)$ *for* $I_1, I_2 \in \mathcal{I}$ *satisfying* $I_1 \preceq I_2$*, where $\preceq$ means the fuzzy max order;*

(g.iii) $\lim_{n\to\infty} g(I_n) = g(I)$ *for* $\{I_n\}_{n=0}^{\infty} \subset \mathcal{I}, I \in \mathcal{I}$ *satisfying* $\lim_{n\to\infty} I_n = I$ *in the sense of Hausdorf metric.*

In this paper, the fuzziness is evaluated by λ-mean functions and evaluation measures. Let $g : \mathcal{I} \mapsto \mathbb{R}$ be a map such that

$$g([x, y]) := \lambda x + (1 - \lambda)y, \quad [x, y] \in \mathcal{I}, \tag{2}$$

where λ is a constant satisfying $0 \leq \lambda \leq 1$. This scalarization is used for the estimation of fuzzy numbers to give a mean value of the interval $[x, y]$ with a weight λ, and λ is called a *pessimistic-optimistic index* and means the pessimistic degree in decision making ([3]). Then, g is called a λ*-mean function* and $g([x, y])$ is called a λ*-mean value* of the interval $[x, y]$. Let $\mathcal{R}_c$ denote the set of fuzzy numbers with a continuous membership function. We introduce mean values of a fuzzy number $\tilde{a} \in \mathcal{R}_c$ with respect to λ-mean functions g and a fuzzy measure $M_{\tilde{a}}$, which depends on $\tilde{a}$ and is called *an evaluation measure* in this paper, as follows

$$\tilde{E}(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha)\, M_{\tilde{a}}(\tilde{a}_\alpha)\, \mathrm{d}\alpha \Bigg/ \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)\, \mathrm{d}\alpha, \tag{3}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ is the α-cut of the fuzzy number $\tilde{a}$. We note that (3) is normalized by $M(\tilde{a}_\alpha)(\alpha \in [0, 1])$. In a comparison with (1), $h(\alpha)$ is replaced with $g(\tilde{a}_\alpha)$ and the measure $\mathrm{d}m(\alpha)$ is taken as $M_{\tilde{a}}(\tilde{a}_\alpha)\, \mathrm{d}\alpha$. In (3), $M_{\tilde{a}}(\tilde{a}_\alpha)$ means a *confidence degree that the fuzzy number $\tilde{a}$ takes values in the interval $\tilde{a}_\alpha$ at each grade* α(see Example 1).

Example 1. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. An evaluation measure $M_{\tilde{a}}$ is called the *possibility evaluation measure* and the *necessity evaluation measure* induced from the fuzzy number $\tilde{a}$ if it is given by the following (4) and (5) respectively:

$$M_{\tilde{a}}^P(I) := \sup_{x\in I} \tilde{a}(x), \tag{4}$$

$$M_{\tilde{a}}^N(I) := 1 - \sup_{x\notin I} \tilde{a}(x) \tag{5}$$

for a Borel set I. We note that $M_{\tilde{a}}^P$ and $M_{\tilde{a}}^N$ satisfy the definition of fuzzy measures since $\tilde{a}$ is continuous and has a compact support. Since $M_{\tilde{a}}^P(\tilde{a}_\alpha) = 1$ and $M_{\tilde{a}}^N(\tilde{a}_\alpha) = 1 - \alpha$ from (4) and (5), the corresponding mean values $\tilde{E}(\tilde{a})$ are reduced to

$$\tilde{E}^P(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha)\, \mathrm{d}\alpha, \tag{6}$$

$$\tilde{E}^N(\tilde{a}) := \int_0^1 g(\tilde{a}_\alpha)\, (2 - 2\alpha)\, \mathrm{d}\alpha. \tag{7}$$

They are called a *possibility mean* and a *necessity mean* of the fuzzy number $\tilde{a}$ respectively. (6) has been discussed in Fortemps and Roubens [3] and so on, however an evaluation method $\int_0^1 g(\tilde{a}_\alpha)(2\alpha)\mathrm{d}\alpha = \int_0^1 g(\tilde{a}_\alpha)\, \alpha\, \mathrm{d}\alpha / \int_0^1 \alpha\, \mathrm{d}\alpha$, which has been studied by Goetshel and Voxman [4] and Carlsson and Fullér [2], is different from our method (3) since $M_{\tilde{a}}(\tilde{a}_\alpha)$ in (3) is non-increasing in $\alpha \in [0, 1]$ from the definition and the property of α-cuts.

Under the following regularity assumption, we extend the estimation (3) to the mean value of a general fuzzy number $\tilde{a}$ whose membership function is upper-semicontinuous but is not necessarily continuous.

Assumption M. There exists a nonincreasing function $w : [0, 1] \mapsto [0, 1]$ such that

$$M_{\tilde{a}}(\tilde{a}_\alpha) = w(\alpha), \quad \alpha \in [0, 1] \quad \text{for all } \tilde{a} \in \mathcal{R}_c. \tag{8}$$

We note that w is independent of $\tilde{a} \in \mathcal{R}_c$ in (8) of Assumption M. Regarding the possibility evaluation measure and the necessity evaluation measure, we may take $w(\alpha)$ in Assumption M as $w(\alpha) = M_{\tilde{a}}^P(\tilde{a}_\alpha) = 1$ and $w(\alpha) = M_{\tilde{a}}^N(\tilde{a}_\alpha) = 1 - \alpha$ respectively(see (4) and (5)). From now on, we suppose Assumption M holds.

Let $\tilde{a}$ be a fuzzy number. We define the *mean values* for the general fuzzy number $\tilde{a}$ by

$$\tilde{E}(\tilde{a}) := \lim_{n\to\infty} \tilde{E}(\tilde{a}^n), \tag{9}$$

where $\tilde{E}(\tilde{a}^n)$ are defined by (3) and $\{\tilde{a}^n\}_{n=1}^{\infty} (\subset \mathcal{R}_c)$ is a sequence of fuzzy numbers whose membership functions are continuous and satisfy that $\tilde{a}^n \downarrow \tilde{a}$ pointwise as $n \to \infty$. The limiting value (10) is called well-defined if it is independent of the selection of the sequences $\{\tilde{a}^n\}_{n=1}^{\infty} \subset \mathcal{R}_c$ (Yoshida [11]). From (6) and (7), by the bounded convergence theorem we obtain the mean values defined by the *possibility evaluation measure* and the *necessity evaluation measure* as follows.

Lemma 1. *For general fuzzy numbers $\tilde{a}$, it holds that*

$$\tilde{E}^P(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha)\,\mathrm{d}\alpha, \tag{10}$$

$$\tilde{E}^N(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha)\,(2 - 2\alpha)\,\mathrm{d}\alpha. \tag{11}$$

Similarly to (10) and (11), under Assumption M we obtain the following representation regarding a general mean value (12) through the dominated convergence theorem.

Lemma 2. *For general fuzzy numbers $\tilde{a}$, it holds that*

$$\tilde{E}(\tilde{a}) = \int_0^1 g(\tilde{a}_\alpha)\,w(\alpha)\,\mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\,\mathrm{d}\alpha. \tag{12}$$

The mean value $\tilde{E}(\cdot)$ has the following natural properties for fuzzy numbers regarding the linearity and the monotonicity for the fuzzy max order.

Theorem 1. *Suppose Assumption M holds. For fuzzy numbers $\tilde{a}, \tilde{b}$ and real numbers $\theta, \zeta \in \mathbb{R}$ such that $\zeta \geq 0$, the following (i) – (iv) hold.*

(i) $\tilde{E}(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}(\tilde{a}) + \theta$, *where 1_B means the characteristic function of a set B.*

(ii) $\tilde{E}(\zeta\tilde{a}) = \zeta\tilde{E}(\tilde{a})$.

(iii) $\tilde{E}(\tilde{a} + \tilde{b}) = \tilde{E}(\tilde{a}) + \tilde{E}(\tilde{b})$.

(iv) *If $\tilde{a} \succeq \tilde{b}$, then $\tilde{E}(\tilde{a}) \geq \tilde{E}(\tilde{b})$ holds, where $\succeq$ is the fuzzy max order.*

3. Measurement of fuzziness

The concept of the degree of fuzziness is given by the distance between fuzzy data and their nearest crisp data (Wang and Klir [7]). By using fuzzy measures, we present a method to measure the size of fuzziness regarding fuzzy numbers. Let $\tilde{a} \in \mathcal{R}_c$ be a fuzzy number. A *measurement of fuzziness* $\tilde{F}(\tilde{a})$ of the fuzzy number $\tilde{a}$ is given as follows: Let $\alpha \in [0, 1]$. For an interval $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ as a number with fuzziness, let $y \in \tilde{a}_\alpha$ be a real number without fuzziness, which is taken temporarily as a true value estimated for $\tilde{a}_\alpha$. Then, a size of fuzziness should be given by the distance between y and $\tilde{a}_\alpha$:

$$\max\{\tilde{a}_\alpha^+ - y, y - \tilde{a}_\alpha^-\}. \tag{13}$$

Therefore, the upper/lower measurements of fuzziness should be given by

$$m^U(\tilde{a}_\alpha) := \max_{y \in \tilde{a}_\alpha} \{\max\{\tilde{a}_\alpha^+ - y, y - \tilde{a}_\alpha^-\}\} = \tilde{a}_\alpha^+ - \tilde{a}_\alpha^-, \tag{14}$$

$$m^L(\tilde{a}_\alpha) := \min_{y \in \tilde{a}_\alpha} \{\max\{\tilde{a}_\alpha^+ - y, y - \tilde{a}_\alpha^-\}\} = \frac{\tilde{a}_\alpha^+ - \tilde{a}_\alpha^-}{2}. \tag{15}$$

We note that m^U and m^L could be understood as the width of the α-cut $\tilde{a}_\alpha$ and its half length respectively. Therefore, from the idea of previous sections, for $m = m^U$ or $m = m^L$ a measurement of fuzziness $\tilde{F}(\tilde{a})$ is given by

$$\tilde{F}(\tilde{a}) := \int_0^1 m(\tilde{a}_\alpha)\,M_{\tilde{a}}(\tilde{a}_\alpha)\,\mathrm{d}\alpha \Big/ \int_0^1 M_{\tilde{a}}(\tilde{a}_\alpha)\,\mathrm{d}\alpha. \tag{16}$$

Suppose that Assumption M holds. Let a fuzzy number $\tilde{a} \in \mathcal{R}_c$. Then, the measurement of fuzziness is represented as follows.

$$\tilde{F}(\tilde{a}) = \int_0^1 m(\tilde{a}_\alpha)\,w(\alpha)\,\mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\,\mathrm{d}\alpha, \tag{17}$$

where $\tilde{a}_\alpha$ is the α-cut of the fuzzy number $\tilde{a} \in \mathcal{R}_c$. In similar arguments in Section 2, we define the measurement of fuzziness of the general fuzzy number $\tilde{a}$ by

$$\tilde{F}(\tilde{a}) := \lim_{n \to \infty} \tilde{F}(\tilde{a}^n), \tag{18}$$

where $\tilde{F}(\tilde{a}^n)$ is defined in (16) and $\{\tilde{a}^n\}_{n=1}^{\infty} (\subset \mathcal{R}_c)$ is a sequence of fuzzy numbers whose membership functions are continuous and satisfy that $\tilde{a}^n \downarrow \tilde{a}$ pointwise as $n \to \infty$. Then, the following lemma is trivial from Assumption M and (17).

Lemma 3. *Let $\tilde{a}$ be a general fuzzy number. Under Assumption M, the measurement of fuzziness is represented as follows.*

$$\tilde{F}(\tilde{a}) = \int_0^1 m(\tilde{a}_\alpha)\,w(\alpha)\,\mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\,\mathrm{d}\alpha. \tag{19}$$

The following lemma is trivial but convenient for numerical calculations.

Lemma 4. *Let $\tilde{a}$ be a general fuzzy number. Then, the measurement of fuzziness in the possibility case and the necessity case are as follows.*

$$\begin{aligned}\tilde{F}^P(\tilde{a}) &:= \int_0^1 m^U(\tilde{a}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\, \mathrm{d}\alpha \\ &= \int_0^1 (\tilde{a}_\alpha^+ - \tilde{a}_\alpha^-)\, \mathrm{d}\alpha, \qquad (20)\\ \tilde{F}^N(\tilde{a}) &:= \int_0^1 m^L(\tilde{a}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\, \mathrm{d}\alpha \\ &= \int_0^1 (\tilde{a}_\alpha^+ - \tilde{a}_\alpha^-)\,(1-\alpha)\, \mathrm{d}\alpha. \qquad (21)\end{aligned}$$

Now we obtain the following natural results about the measurement of fuzziness $\tilde{F}(\cdot) = \tilde{F}^P(\cdot)$ or $\tilde{F}(\cdot) = \tilde{F}^N(\cdot)$.

Theorem 2. *Suppose Assumption M holds. Let $\tilde{F} = \tilde{F}^P$ or $\tilde{F} = \tilde{F}^N$. For general fuzzy numbers $\tilde{a}, \tilde{b}$ and real numbers $\theta, \zeta \in \mathbb{R}$, the following (i) – (iv) hold.*

(i) $\tilde{F}(\tilde{a} + 1_{\{\theta\}}) = \tilde{F}(\tilde{a})$.

(ii) $\tilde{F}(\zeta\tilde{a}) = |\zeta|\tilde{F}(\tilde{a})$.

(iii) $\tilde{F}(\tilde{a} \pm \tilde{b}) = \tilde{F}(\tilde{a}) + \tilde{F}(\tilde{b})$.

(iv) *If $\tilde{a} \supset \tilde{b}$, then $\tilde{F}(\tilde{a}) \geq \tilde{F}(\tilde{b})$ holds, where $\supset$ means the inclusion in the sense of fuzzy sets.*

4. Conclusion

Regarding two types of uncertainty, i.e. fuzziness and randomness, the measurement of fuzziness is related to the imprecision of data and the variance is based on the randomness of data. The variance is defined with the mean value which is the theoretical center value of data and which is derived with some probabilities([2]). However, the concept of the measurement of fuzziness has essentially no relation with the mean values in imprecise data, and it is natural to derive a criterion in comparison with crisp data. Therefore, when we deal with fuzzy data, the measurement of fuzziness $\tilde{F}$ presented in this paper is more reasonable than the variance as a criterion to measure uncertainty.

The results represented in this paper are also easily applicable to fuzzy random variables and fuzzy stochastic processes in decision making modeling.

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