Applied Mechanics and Engineering Model on Raindrops falling

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Keywords: atmosphere science, raindrops falling, buoyancy, viscous resistance, Coriolis force

Abstract. The theoretical research on raindrops movement should promote water conservation in the Loess Plateau. Four influence factors on the raindrops falling in the Earth atmosphere were studied, which include gravitational force, buoyancy, viscous resistance and Coriolis force. Three mathematical models on raindrops were established with Newton's dynamic equation. Simulated to the equations, all characteristics of raindrops falling were obtained. The research shows that the raindrops have different terminal falling speed as different size due to the viscous resistance. The buoyancy is almost no influence to all characteristics of raindrop falling. The Coriolis force leads raindrops to move eastward and southward, as well as, the eastward migration is more obvious than the southward. The raindrops falling migration from the viscous resistance action is very significant.

Introduction

Over the last decade, as aware of the severe hazards of soil erosion, much attention and interests were paid to the research on soil erosion for terminal speed, kinetic energy and influence factors on the raindrops falling in the Loess Plateau slope land[1-2]. But the detailed researching materials about the characters or laws of raindrop movement are short and little, only some low precision research and the ideal model [3-7]. Therefore, the studies on quasi-real raindrops movement mathematical model and its computer simulation will support not only the theoretical research of the dynamic model in the soil and water conservation(SWC) in Loess Plateau, but also enrich and perfect the detailed data of the quasi-real raindrop movement in theory and experiment. Further more, the classification standards of the convective cloud system and the stratiform precipitation cloud system should be put forward according to the different characteristics of the precipitation cloud system. Thus an innovative research idea will be proposed by the dynamics based on the statistical method.

In this research, taking the Xi'an area for example, we analyzed influencing factors (i.e. gravitational force, buoyancy, viscous resistance and Coriolis force) on quasi-real raindrops in the Earth's atmosphere, established the dynamics mathematical model of raindrop movement. Comparing all numerical solutions with the experimental data of the SWC, we proposed detailed model and images of raindrops movement and four influence weighting factors.

Parameters of Raindrops

Height of Rainfall

According to the prediction models used into the radio space communications systems design and the stratospheric height information at temperature 0°C under the rainfall conditions, the Consultative Committee of International Radio usually uses the following model to estimate the rainfall height [8]

$$h_{\mathrm{FR}}(\mathrm{km}) = \begin{cases} 5-0.075(\theta-23) & \theta > 23, \text{ in the Northern Hemisphere} \\ 5 & 0 \le \theta \le 23, \text{ in the Northern Hemisphere} \\ 5 & 0 \ge \theta \ge -21, \text{ in the Southern Hemisphere} \\ 5+0.1(\theta+21) & -71 \le \theta < -21, \text{ in the Southern Hemisphere} \\ 0 & \theta < -71, \text{ in the Southern Hemisphere} \end{cases} \quad (1)$$

where θ is the latitude, the name of the unit of θ is "degree". In the northern hemisphere, making use of the Xi'an latitude 34°, the calculation value of the rainfall height in Xi'an zone is 4175m.

Temperature Distribution of Convection Layer of the Atmosphere

According to the theory of atmospheric physics[9], within the range of the convection layer in the mid-latitude zone, the gradient of temperature is -6.5°C/km, so that the expression of the air temperature with respect to height h is T=288.15-0.0065h(SI).

Density Distribution of the Air

Due to the action of the Earth's gravitational field, the molecular number density n in the atmosphere around the Earth decreases while height increases. Based on the laws of the Boltzmann molecular distribution, the density distribution of air can be expressed as $\rho(z)=\rho_0\exp(-m_0gz/kT)$. Where ρ_0=1.225kg/m^3 at the height of 0, $k=1.38\times10^{-23}$(SI), $m_0=(0.029/6.02)\times10^{-23}$kg.

Diameter of Raindrops

The diameters of raindrops from the clouds to land vary generally within 0.5-6mm. Because drizzle is a special form with small and uniform rainfall, which diameters spread for 0.2-0.5mm, the drizzle is not considered in our work. Our studies include small, middle and big raindrops with diameter within 0.5-1.5mm, 1.5-4.0mm and 4.0-6.0mm respectively. When a raindrop diameter is greater than 6mm, air resistance will be more than its molecular cohesion which keeps raindrop a sphere, so big raindrop will be separated into many small raindrops [9].

Influence Factors of Raindrops Falling

Gravitational Attractive Force

The expression of the universal gravitational force is $F_g = GMm(R+z)^2$. Where $G=6.67\times10^{-11}$Nm^2/kg, M and R are the mass and the radius of the Earth respectively, z is the height of the raindrops from the ground, m is the mass of a raindrop. Because of the Earth's self-rotation, the gravitational force can be decomposed into gravity and a centripetal force perpendicular to axis making uniform circular motion. The centripetal force is $F = m\omega^2(R+h)\cos\theta$, which is very small about 0.3% compared with the gravity. Where ω is angular velocity of the Earth's self-rotation, θ is the rainfall latitude. While raindrops are subject to constant force in later study, the gravitational force is equal to the gravity.

Buoyancy

According to the Archimedes buoyancy law, the expression of floating force of raindrops is $F_f=(4/3)\rho\pi gr^3$. Where $\rho=\rho_0\exp(-m_0gz/kT)$ is the air density as shown in section 1.3.

Viscous Resistance

Any fluid flowing always possesses viscous characteristic, the viscous resistance is known as the "internal friction". In falling process, the viscous resistance acting on the raindrop is a variable force depending on the size and the shape, the falling speed, the air temperature, the density and the viscosity coefficient, and so on. The force is expressed as $F_r = \rho C_d S v^2/2$. Where ρ is air density, S is the area of the object projected to a plane which is perpendicular to the velocity vector, C_d is the resistance coefficient related to its size and geometry shape, and so on. R_e is Reynolds number of fluid. After the studying the range [7-9] of Reynolds number of raindrops regarded as spherical bodies, we find the expressions of viscous resistance of raindrops as follows

$$F_r=0.023\rho^{0.5624}\pi r^{1.5624}\upsilon^{1.5624} \quad \text{(small raindrop)} \qquad (2)$$

$$F_r=0.250\rho\pi r^2\upsilon^2 \quad \text{(middle raindrop)} \qquad (3)$$

$$F_r=11.056\rho^{1.929}\pi r^{2.929}\upsilon^{2.929} \quad \text{(big raindrop)} \qquad (4)$$

Coriolis Force

Coriolis force is an inertia force when object moves relative to the rotating Earth reference frame. As shown in Fig.1, a standard coordinate system or Cartesian coordinate system fixed on the ground is oxyz. The raindrop position projects to the origin "o" at initial moment, the ox axis pointing to the South is tangent to the meridian, the oy axis pointing to the East is tangent to the latitude line, the oz axis is the line jointing the Earth center and the origin, which direction is vertical upward. The latitude of the origin point θ is also the angle between oz axis and equatorial plane. The unit vectors in three directions are $\vec{i}$ and $\vec{j}$ and $\vec{k}$ respectively, so the Coriolis force acting on raindrop can be written as

$$\vec{F}_c = -2m\vec{\omega}\times\vec{v} = 2m\omega\sin\theta\frac{dy}{dt}\vec{i} - 2m(\sin\theta\frac{dx}{dt}+\cos\theta\frac{dz}{dt})\vec{j} + 2m\omega\cos\theta\frac{dy}{dt}\vec{k} \quad (5)$$

where m is the mass of a raindrop, $m=(4/3)\rho_{water}\pi r^3$, ω is angular velocity of the Earth's self-rotation, $\omega=7.27\times10^{-5}$rad/s, and $\vec{v}$ is the velocity of the raindrops.

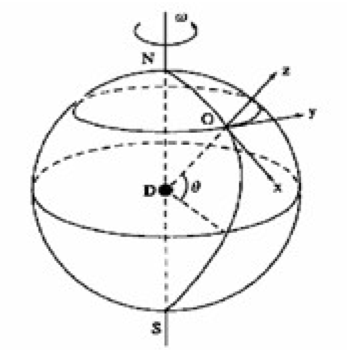

Fig.1. The Cartesian coordinate system of rainfall

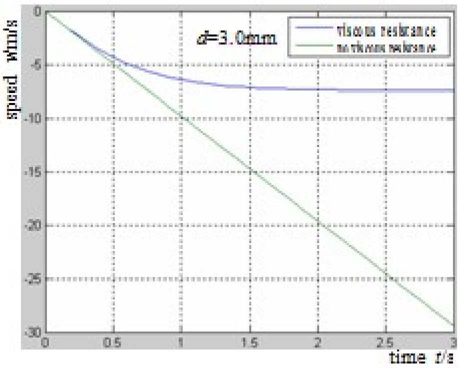

Fig.2. Influence to velocity from viscous resistance

Dynamics Equations and Mathematical Models of Raindrops Motion

Taking the Earth as reference system, substituting the components of the gravitational force, the buoyancy, the viscous resistance and the Coriolis force into the expression of Newton's second law, we establish three groups equations of raindrops in above coordinate system, namely, from left to right these dynamic equations of small, middle and big raindrops are as follows respectively

$$\begin{cases} m\dfrac{d^2x}{dt^2} = 2m\omega\sin\theta\dfrac{dy}{dt} \\ m\dfrac{d^2y}{dt^2} = -2m\omega(\dfrac{dx}{dt}\sin\theta+\dfrac{dz}{dt}\cos\theta) \\ m\dfrac{d^2z}{dt^2} = -G\dfrac{Mm}{(R+z)^2}+\dfrac{4}{3}\pi r^3 g\rho \\ +0.023\rho^{0.5624}\pi r^{1.5624}(\dfrac{dz}{dt})^{1.5624} \\ +2m\omega\cos\theta\dfrac{dy}{dt} \\ \rho=\rho_0\exp(-m_0gz/kT) \\ T=288.15-0.0065z \end{cases} \quad \begin{cases} m\dfrac{d^2x}{dt^2} = 2m\omega\sin\theta\dfrac{dy}{dt} \\ m\dfrac{d^2y}{dt^2} = -2m\omega(\sin\theta\dfrac{dx}{dt}+\cos\theta\dfrac{dz}{dt}) \\ m\dfrac{d^2z}{dt^2} = -G\dfrac{Mm}{(R+z)^2}+0.25\rho\pi r^2(\dfrac{dz}{dt})^2 \\ +\dfrac{4}{3}\pi r^3 g\rho+2m\omega\cos\theta\dfrac{dy}{dt} \\ \rho=\rho_0\exp(-m_0gz/kT) \\ T=28815-0.0065z \end{cases} \quad \begin{cases} m\dfrac{d^2x}{dt^2} = 2m\omega\sin\theta\dfrac{dy}{dt} \\ m\dfrac{d^2y}{dt^2} = -2m\omega(\sin\theta\dfrac{dx}{dt}+\cos\theta\dfrac{dz}{dt}) \\ m\dfrac{d^2z}{dt^2} = -G\dfrac{Mm}{(R+z)^2}+\dfrac{4}{3}\pi r^3 g\rho \\ +11.056\rho^{1.929}\pi r^{2.929}(\dfrac{dz}{dt})^{2.929} \\ +2m\omega\cos\theta\dfrac{dy}{dt} \\ \rho=\rho_0\exp(-m_0gz/kT) \\ T=288.15-0.0065z \end{cases} \quad (6)$$

Analysis and Discussion on all Influencing Factors

The Influence to Model from Constant Gravity

Let the gravity be a constant force mg, g=9.8m/s^2, i.e. $F_g=GMm/(R+z)^2=mg$. Taking middle raindrops for example(d=3mm), we calculate the dynamic characteristics as shown in Tab.1.

Table 1. The influences to the model from constant gravity and vary gravity for d=3.0mm.

Events	Speed $v(t)$ [m/s]	Height $z(v)$ [m]	Height $z(t)$ [m]	Eastward migration $y(t)$[m]	Southward migration $x(t)$ [m]
Observation point	at t=2.33s	at the height as v=7.39m/s	at the height z(639.3s)	at the migration y(639.3s)	at the migration x(639.3s)
Vary gravity	7.39	4161.8	4.33 (4175-4.33)	110.36	1.52
Constant gravity	7.38	4160.2	1.25 (4175-1.25)	111.28	1.56
absolute deviation	0.01	1.6	3.08	0.92	0.04
relative deviation	0.1353%	0.0380%	0.07385%	0.8336%	2.631%

From the above five relative deviations, if the vary gravity is regarded as constant gravity, the influence to raindrops model from constant gravity is obviously little. These reasons include that the

Earth's radius is much larger than raindrops falling height, i.e.6370km>>4715m, the acceleration g varies little with height changes in the 4715m range. This poor variety has also little effect on the raindrops falling compared with the influence to the model from viscous resistance. Thus we unworriedly choose the gravity acceleration to be 9.8m/s^2 in the related research.

The Influences to Model from Coriolis Force and Floating Force

Assuming the Coriolis force no existence, the raindrops are subject to gravitational force, viscous resistance and buoyancy. Taking middle raindrop for example, we have the data as shown in Table 2.

The Coriolis force effects on falling are negligible, except for eastward and southward migration. At one and two level approximation of ω, the falling effect by F_c along the z axis is also very small.

As per Eq.(5), because of the relation $\vec{v} \perp \vec{F}_c$, the Corciolis force F_c does not change the speed of moving raindrop. This simulation experiment results are highly consistent to the theoretical analysis.

Based on the dynamic data obtained with the MATLAB, if the buoyancy is no existence, the influences to three sorts of raindrops model from the floating force are both small.

Table 2. The influences to the model from no Coriolis force and Coriolis force for d=3.0mm.

Events	Speed $v(t)$ [m/s]	Height $z(v)$ [m]	Height $z(t)$ [m]	Eastward migration $y(t)$[m]	Southward migration $x(t)$ [m]
Observation point	v=terminal velocity	at the terminal velocity	at the height z(639.3s)	at the migration y(639.3s)	at the migration x(639.3s)
Coriolis force	7.39	4161.8	4.33(4175-4.33)	110.36	1.52
No Coriolis F_c	7.39	4161.6	4.32(4175-4.32)	0.00	0.00
absolute deviation	0.00	0.2	0.01		
relative deviation	0.00%	0.0048%	0.00024%		

The Influence to Model from Viscous Resistance

Taking the middle raindrop for example (d=3.0mm), we obtain the graphs of velocity with respect to time under two scenes of viscous resistance and no viscous resistance (in Fig.2).

The results show that raindrop movement is approximatiely to a free falling without viscous resistance, which final speed is very large, as well as a very short falling time. Raindrop subjected to the viscous resistance will get to a uniform motion state with acceleration zero, its falling time is long.

Obviously, viscous resistance is the most important factor influencing rainfall. The terminal uniform velocity of raindrop subjected to the force is very small. So the viscous resistance is a key aspect on study the SWC.

Conclusion

Because of the very small changes of gravity acceleration caused by the inertia centrifugal force, the changes of terminal velocity are ignored. All analysis results on raindrop falling in the case of the constant gravity acceleration may be accepted.

Viscous resistance is the most important factor influencing factor in raindrop falling. So the raindrop must get a terminal velocity before reaching the ground. The falling time of raindrops increases greatly, and its terminal velocity increases with increasing of the diameter of raindrop, as well as the larger diameter the larger terminal velocity.

The buoyancy is approximately 1/1000 of the gravity, so that the buoyancy has no effect on terminal velocity nearly, has a little effect on other parameters of the model. Compared to the middle raindrops and the big raindrop, the small raindrop influence subjected to the buoyancy is smaller.

The Coriolis force is only factor on eastward and southward falling of raindrops. The eastward migration is more obvious than the southward case. The greater falling height and the longer fall time may bring about the greater migration. Moreover, the larger diameter of the raindrop and the shorter falling time lead to the smaller migration. The Coriolis force only changes the velocity direction of the raindrop movement, does not effect on its speed. All above conclusions are obtained for Xi'an region.

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