Solving Linear PDEs with the Aid of Two-Dimensional Legendre Wavelets

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***Abstract*—In this paper, we develop a method, which using two-dimensional Legendre wavelets, to solve linear PDEs. Based on the properties of shifted Legendre polynomials, we give a brief proof about the general procedure of two-dimensional operational matrices of integration, and then employ aforementioned matrices to find the solution of the PDEs. The proposed method is mathematically simple and fast. To demonstrate the efficiency of the method, two test problems (solution of the diffusion, Poisson) are discussed. The experimental results showed that the accuracy of the method is very high and only need a small number of collocation points.**

Keywords-two-dimensional, Legendre wavelets, operational matrix, integration, PDEs

I. Introduction

Wavelets theory, as a relatively new and an emerging area in mathematical research, has received considerable attention in dealing with PDEs [1]. Wavelets analysis possesses several useful properties, such as orthogonality, compact support, exact representation of polynomials to a certain degree, and multi-resolution analysis (MRA) [2]. Moreover, wavelets establish a connection with fast numerical algorithms [3]. Therefore the wavelet is successfully used in many fields. So the wavelet analysis has application advantage on many fields such as signal analysis, data compression, image manipulation and numerical computing [4].

In most case the wavelet coefficients were calculated by Galerkin or collocation method, by it we have to evaluate integral of some combinations of wavelet functions (called also connection coefficients)[5]. The fundamental idea of Legendre wavelet method is, by using the operational matrices, the PDE problems which satisfies the boundary conditions and initial condition can be converted into a set of algebraic equations which involves a finite number of variables those of solving a system of algebraic equations, thus greatly simplify the problem and reduce the computation cost [6]. The Large systems of algebraic equations may lead to greater computational complexity and large storage requirements. However, the operational matrix of the Legendre wavelets is sparse, has lower dimension and most importantly is equal on every subinterval [7]. These features can decrease the saving and computational complexity when solving the system of algebraic equations converted by the computational operators.

The main purpose of this work is to develop a effective 2-D Legendre wavelets method combing with collocation method for solving PEDs, which is fast, mathematically simple and only needs a small number of grid points to guarantees the necessary accuracy. The remainder of the paper is organized as follows. Section 2 introduces the two-dimensional Legendre wavelets and the properties of shifted Legendre polynomials. We give a proof of operational matrices of integration in Section 3. Section 4 presents the methods that utilize two dimensional Legendre wavelets operational matrices to solve second order linear PDEs. Two illustrative examples are also given to demonstrate the validity and applicability of proposed method. Finally, a brief summary is presented.

II. Preliminaries and Notations

A. *The Properties of Shifted Legendre Polynomials*

Shifted Legendre polynomials and their properties are described in [8, 9]. In this article, we only pay close attention to two important properties of shifted Legendre polynomials. The relations between shifted Legendre wavelets and their integration and derivative have been derived in [8] and [9] respectively, which play an important role in deriving the Legendre wavelet operational matrix.

The well-known Legendre differential equation is

$$\left[\left(1-x^2\right)\bar{P}_n'(x)\right]'+n(n+1)\bar{P}_n(x)=0,-1\le x\le 1$$

here $\bar{P}_n(x)$ is Legendre polynomials of order n defined over the interval $[-1,1]$. The so-called shifted Legendre differential equation is obtained directly from the above equation by letting $x=2^k\tau-2n+1$.

Theorem 1. Let $\bar{P}_m(x)$ be the Legendre polynomials shifted into $[-1,1]$, then we have

$$(2m+1)\bar{P}_m(x)=\bar{P}_{m+1}'(x)-\bar{P}_{m-1}'(x) \qquad (1)$$

Theorem 1 is a useful property of Legendre polynomials, which can be seen in [10].

Corollary 1. Let $x = 2^k \tau - 2n + 1$, and $P_m(\tau)$ be the Legendre polynomials shifted into $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$, then we have

$$2^k (2m+1) P_m(\tau) = P'_{m+1}(\tau) - P'_{m-1}(\tau) \tag{2}$$

Proof.

From the definition of shifted Legendre polynomials and using the law of derivate, we have

$$\bar{P}'_n(x) = \frac{d\bar{P}_n(x)}{dx} = \frac{dP_n(\tau)}{dx} = \frac{dP_n(\tau)}{d\tau}\frac{d\tau}{dx} = \frac{1}{2^k} P'_n(\tau), \tag{3}$$

After substituting (3) in (1), we can get (2) directly.

B. Two-Dimensional Legendre Wavelets

Two-dimensional Legendre wavelets in $L^2(R)$ over the interval $[0,1]\times[0,1]$ as the form [11]:

$$\psi_{n,m,n',m'}(x,y) = \begin{cases} \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)} 2^{\frac{k+k'}{2}} P_m(x) P_{m'}(y), \\ \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \frac{n'-1}{2^{k'-1}} \le y \le \frac{n'}{2^{k'-1}}; \\ 0, \qquad \text{otherwise.} \end{cases}, \tag{4}$$

and $m = 0,1,2,\cdots, M-1, m' = 0,1,2,\cdots, M'-1,\ \ n = 1,2,\cdots,2^{k-1}$, $n' = 1,2,\cdots,2^{k'-1}$.

where $P_m(x) = \bar{P}_m(2^k x - 2n + 1)$, $P_{m'}(y) = \bar{P}_{m'}(2^{k'} y - 2n' + 1)$, $\bar{P}_m$ are Legendre functions of order m defined over the interval $[-1,1]$.

Two dimensions Legendre wavelets are an orthonormal set over $[0,1]\times[0,1]$

$$\int_0^1\int_0^1 \psi_{n,m,n',m'}(x,y)\psi_{n_1,m_1,n'_1,m'_1}(x,y)\,dxdy = \delta_{n,n_1}\delta_{m,m_1}\delta_{n',n'_1}\delta_{m',m'_1} \tag{5}$$

The function $u(x,y) \in L^2(R)$ defined over $[0,1]\times[0,1]$ may be expanded as

$$u(x,y) = X(x)(y) \cong \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} c_{n,m,n',m'}\psi_{n,m,n',m'}(x,y) \tag{6}$$

If the infinite series in (6) is truncated, then (6) can be written as

$$u(x,y) = X(x)(y) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} c_{n,m,n',m'}\psi_{n,m,n',m'}(x,y) \tag{7}$$

where $c_{n,m,n',m'} = \int_0^1\int_0^1 X(x)Y(y)\psi_{n,m,n',m'}(x,y)\,dxdy$ .The Eq. (7) can be expressed as the form

$$u(x,y) = C^T \cdot \Psi(x,y). \tag{8}$$

where C and $\Psi(x,y)$ (more details see[11]) are coefficients matrix and wavelets vector matrix respectively. The number of dimensions of C and $\Psi(x,y)$ are $2^{k-1}2^{k'-1}MM'\times 1$, and given by

$$\begin{aligned} C = [&c_{1,0,1,0},\cdots,c_{1,0,1,M'-1},c_{1,0,2,0},\cdots,c_{1,0,2,M'-1},\cdots,c_{1,0,2^{k'-1},0},\cdots,c_{1,0,2^{k'-1},M'-1}, \\ &\cdots c_{1,M-1,1,0},\cdots,c_{1,M-1,1,M'-1},c_{1,M-1,2,0},\cdots,c_{1,M-1,2,M'-1},\cdots,c_{1,M-1,2^{k'-1}0},\cdots,c_{1,M-1,2^{k'-1},M'-1}, \\ &\cdots c_{2,0,1,0},\cdots,c_{2,0,1,M'-1},c_{2,0,2,0},\cdots,c_{2,0,2,M'-1},\cdots,c_{2,0,2^{k'-1},0},\cdots,c_{2,0,2^{k'-1},M'-1}, \\ &\cdots c_{2,M-1,1,0},\cdots,c_{2,M-1,1,M'-1},c_{2,M-1,2,0},\cdots,c_{2,M-1,2,M'-1},\cdots,c_{2,M-1,2^{k'-1},0},\cdots,c_{2,M-1,2^{k'-1},M'-1}, \\ &\cdots c_{2^{k-1},0,1,0},\cdots,c_{2^{k-1},0,1,M'-1},c_{2^{k-1},0,2,0},\cdots,c_{2^{k-1},0,2,M'-1},\cdots,c_{2^{k-1},0,2^{k'-1},0},\cdots,c_{2^{k-1},M-1,2^{k'-1},M'-1}]^T \end{aligned} \tag{9}$$

$$\begin{aligned} \Psi = [&\psi_{1,0,1,0},\cdots,\psi_{1,0,1,M'-1},\psi_{1,0,2,0},\cdots,\psi_{1,0,2,M'-1},\cdots,\psi_{1,0,2^{k'-1},0},\cdots,\psi_{1,0,2^{k'-1},M'-1}, \\ &\cdots\psi_{1,M-1,1,0},\cdots,\psi_{1,M-1,1,M'-1},\psi_{1,M-1,2,0},\cdots,\psi_{1,M-1,2,M'-1},\cdots,\psi_{1,M,-1,2^{k'-1},0},\cdots,\psi_{1,M-1,2^{k'-1},M'-1}, \\ &\cdots\psi_{2,0,1,0},\cdots,\psi_{2,0,1,M'-1},\psi_{2,0,2,0},\cdots,\psi_{2,0,2,M'-1},\cdots,\psi_{2,0,2^{k'-1},0},\cdots,\psi_{2,0,2^{k'-1},M'-1}, \\ &\cdots\psi_{2,M-1,1,0},\cdots,\psi_{2,M-1,1,M'-1},\psi_{2,M-1,2,0},\cdots,\psi_{2,M-1,2,M'-1},\cdots,\psi_{2,M-1,2^{k'-1},0},\cdots,\psi_{2,M-1,2^{k'-1},M'-1}, \\ &\cdots\psi_{2^{k-1},0,1,0},\cdots,\psi_{2^{k-1},0,1,M'-1},\psi_{2^{k-1},0,2,0},\cdots,\psi_{2^{k-1},0,2,M'-1},\cdots,\psi_{2^{k-1},0,2^{k'-1},0},\cdots,\psi_{2^{k-1},M-1,2^{k'-1},M'-1}]^T \end{aligned} \tag{10}$$

The integration of the product of two Legendre wavelet function vectors is obtained as

$$\int_0^1\int_0^1 \Psi(x,y)\Psi^T(x,y)\,dxdy = I \tag{11}$$

where I is identity matrix.

III. Two-Dimensional Operational Matrix of Integration

Although a general procedure for forming two-dimensional operational matrix of integration has been presented in [11], there is no detailed proof. In this section, we will give a brief proof about the general procedure of two-dimensional operational matrix of integration by applying corollary 1.

A. Operational matrix of integration for x variable

Theorem 2. Let $\Psi(x,y)$ be the two-dimensional Legendre wavelets vector defined in (10), we have

$$\int_0^x \Psi(\tau,y)\,d\tau = P_x\Psi(x,y), \tag{12}$$

where P_x is $2^{k-1}2^{k'-1}MM'\times 2^{k-1}2^{k'-1}MM'$ operational matrix for integration and is given as.

$$P_x = \frac{1}{M'2^{k'+k-1}}\begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ O & O & L & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & F \\ O & O & O & \cdots & L \end{bmatrix}$$

O is $2^{k-1}MM' \times 2^{k-1}MM'$ matrix. F and L are $2^{k-1}MM' \times 2^{k-1}MM'$ matrices that define as follow

$$F = \begin{bmatrix} 2D & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ O' & O' & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & O' \\ O' & O' & O' & \cdots & O' \end{bmatrix} D = \begin{bmatrix} I & I & I & \cdots & I \\ I & I & I & \cdots & I \\ I & I & I & \cdots & I \\ \vdots & \vdots & \vdots & \ddots & I \\ I & I & I & \cdots & I \end{bmatrix},$$

$$L = \begin{bmatrix} D & \frac{1}{\sqrt{3}}D & O' & \cdots & O' \\ -\frac{1}{\sqrt{3}}D & O' & \frac{1}{\sqrt{3}\sqrt{5}}D & \cdots & O' \\ O' & \frac{-1}{\sqrt{3}\sqrt{5}}D & O' & \cdots & O' \\ \vdots & \vdots & \vdots & \ddots & O' \\ O' & O' & O' & \cdots & O' \end{bmatrix},$$

O' is $2^{k'-1}M' \times 2^{k'-1}M'$ full zero matrix and I is $M' \times M'$ identity matrix, and D is $2^{k'-1}M' \times 2^{k'-1}M'$ matrix given by

Proof.

When $m = 0$, we obtain

$$\int_0^x \psi_{n,0,n',m'}(\tau, y)\, d\tau = A_{0,m'}P_{m'}(y)\begin{cases} \int_{\frac{n-1}{2^{k-1}}}^{x} P_0(x)\,dx, & x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]; \\ 0, & \hat{n} = 1,\cdots,n-1, x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]; \\ \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} P_0(x)\,dx, \hat{n} = n+1,\cdots,2^{k-1}, & x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]. \end{cases} \tag{13}$$

From (13) and completing the integration, then one has

$$\int_0^x \psi_{n,0,n',m'}(\tau, y)\, d\tau = \begin{cases} \frac{1}{2^k}\left[\frac{1}{\sqrt{3}}\psi_{n,1,n',m'}(x,y) + \psi_{n,0,n',m'}(x,y)\right], & x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]; \\ 0, \quad \hat{n} = 1,\cdots,n-1, & x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]; \\ \frac{1}{2^{k-1}}\psi_{\hat{n},0,n',m'}(x,y), \quad \hat{n} = n+1,\cdots,2^{k-1}, & x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]. \end{cases} \tag{14}$$

When $m \neq 0$, by integrating with respect to x in (4) we have

$$\int_0^x \psi_{n,m,n',m'}(\tau, y)\, d\tau = A_{m,m'}P_{m'}(y)\begin{cases} \int_{\frac{n-1}{2^{k-1}}}^{x} P_m(\tau)\,d\tau, & x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]; \\ 0, \quad \hat{n} = 1,\cdots,n-1, & x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]; \\ \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} P_m(\tau)\,d\tau, \hat{n} = n+1,\cdots,2^{k-1}, & x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]. \end{cases} \tag{15}$$

Using Corollary 1, and integrating with respect to x in (2) from $\frac{n-1}{2^{k-1}}$ to $\frac{n}{2^{k-1}}$, then

$$\int_0^x \psi_{n,m,n',m'}(\tau, y)\, d\tau = A_{m,m'}P_{m'}(y)\begin{cases} \frac{1}{2^k(2m+1)}\left(P_{m+1}(\tau) - P_{m-1}(\tau)\right)\Big|_{\tau=\frac{n-1}{2^{k-1}}}^{\tau=x}, x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]; \\ 0, \quad \hat{n} = 1,\cdots,n-1, \quad x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]; \\ \frac{1}{2^k(2m+1)}\left(P_{m+1}(\tau) - P_{m-1}(\tau)\right)\Big|_{\tau=\frac{n-1}{2^{k-1}}}^{\tau=\frac{n}{2^{k-1}}}, \hat{n} = n+1,\cdots,2^{k-1}, \\ x \in \left[\frac{\hat{n}-1}{2^{k-1}}, \frac{\hat{n}}{2^{k-1}}\right]. \end{cases} \tag{16}$$

Because $\left(P_{m+1}(\tau) - P_{m-1}(\tau)\right)\Big|_{\tau=\frac{n-1}{2^{k-1}}} = \bar{P}_{m+1}(-1) - \bar{P}_{m-1}(-1) = 0$, and $\left(P_{m+1}(\tau) - P_{m-1}(\tau)\right)\Big|_{\tau=\frac{n}{2^{k-1}}} = \bar{P}_{m+1}(1) - \bar{P}_{m-1}(1) = 0$, thus we have

$$\int_0^x \psi_{n,m,n',m'}(\tau, y)\, d\tau = \begin{cases} \frac{1}{2^k\sqrt{2m+1}}\left[\frac{1}{\sqrt{2m+3}}\psi_{n,m+1,n,m} - \frac{1}{\sqrt{2m-1}}\psi_{n,m-1,n,m}\right], \\ x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]; \\ 0, \quad \textit{otherwise.} \end{cases} \tag{17}$$

Writing $\int_0^y \Psi(x,\tau)\, d\tau$ as a vector whose element is $\int_0^x \psi_{n,m,n',m'}(\tau, y)\, d\tau$, we could get matrix form as $\int_0^x \Psi(\tau, x)\, d\tau = P_x \Psi(x, y)$.

□

B. Operational matrix of integration for y variable

Theorem 3. Let $\Psi(x,y)$ be the two dimensions Legendre wavelets vector defined in (10), then we have

$$\int_0^y \Psi(x,\tau)\, d\tau = P_y \Psi(x,y), \tag{18}$$

in which

$$P_y = \begin{bmatrix} P & P & \cdots & P \\ P & P & \cdots & P \\ \vdots & \vdots & \ddots & \vdots \\ P & P & \cdots & P \end{bmatrix}, P = \begin{bmatrix} L & F & \cdots & F \\ O & L & \cdots & F \\ O & O & \ddots & F \\ O & O & O & L \end{bmatrix}$$

P_y is a $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix. P is $2^{k-1}M \times 2^{k'-1}M'$ matrix, L, F and O is $M \times M'$ matrix.

$$F=\frac{1}{2^k}\begin{bmatrix} a & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}, a=\begin{cases}2, k>1\\ 0, k=1\end{cases}$$

$$L=\frac{1}{2^k}\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0\\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}\sqrt{5}} & \cdots & 0\\ 0 & -\frac{1}{\sqrt{3}\sqrt{5}} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

Proof.

When $m'=0$, by integrating with respect to y in (4), then

$$\int_0^y \psi_{n,m,n',0}(x,\tau)d\tau = A_{m,0}P_m(x)\begin{cases}\int_{\frac{n'-1}{2^{k'-1}}}^{y} P_0(\tau)d\tau, & y\in\left[\frac{n'-1}{2^{k'-1}},\frac{n'}{2^{k'-1}}\right]\\ 0, & \hat{n}'=1,\cdots,n'+1, y\in\left[\frac{\hat{n}'-1}{2^{k'-1}},\frac{\hat{n}'}{2^{k'-1}}\right]\\ \int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}} P_0(\tau)d\tau, \hat{n}'=n'+1,\cdots,2^{k'-1}, & y\in\left[\frac{\hat{n}'-1}{2^{k'-1}},\frac{\hat{n}'}{2^{k'-1}}\right]\end{cases} \quad (19)$$

$$\int_0^y \psi_{n,m,n',0}(x,\tau)d\tau = \begin{cases}\frac{1}{2^{k'}}\left[\frac{1}{\sqrt{3}}\psi_{n,m,n',1}+\psi_{n,m,n',0}\right], & y\in\left[\frac{n'-1}{2^{k'-1}},\frac{n'}{2^{k'-1}}\right]\\ 0, & \hat{n}'=1,\cdots,n'-1, y\in\left[\frac{n'-1}{2^{k'-1}},\frac{n'}{2^{k'-1}}\right]\\ \frac{1}{2^{k'-1}}\psi_{n,m,\hat{n}',0}, \hat{n}'=n'+1,\cdots,2^{k'-1}, & y\in\left[\frac{n'-1}{2^{k'-1}},\frac{n'}{2^{k'-1}}\right],\end{cases} \quad (20)$$

When $m'\neq 0$, by integrating with respect to y in (4), then

$$\int_0^y \psi_{n,m,n',m'}(x,\tau)d\tau = \begin{cases}\frac{1}{2^{k'}\sqrt{2m'+1}}\left[\frac{1}{\sqrt{2m'+3}}\psi_{n,m,n',m'+1}-\frac{1}{\sqrt{2m'-1}}\psi_{n,m,n',m'-1}\right],\\ y\in\left[\frac{n'-1}{2^{k'-1}},\frac{n'}{2^{k'-1}}\right];\\ 0, \quad \textit{otherwise.}\end{cases} \quad (21)$$

writing $\int_0^y \Psi(x,\tau)d\tau$ as a vector whose element is $\int_0^y \psi_{n,m,n',m'}(x,\tau)d\tau$, we could get matrix form as $\int_0^y \Psi(x,\tau)d\tau = P_y\Psi(x,y)$.

IV. Problem Statement and Method of Solution

Consider the following form PDEs

$$u=F\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}\right). \quad (22)$$

with the Dirichlet boundary conditions

In this section, we let $\cdot$ and $'$ denotes differentiation with respect to x and y, respectively. According to the Legendre wavelet method, it generally assumes that $\ddot{u}''(x,y)$ can be expanded in terms of Legendre wavelets as formula

$$\ddot{u}''=C^T\Psi(x,y), \quad (23)$$

Integrating formula Eq. (23) with respect to x twice from 0 to x and with respect to y once from 0 to y, we obtain

$$\dot{u}''(x,y)=C^TP_x\Psi(x,y)+\dot{u}''(0,y), \quad (24)$$

$$u''(x,y)=C^TP_x^2\Psi(x,y)+x\dot{u}''(0,y)+u''(0,y), \quad (25)$$

$$\begin{aligned}u'(x,y)&=C^TP_x^2P_y\Psi(x,y)+x\left[\dot{u}'(0,y)-\dot{u}'(0,0)\right]\\&+u'(0,y)-u'(0,0)+u'(x,0),\end{aligned} \quad (26)$$

Putting $x=1$ in the Eq. (25) and (26), we have

$$\dot{u}''(0,y)=-C^TP_x^2\Psi(1,y)+u''(1,y)-u''(0,y), \quad (27)$$

$$\dot{u}'(0,y)-\dot{u}'(0,0)=-C^TP_x^2P_y\Psi(1,y)+g_1(y), \quad (28)$$

where $g_1(y)=u'(1,y)-u'(0,y)+u'(0,0)-u'(1,0)$.

Substituting Eq. (27) into (25) and Eq. (28) into (26), we have

$$\begin{aligned}u''(x,y)&=C^TP_x^2\left[\Psi(x,y)-x\Psi(1,y)\right]\\&+x\left[u''(1,y)-u''(0,y)\right]+u''(0,y),\end{aligned} \quad (29)$$

$$\begin{aligned}u'(x,y)&=C^TP_x^2P_y\left[\Psi(x,y)-x\Psi(1,y)\right]\\&+g_2(x,y)-u'(0,0)+u'(x,0),\end{aligned} \quad (30)$$

where

$g_2(x,y)=x\left[u'(1,y)-u'(0,y)+u'(0,0)-u'(1,0)\right]+u'(0,y)$.

Integrating formula Eq. (23) with respect to x from 0 to x we obtain

$$\begin{aligned}u(x,y)&=C^TP_x^2P_y^2\left[\Psi(x,y)-x\Psi(1,y)\right]+g_3(x,y)\\&+y\left[u'(x,0)-u'(0,0)\right]+u(x,0),\end{aligned} \quad (31)$$

in which

$$\begin{aligned}g_3(x,y)&=x\left[u(1,y)-u(1,0)-u(0,y)+u(0,0)\right]\\&+x\left[yu'(0,0)-yu'(1,0)\right]+u(0,y)-u(0,0)\end{aligned}.$$

Putting $y=1$ in the Eq. (30), we have

$$\begin{aligned}u'(x,0)-u'(0,0)&=-C^TP_x^2P_y^2\left[\Psi(x,1)-x\Psi(1,1)\right]\\&-g_3(x,1)+u(x,1)-u(x,0),\end{aligned} \quad (32)$$

Substituting Eq. (32) into (30) and (31), we have

$$u'(x,y)=C^TP_x^2P_y\left[\Psi(x,y)-x\Psi(1,y)\right]$$
$$+C^TP_x^2P_y^2\left[-\Psi(x,1)+x\Psi(1,1)\right] \quad (33)$$
$$+g_2(x,y)-g_3(x,1)+u(x,1)-u(x,0),$$
$$u(x,y)=C^TP_x^2P_y^2\left[\Psi(x,y)-x\Psi(1,y)\right]$$
$$-C^TP_x^2P_y^2\left[y\Psi(x,1)+xy\Psi(1,1)\right] \quad (34)$$
$$+g_3(x,y)-yg_3(x,1)+g_4(x),$$
where $g_4(x)=y\left[u(x,1)-u(x,0)\right]+u(x,0)$.

Integrating Eq. (23) with respect to y from 0 to y and with respect to x from 0 to x, we obtain

$$\ddot{u}'(x,y)=C^TP_y\Psi(x,y)+\ddot{u}'(x,0), \quad (35)$$
$$\ddot{u}(x,y)=C^TP_y^2\Psi(x,y)+y\ddot{u}'(x,0)+\ddot{u}(x,0), \quad (36)$$
$$\dot{u}(x,y)=C^TP_y^2P_x\Psi(x,y)+y\left[\dot{u}'(x,0)-\dot{u}'(0,0)\right]$$
$$+\dot{u}(x,0)-\dot{u}(0,0)+\dot{u}(0,y), \quad (37)$$

Putting $y=1$ in the Eq. (36) and (37), we have

$$\ddot{u}'(x,0)=-C^TP_y^2\Psi(x,1)+\ddot{u}(x,1)-\ddot{u}(x,0), \quad (38)$$
$$\dot{u}'(x,0)-\dot{u}'(0,0)=-C^TP_y^2P_x\Psi(x,1)+g_5(x), \quad (39)$$

where $g_5(x)=\dot{u}(x,1)-\dot{u}(x,0)+\dot{u}(0,0)-\dot{u}(0,1)$.

Substituting Eq. (38) into (36) and (39) into (37), we have

$$\ddot{u}(x,y)=C^TP_y^2\left[\Psi(x,y)-y\Psi(x,1)\right]$$
$$+y\left[\ddot{u}(x,1)-\ddot{u}(x,0)\right]+\ddot{u}(x,0), \quad (40)$$
$$\dot{u}(x,y)=C^TP_y^2P_x\left[\Psi(x,y)-y\Psi(x,1)\right]$$
$$+yg_5(x)+\dot{u}(x,0)-\dot{u}(0,0)+\dot{u}(0,y), \quad (41)$$

Then, integrating formula (41) with respect to x from 0 to x, we obtain

$$u(x,y)=C^TP_y^2P_x^2\left[\Psi(x,y)-y\Psi(x,1)\right]$$
$$+g_6(x,y)+x\left[\dot{u}(0,y)-\dot{u}(0,0)\right]+u(0,y), \quad (42)$$

where

$$g_6(x,y)=y\left[u(x,1)-u(0,1)-u(x,0)+u(0,0)\right]$$
$$+xy\left[\dot{u}(0,0)-\dot{u}(0,1)\right]+u(x,0)-u(0,0).$$

At $x=1$ in the Eq. (42), we have

$$\dot{u}(0,y)-\dot{u}(0,0)=-C^TP_y^2P_x^2\left[\Psi(1,y)-y\Psi(1,1)\right]$$
$$-g_6(1,y)+u(1,y)-u(0,y), \quad (43)$$

Substituting Eq. (43) into (41), we have

$$\dot{u}(x,y)=yg_5(x)+\dot{u}(x,0)-g_6(1,y)+u(1,y)-u(0,y)$$
$$+C^TP_y^2P_x\left[\Psi(x,y)-y\Psi(x,1)-P_x\Psi(1,y)+yP_x\Psi(1,1)\right], \quad (44)$$

Substituting Eq. (29), (33), (34), (40) and (44) into Eq. (22), and collocate Eq. (22) at $2^{k-1}2^{k'-1}MM'$ points, we can get a set of algebraic equations which can be solved for C.

V. ILLUSTRATIVE EXAMPLES

In this section, we will demonstrate the effectiveness of the proposed two-dimensional Legendre wavelets method with two illustrative examples.

A. *Diffusion equation*

Consider the following diffusion equation

$$\frac{\partial u}{\partial t}-0.1\frac{\partial u}{\partial x}=0.01\frac{\partial^2 u}{\partial x^2},\quad (x,t)\in[0,1] \quad (45)$$

Subject to the Dirichlet boundary conditions as

$$u(x,0)=e^{-x},u(x,1)=e^{-x-0.09}, \quad (46)$$
$$u(0,t)=e^{-0.09t},u(1,t)=e^{-1-0.09t}, \quad (47)$$

and the exact solution is $u(x,t)=e^{-x-0.09t}$.

We solve the above problem by applying the technique described in Section III and have

$$\ddot{u}(x,t)=C^TP_t\Psi(x,t)+\ddot{u}(x,0)$$
$$\dot{u}(x,t)=C^TP_tP_x\left[\Psi(x,t)-P_x\Psi(1,t)\right]+g_1(x,t), \quad (48)$$
$$u'(x,t)=C^TP_x^2\left[\Psi(x,t)-x\Psi(1,t)\right]+g_2(x,t)$$

$$u(x,t)=C^TP_tP_x^2\left[\Psi(x,t)-x\Psi(1,t)\right]+u(x,0)-u(0,0)$$
$$+x\left[u(1,t)-u(1,0)+u(0,0)-u(0,t)\right]+u(0,t); \quad (49)$$

in which $g_1(x,t)=\dot{u}(x,0)-u(1,0)-u(0,t)+u(1,t)+1$, and

$g_2(x,t)=x\left[u'(1,t)-u'(0,t)\right]+u'(0,t)$.

Substituting (48) and (49) into (45), we obtain

$$C^T\Lambda=g(x,t) \quad (50)$$

where $\Lambda=\begin{array}{l}\left(P_x^2-0.1P_tP_x-0.01P_t\right)\Psi(x,t)\\+\left(xI-0.1P_t\right)P_x^2\Psi(1,t)\end{array}$.

From formula (50) the wavelet coefficients C^T can be calculated. When $M=M'=12$ and $k=k'=1$, the absolute error is plotted in figure 1.

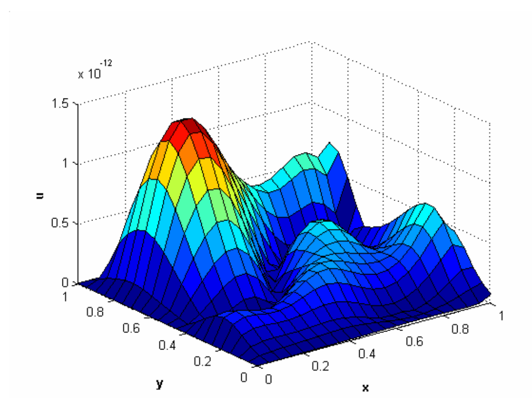

Figure 1. the absolute error of diffusion equation

From figure 1, we can see that numerical solution obtained by our method is good agrees with the exact solution and the accuracy of the method is very high.

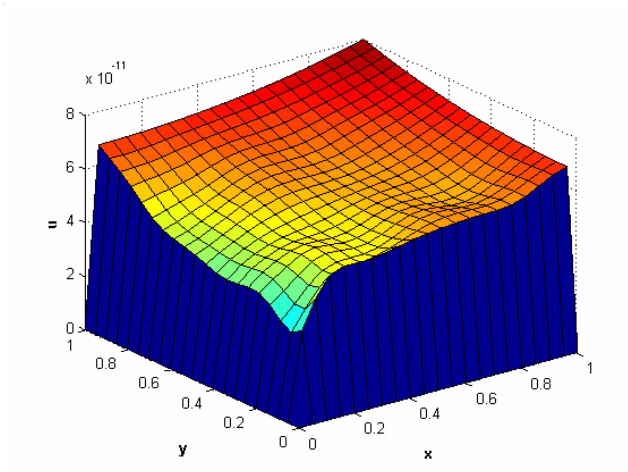

Figure 2. the absolute error of Poisson equation

B. Poisson equation

Consider the following Poisson equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-2u,(x,y)\in[0,1]\times[0,1] \tag{51}$$

with the Dirichlet boundary conditions

$$\begin{aligned}&u(0,y)=0,u(1,y)=\sin(1)\sin(y),\\&u(x,0)=0,u(x,1)=\sin(x)\sin(1)\end{aligned} \tag{52}$$

has the exact solution is $u=\sin(x)\sin(y)$.

We solve the above problem by applying the technique described in Section III and have

$$\begin{aligned}\ddot{u}(x,y)&=C^T P_y^2\left[\Psi(x,y)-y\Psi(x,1)\right]\\&\quad+y\left[\ddot{u}(x,1)-\ddot{u}(x,0)\right]+\ddot{u}(x,0),\\u''(x,y)&=C^T P_x^2\left[\Psi(x,y)-x\Psi(1,y)\right]\\&\quad+x\left[u''(1,y)-u''(0,y)\right]+u''(0,y),\end{aligned} \tag{53}$$

$$\begin{aligned}u(x,y)&=C^T P_x^2 P_y^2\left[\Psi(x,y)-x\Psi(1,y)-y\Psi(x,1)\right]\\&\quad-xyC^T P_x^2 P_y^2\Psi(1,1)+g_3(x,y)-yg_3(x,1)+g_4(x,y),\end{aligned} \tag{54}$$

where $g_3(x,y)=x\left[u(1,y)-u(1,0)-u(0,y)+u(0,0)\right] + xy\left[u'(0,0)-u'(1,0)\right]+u(0,y)-u(0,0)$ and

$g_4(x,y)=y\left[u(x,1)-u(x,0)\right]+u(x,0)$.

Substituting (53) and (54) into (51), we obtain

$$C^T\Lambda=g(x,y) \tag{55}$$

where $\Lambda=P_y^2\left[\Psi(x,y)-y\Psi(x,1)\right]+P_x^2\left[\Psi(x,y)-x\Psi(1,y)\right] +2P_x^2P_y^2\left[\Psi(x,y)-x\Psi(1,y)-y\Psi(x,1)-xy\Psi(1,1)\right]$

and

$$\begin{aligned}g(x,y)=&-x\left[u''(1,y)-u''(0,y)\right]-u''(0,y)\\&-y\left[\ddot{u}(x,1)-\ddot{u}(x,0)\right]-\ddot{u}(x,0)\\&-2\left[g_3(x,y)-yg_3(x,1)+g_4(x)\right]\end{aligned}.$$

From figure 2, we can see that numerical solution obtained by our method is full agrees with the exact solution.

CONCLUSION

In this paper, we give a brief proof about the general procedure of two-dimensional operational matrices of integration, and then develop a solution of PDEs by using the two-dimensional operational matrices of integration. The main benefits of the proposed method are its computation-effective (only need a small number of collocation points guarantees the necessary accuracy) and universality (the same approach is applicable for a wide class of linear PDEs).

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