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# Geometric approach to BRST-symmetry and $\mathbb{Z}_N$-generalization of superconnection

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**Abstract**

We propose a geometric approach to the BRST-symmetries of the Lagrangian of a topological quantum field theory on a four dimensional manifold based on the formalism of superconnections. Making use of a graded $q$-differential algebra, where $q$ is a primitive $N$-th root of unity, we also propose a notion of $\mathbb{Z}_N$-connection which is a generalization of a superconnection. In our approach the Lagrangian of a topological field theory is presented as the value of the curvature of a superconnection evaluated at an appropriate section of a vector bundle. Since this value of the curvature satisfies the Bianchi identity and representing the Bianchi identity in this case in the form of an operator applied to the mentioned above value of the curvature we obtain an operator which gives zero when applied to the Lagrangian. We show that this operator generates the BRST-transformations of the fields of a topological field theory on a four dimensional manifold.

## 1 Introduction

The topological quantum field theory on a four dimensional Riemannian manifold ($\mathrm{TQFT}_4$) proposed by E. Witten [9] leads to a field theoretic interpretation for the Donaldson invariants. According to the classification scheme [5] $\mathrm{TQFT}_4$ is a Witten type topological field theory. The distinguishing feature of a Witten type theory is that the Lagrangian $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \{\mathfrak{Q}, V\}, \tag{1.1}$$

where $\mathfrak{Q}$ is the BRST charge and $V$ is the functional depending on the multiplet of the fields of a theory. The BRST charge $\mathfrak{Q}$ is nilpotent (up to gauge transformation) and the Lagrangian $\mathcal{L}$ is gauge invariant. Applying the BRST charge $\{\mathfrak{Q},\ \}$ to both sides of (1.1) we obtain the relation

$$\{\mathfrak{Q}, \mathcal{L}\} = 0, \tag{1.2}$$

which shows that the Lagrangian of $TQFT_4$ is supersymmetric. The purpose of this paper is to show that from a geometric point of view the invariance of the Lagrangian (1.2) originates from a Bianchi identity if we interpret the Lagrangian $\mathcal{L}$ as the curvature of a superconnection and the BRST charge $\mathfrak{Q}$ as a covariant differential on an appropriate fibre bundle. It should be mentioned that in the case of a $TQFT_4$ this fibre bundle is the infinite dimensional principal bundle of all irreducible connections on a finite dimensional principal bundle over a four dimensional smooth Riemannian manifold, and the structure group of this infinite dimensional principal bundle of the irreducible connections is the group of gauge transformations or, by other words, the group of all preserving fibres automorphisms of a finite dimensional principal bundle.

The geometric approach to the Lagrangian of a $TQFT_4$ proposed in [3] is based on the theory of superconnections [4, 8]. In this approach the partition functions of $TQFT_4$ can be considered as the generalized Thom classes [8] of the infinite dimensional principal bundle of all irreducible connections. This is the reason why the partition functions of $TQFT_4$ do not depend on the metric of a four-dimensional Riemannian manifold, which means that they are topological invariants. It was shown [3] within the framework of a geometric approach to $TQFT_4$ that the Lagrangian of $TQFT_4$ can be considered as the curvature of a superconnection on a infinite dimensional vector bundle. In this paper using the same geometric approach we show that the invariance of the Lagrangian $\mathcal{L}$ with respect to BRST-supersymmetry (1.2) originates from the Bianchi identity for the curvature and we construct the operator of BRST-supersymmetry $\mathfrak{Q}$ extracting it from the Bianchi identity. We stress on the point that our geometric approach to the fields and the BRST-supersymmetry unlike the geometric approaches based on the structures of a finite dimensional fibre bundles uses the geometric structures of the infinite dimensional principal fibre bundle of all irreducible connections. The advantage of our approach is that it is consisent with the nature of the ghost fields which are the generators of an infinite dimensional Grassmann algebra and they anticommute not only with respect to the discrete indices but also at different points of a manifold. Consequently a point of a manifold may be viewed as a continuous index of a ghost field. For instant the field $\psi$ of a topological field theory in our approach is a 1-form on the infinite dimensional principal fibre bundle of all irreducible connections (section 5). The second advantage of our approach is that it clearly demonstrates that the Lagrangian of a theory is invariant with respect to the supersymmetry because it can be obtained from the curvature (which is a functional form on an infinite dimensional space) which in turn satisfies the infinite dimensional analog of the Bianchi identity. This part of our work is based on the preprint [1].

We propose a notion of a $\mathbb{Z}_N$-connection, where $N \geq 2$, which can be viewed as a generalization of a notion of $\mathbb{Z}_2$-connection or superconnection [2]. We use the algebraic approach to the theory of connections to give the definition of a $\mathbb{Z}_N$-connection and to explore its structure. It is well known that one of the basic structures of the algebraic approach to the theory of connections is a graded differential algebra with differential $d$ satisfying $d^2 = 0$. In order to construct a $\mathbb{Z}_N$-generalization of a superconnection for any $N > 2$ we make use of a $\mathbb{Z}_N$-graded $q$-differential algebra, where $q$ is a primitive $N$-th root of unity, with $N$-differential $d$ satisfying $d^N = 0$.

## 2 $\mathbb{Z}_N$-connection, curvature and Bianchi identity.

In this section we propose a notion of a $\mathbb{Z}_N$-connection which is based on the notion of a graded $q$-differential algebra ([7]), where $q$ is a primitive $N$-th root of unity. A notion of a $\mathbb{Z}_N$-connection is a generalization of the notion of a superconnection or $\mathbb{Z}_2$-connection.

Let $\mathcal{B} = \oplus_{k\in\mathbb{Z}}\mathcal{B}^k$ be an associative unital $\mathbb{Z}$-graded algebra over $\mathbb{C}$. We shall denote the grading of a homogeneous element $\omega \in \mathcal{B}$ by $|\omega|$, i.e. if $\omega \in \mathcal{B}^k$ then $|\omega| = k$. An algebra $\mathcal{B}$ is said to be a graded $q$-differential algebra, where $q$ is a primitive $N$-th root of unity ($N \geq 2$), if it is endowed with a linear mapping $d : \mathcal{B}^k \longrightarrow \mathcal{B}^{k+1}$ of degree 1 satisfying the graded $q$-Leibniz rule $d(\omega\,\omega') = d(\omega)\,\omega' + q^{|\omega|}\omega\, d(\omega'), \omega, \omega' \in \mathcal{B}$, and $d^N(\omega) = 0, \forall w \in \mathcal{B}$. A mapping $d$ is called a $N$-differential of a graded $q$-differential algebra. A graded differential algebra is a particular case of a graded $q$-differential algebra when $N = 2$ and $q = -1$.

The subspace $\mathcal{B}^0 \subset \mathcal{B}$ of elements of grading zero is the subalgebra of an algebra $\mathcal{B}$. A $N$-differential calculus over a unital associative algebra $\mathcal{A}$ is a pair $(\mathcal{B}, d)$, where $\mathcal{B}$ is a graded $q$-differential algebra with $N$-differential $d$ and $\mathcal{A} = \mathcal{B}^0$. For any $k \in \mathbb{Z}$ a subspace $\mathcal{B}^k$ of elements of grading $k$ has a structure of a bimodule over the algebra $\mathcal{B}^0$ and a graded $q$-differential algebra can be viewed as a $N$-differential complex ([6])

$$\ldots \xrightarrow{d} \mathcal{B}^{k-1} \xrightarrow{d} \mathcal{B}^k \xrightarrow{d} \mathcal{B}^{k+1} \xrightarrow{d} \ldots, \tag{2.1}$$

with differential $d$ satisfying the graded $q$-Leibniz rule.

Let $\mathcal{A}$ be a unital associative $\mathbb{C}$-algebra, $(\mathcal{B}, d)$ be a $N$-differential calculus over $\mathcal{A}$ and $\mathcal{E} = \oplus_{k\in\mathbb{Z}_N}\mathcal{E}^k$ be a $\mathbb{Z}_N$-graded left $\mathcal{A}$-module. Since a graded $q$-differential algebra $\mathcal{B}$ has a structure of a $(\mathcal{B}, \mathcal{B})$-bimodule the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ of the right $\mathcal{A}$-module $\mathcal{B}$ and the left $\mathcal{A}$-module $\mathcal{E}$ has a structure of a left $\mathcal{B}$-module. We denote this left $\mathcal{B}$-module by $\mathcal{E}_{\mathcal{B}}$, i.e. $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$.

The left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$ is a $\mathbb{Z}_N$-graded left $\mathcal{B}$-module if we consider $\mathcal{B}$ as a $\mathbb{Z}_N$-graded algebra and define the grading of an element $\omega \otimes \xi \in \mathcal{E}_{\mathcal{B}}$ by $|\omega \otimes \xi| = |\omega| + |\xi|$. Then

$$\mathcal{E}_{\mathcal{B}} = \oplus_{k\in\mathbb{Z}_N}\mathcal{E}^k_{\mathcal{B}}, \quad \mathcal{E}^k_{\mathcal{B}} = \oplus_{m+l=k}\, \mathcal{B}^m \otimes_{\mathcal{A}} \mathcal{E}^l, \tag{2.2}$$

where $k, l, m \in \mathbb{Z}_N$.

**Definition 1.** A $\mathbb{Z}_N$-graded $\mathcal{B}$-connection on a $\mathbb{Z}_N$-graded left $\mathcal{A}$-module $\mathcal{E}$ is a mapping $\nabla : \mathcal{E}^k_{\mathcal{B}} \longrightarrow \mathcal{E}^{k+1}_{\mathcal{B}}$ of grading 1 satisfying the condition

$$\nabla(\omega\,\zeta) = d(\omega)\,\zeta + q^{|\omega|}\omega\,\nabla(\zeta), \tag{2.3}$$

where $\omega \in \mathcal{B}$, $\zeta \in \mathcal{E}_{\mathcal{B}}$, and $d$ is the $N$-differential of a $\mathbb{Z}_N$-graded $q$-differential algebra $\mathcal{B}$.

A $\mathbb{Z}_N$-connection $D$ can be extended to act on the $\mathbb{Z}_N$-graded algebra $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ in a way consistent with the graded $q$-Leibniz rule if we define

$$D(A) = [D, A]_q = D \circ A - q^{|A|}A \circ D, \quad A \in \mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}). \tag{2.4}$$

It is evident that $D : \mathrm{End}^k_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}) \longrightarrow \mathrm{End}^{k+1}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ and

$$D(A\,B) = D(A) \circ B + q^{|A|}A \circ D(B). \tag{2.5}$$

It can be proved [2] that the $N$-th power of an endomorphism $D \in \mathrm{End}^1_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is the grading zero endomorphism of the left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$, i.e. $D^N \in \mathrm{End}^0_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$.

**Definition 2.** The curvature $F_D$ of a $\mathbb{Z}_N$-connection $D$ is the endomorphism $D^N$ of grading zero of the left $\mathbb{Z}_N$-graded $\mathcal{B}$-module $\mathcal{E}_\mathcal{B}$, i.e. $F_D = D^N \in \mathrm{End}^0_\mathcal{B}(\mathcal{E}_\mathcal{B})$.

**Theorem 1.** *For any $\mathbb{Z}_N$-connection $D$ the curvature $F_D$ of this connection satisfies the Bianchi identity $D(F_D) = 0$.*

**Proof.** We have $D(F_D) = [D, F_D]_q = D \circ F_D - F_D \circ D = D^{N+1} - D^{N+1} = 0$. ■

The notion of a $\mathbb{Z}_N$-connection, as it is given above, is based on the algebraic structures such as differential algebras and modules. It is a good question to ask whether a $\mathbb{Z}_N$-connection may be constructed geometrically as a connection on a fibre bundle. We hope that for $N > 2$ this can be done on a non-commutative analog of a fibre bundle. In the case of $N = 2$ (and $q = -1$) a $\mathbb{Z}_N$-connection can be realized as a connection on a classical (commutative) fibre bundle giving a well-known notion of a superconnection [8, 4]. Indeed let $\pi : E = E^+ \oplus E^- \longrightarrow M$ be a superbundle with a base $M^n$, where $M^n$ is a $n$-dimensional smooth manifold. In this case let $\mathcal{B} = \oplus_r \Omega^r(M^n)$ be the algebra of differential forms on a manifold $M^n$ and $d$ be the exterior differential of this algebra. It is evident that $\mathcal{B}$ is a $\mathbb{Z}_2$-graded algebra, where the grading of a homogeneous differential form is equal to its degree modulo 2. Let $\mathcal{B}^0 = \Omega^0(M^n) = \Gamma(M^n)$ be the algebra of smooth functions on a manifold $M^n$, and $\Gamma(E) = \Gamma(E)^+ \oplus \Gamma(E)^-$ be the left $\mathbb{Z}_2$-graded $\Gamma(M^n)$-module of smooth sections of a superbundle $E$. Then the tensor product $\mathcal{B} \otimes_\mathcal{A} \mathcal{E}$ is the space of $E$-valued smooth differential forms $\Omega(E) = \oplus_r \Omega^r(E)$ on a manifold $M^n$. The total grading of a homogeneous $E$-valued differential form is the sum of two gradings, where first is determined by the graded structure of the algebra of differential forms and the second is determined by the graded structure of a superbundle $E$. The space $\mathrm{End}_\mathbb{C}(\mathcal{E}_\mathcal{B})$ is the space of differential forms on a manifold $M$ with the values in the superbundle $\mathrm{End}(E)$. The $q$-commutator becomes the supercommutator if we take $q = -1$. Finally the definition of a $\mathbb{Z}_N$-connection coincides in this special case with the definition of a superconnection as it is given in [4].

## 3 Superconnection and supersymmetry operator

Let $L$ be an odd endomorphism of a supervector bundle $E$, i.e. $L : E^\pm \longrightarrow E^\mp$. Since $\Omega(E) \cong \Omega(M^n) \otimes_{\Gamma(M^n)} \Gamma(E)$ an endomorphism $L$ can be extended from $E$ to the space of $E$-valued forms $\Omega(E)$ if we put

$$L(\sigma \otimes s) = (-1)^{|\sigma|}\, \sigma \otimes L(s), \tag{3.1}$$

where $\sigma \in \Omega(M^n), s \in \Gamma(E)$, and $|\sigma|$ is the grading of $\sigma$. Let $\mathfrak{D}$ be a connection on a superbundle $E$, i.e. $\mathfrak{D} : \Omega^r(E) \longrightarrow \Omega^{r+1}(E)$. Then the linear operator $D = \mathfrak{D} + L$ is the superconnection on a superbundle $E$. Locally a superconnection $D$ has the form

$$D = d + \theta + L, \tag{3.2}$$

where $\theta$ is the matrix of 1-forms of a connection $\mathfrak{D}$. Using the definition 2 we find the local form of the curvature $F_D \in \Omega^2(\mathrm{End}(E))$ of the superconnection $D$

$$F_D = D^2 = \Theta + d\,L + [\theta, L] + L^2, \tag{3.3}$$

where $\Theta$ is the local curvature 2-form of a connection $\mathfrak{D}$. From the theorem 1 it follows that the curvature $F_D$ satisfies the Bianchi identity which locally can be written as follows

$$d\,F_D = [F_D, \theta + L]. \tag{3.4}$$

Let $Q$ be a principal fibre bundle over a manifold $M^n$ with structure group $G$ and $r : G \longrightarrow GL^+(W)$ be a representation of $G$ on a supervector space $W$ which preserves the graded structure of $W$, i.e. $r(g) : W^{\pm} \longrightarrow W^{\pm}, \forall g \in G$. The associated vector bundle $E = Q \times_G W$ is a superbundle. The space $\Omega(E)$ of $E$-valued forms is isomorphic to the space $\Omega(Q \times W)$ of equivariant $(Q \times W)$-valued forms of the trivial superbundle $Q \times W$, i.e. $\Omega(E) \cong \Omega(Q \times W)$. Let $\omega$ be a connection on a principal bundle $Q$ and $\nabla_\omega(\sigma) = d\sigma + r'(\omega) \wedge \sigma$ be the covariant differential induced by this connection on the associated superbundle $E$, where $r' : \underline{G} \longrightarrow \mathrm{End}^+(W)$ is the infinitesimal representation of the Lie algebra $\underline{G}$ of $G$ and $\sigma \in \Omega(Q \times W)$.

Now let $L$ be an odd equivariant endomorphism of the trivial superbundle $Q \times W$, that is, $L \in \mathrm{End}^-(Q \times W)$ and $L_{pg} = r(g)^{-1} L_p\, r(g), p \in Q, g \in G$. Having extended $L$ to the space of equivariant $(Q \times W)$-valued forms by means of (3.1) we construct the superconnection $D_\omega$ which locally can be expressed in the terms of a connection $\omega$ and endomorphism $L$ as follows $D_\omega = \nabla_\omega + L = d + r'(\omega) + L$. Locally the curvature $F_\omega$ of this superconnection $D_\omega$ can be written as follows

$$F_\omega = r'(\Theta_\omega) + dL + [r'(\omega), L] + L^2, \tag{3.5}$$

where $\Theta_\omega$ is the curvature 2-form of $\omega$. The Bianchi identity in this case take on the form

$$dF_\omega = [F_\omega, r'(\omega) + L]. \tag{3.6}$$

Let $P$ be a principal fiber bundle over a smooth manifold $M^n$ with the structure group $G = \mathrm{Spin}(p)$, where $p$ is an even integer. Making use of the homomorphism $\mathrm{Spin}(p) \longrightarrow \mathrm{O}(p)$ of the spinor group on to the group of orthogonal matrices one can construct the associated vector bundle $E = P \times_G V$, where $V = \mathbb{R}^p$. The trivial bundle $Q = P \times V$ can be considered as the principal bundle over $E$.

Let $C_p$ be a Clifford algebra over $\mathbb{C}$ generated by $\gamma_1, \gamma_2, ..., \gamma_p$ and $S^2$ be the complex plane $\mathbb{C}^2$. If we associate the well known Pauli matrices to the elements $\gamma_1, \gamma_2, -i\gamma_1\gamma_2$ of the Clifford algebra $C_2$ then $S^2$ becomes the supermodule over the algebra $C_2$. Similarly the tensor product $S^p = S^2 \otimes ... \otimes S^2$ of the supermodules is the supermodule over the algebra $C_p$. Clearly $C_p \cong \mathrm{End}(S^p)$, and we denote this isomorphism by $\nu : C_p \longrightarrow \mathrm{End}(S^p)$. Let $C_p^*$ be the group of invertible elements of the Clifford algebra $C_p$. There is the group homomorphism $\rho : \mathrm{Spin}_{\mathbb{C}}(p) \longrightarrow C_p^*$ such that the infinitesimal homomorphism of the corresponding Lie algebras has the form $\rho'(a) = \imath 4\, \gamma^t a \gamma$, where $a$ is an element of the Lie algebra of $\mathrm{Spin}_{\mathbb{C}}(p)$ (skew-symmetric matrix) and $\gamma$ is the column-matrix, whose entries are the generators of $C_p$. The restriction of $\rho'$ on to the real subspace of the Lie algebra of $\mathrm{Spin}_{\mathbb{C}}(p)$ induces the homomorphism $\rho : \mathrm{Spin}(p) \longrightarrow C_p^*$ between the group of real spinors $\mathrm{Spin}(p)$ and $C_p^*$. This homomorphism gives the representation (preserving the graded structure) of the real spinor group $\mathrm{Spin}(p)$ on the supervector space $S^p$ and we can consider the supervector bundle $\mathcal{W} = Q \times_G S^p = (P \times V) \times_G S^p$ over the vector bundle $E$.

Let $\omega$ be a connection 1-form on a principal bundle $P$. We extend this connection form to the connection 1-form on the bundle $P \times V$ by means of $\omega_{(q)}(X, v) = \omega_p(X)$, where $q = (p, x) \in P \times V,\ X \in T_pP,\ v \in T_xV$. Let $l$ be the $C_p$-valued function on the trivial bundle $Q = P \times V$ defined by $l_q = ix^k\gamma_k$, where $q = (p, x) \in Q, p \in P, x \in V = \mathbb{R}^p$, $x^1, x^2, \ldots, x^p$ are the coordinates of $x$, and $\gamma_1, \gamma_2, \ldots, \gamma_p$ are the generators of the Clifford algebra $C_p$. Since $S^p$ is the supermodule over the Clifford algebra $C_p$, i.e. any element of $C_p$ determines the endomorphism of the supervector space $S^p$, we can extend the $C_p$-valued function $l$ to the odd endomorphism $L$ of the trivial bundle $Q \times S^p$ by means of $L(q, t) = (q, l_q(t))$, where $q \in Q, t \in S^p$ and $l_q : S^p \longrightarrow S^p$. It can be easily checked that $L$ is equivariant under the action of $G$.

The family of locally defined operators

$$D_\omega = d + \omega + L = d + \imath 4\omega_{ij}\gamma_i\gamma_j + ix^i\gamma_i, \tag{3.7}$$

determines the superconnection on the bundle $Q \times_G S^p$ over $E$. The curvature of this superconnection is

$$F_\omega = -x^2 + i\,\nabla_\omega x^i\gamma_i + \frac{1}{4}\Theta^\omega{}_{ij}\gamma_i\gamma_j, \tag{3.8}$$

where $\Theta^\omega{}_{ij}$ is the curvature 2-form of the connection $\omega$, $\nabla_\omega$ is the ordinary covariant differential, and $x^2 = < x, x > = \sum_{i=1}^p (x^i)^2$ is the standard scalar product in $V = \mathbb{R}^p$.

Let $\mathfrak{G}_p$ be a Grassmann algebra with $p$ generators $\xi_1, \xi_2, \ldots, \xi_p$ and the unity element 1. If we replace in (3.8) the generators $\gamma_1, \gamma_2, \ldots, \gamma_p$ of the Clifford algebra by the generators $\xi_1, \xi_2, \ldots, \xi_p$ of the Grassmann algebra, i.e. $\gamma_k \longrightarrow \xi_k$, then we get the polynomial

$$\Phi_\omega = -x^2 + i\nabla_\omega x^k\xi_k + \frac{1}{4}\Theta^\omega{}_{ij}\xi_i\xi_j. \tag{3.9}$$

It can be shown [3] that taking this polynomial as a starting point one can derive the finite dimensional analog of the Lagrangian $\mathcal{L}$ of a topological quantum field theory on a four-dimensional manifold. The fundamental property of the Lagrangian is the BRST-invariance (1.2) which is a crucial tool for deriving the Donaldson invariants of a four-dimensional manifold. In this paper our aim is to show that the origin of this invariance is the Bianchi identity for the curvature. Indeed the polynomial $\Phi_\omega$ is obtained from the curvature $F_\omega$ by the replacement $\gamma_k \longrightarrow \xi_k$ but the curvature satisfies the Bianchi identity $D_\omega(F_\omega) = 0$ which is very similar to the BRST-invariance of the Lagrangian $\{\mathfrak{Q}, \mathcal{L}\} = 0$. Taking into account that the Lagrangian $\mathcal{L}$ can be derived from the curvature $F_\omega$ we expect that the BRST-symmetry of $\mathcal{L}$ can be derived from the connection $D_\omega$ if we consider it as an operator acting on the space of differentials forms.

The basic element of our construction is the representation $\mu : C_p \longrightarrow \mathrm{End}(\mathfrak{G}_p)$ of the elements of the Clifford algebra $C_p$ by the linear operators acting on the Grassmann algebra $\mathfrak{G}_p$ which is determined by

$$\gamma_{2j} = \hat{\xi}_j + \partial_j,\ \gamma_{2j+1} = i(\hat{\xi}_j - \partial_j), \qquad j = 1, 2, \ldots m,\ p = 2m \tag{3.10}$$

where $\partial_j$ is the partial derivative on the Grassmann algebra with respect to a generator $\xi_j$ and $\hat{\xi}_j$ is the operator of multiplication by a generator $\xi_j$. This representation allows us to relate the curvature $F_\omega$ to the polynomial $\Phi_\omega$ not formally by the replacement of generators $\gamma_k \longrightarrow \xi_k$ but geometrically with the help of the superbundle $\mathcal{W} = Q \times_G \mathfrak{G}_p$. In other words the construction used in [8] is based on the sequence of homomorfisms $\mathrm{Spin}(p) \xrightarrow{\rho} C_p \xrightarrow{\nu} \mathrm{End}(S^p)$ but we use the sequence $\mathrm{Spin}(p) \xrightarrow{\rho} C_p \xrightarrow{\mu} \mathrm{End}(\mathfrak{G}_p)$. The local connection $D_\omega$ and the curvature of this connection $F_\omega$ are given in the terms of the bundle $Q \times_G S^p$, and if we wish to express them in the coordinates of the bundle $\mathcal{W} = Q \times_G \mathfrak{G}_p$ we should use the representation (3.10). We calculate

$$D_\omega = d + \frac{1}{2}\omega_{kl}\hat{\xi}_k\partial_l + ix^k\,\partial_k + \frac{1}{4}\omega_{kl}\,\partial_k\,\partial_l + \frac{1}{4}\omega_{kl}\,\hat{\xi}_k\hat{\xi}_l + ix^k\,\hat{\xi}_k, \tag{3.11}$$

$$F_\omega = \frac{1}{2}\Theta^\omega_{kl}\hat{\xi}^k\partial_l + i\,\nabla_\omega x^k\partial_k + \frac{1}{4}\Theta^\omega_{kl}\partial_k\partial_l - x^2 + i\,\nabla_\omega x^k\,\hat{\xi}_k + \frac{1}{4}\Theta^\omega_{kl}\hat{\xi}_k\hat{\xi}_l. \tag{3.12}$$

The trivial fibre bundle $Q \times \mathfrak{G}_p$ has the section $e : Q \longrightarrow Q \times \mathfrak{G}_p$ which is defined by $e(q) = (q, 1)$, where 1 is the unity element of $\mathfrak{G}_p$. Since this section is equivariant with respect to the action of $G$ it induces the section of the bundle $Q \times_G \mathfrak{G}_p$ which will be denoted by the same symbol $e$. Acting on this section by the curvature (3.12) which is now the $\mathrm{End}(Q \times_G \mathfrak{G}_p)$-valued 2-form we obtain $(Q \times_G \mathfrak{G}_p)$-valued 2-form

$$F_\omega(e) = -x^2 + i\nabla_\omega x^k\xi_k + \frac{1}{4}\Theta^\omega_{kl}\xi_k\xi_l. \tag{3.13}$$

Comparing the above expression with (3.9) we see that this form coincides with $\Phi_\omega$ and the first aim of our construction to give $\Phi_\omega$ a geometric meaning in the framework of an appropriate fibre bundle is achieved.

Now our aim is to find an operator which will vanish being applied to the $F_\omega(e)$ (or to the $\Phi_\omega$ which is the same) extracting it from the Bianchi identity for the curvature $F_\omega$. This identity can be written in the form

$$d\,F_\omega + [D_\omega - d, F_\omega] = 0. \tag{3.14}$$

Making use of (3.11),(3.12) and collecting together all terms containing a partial derivative on the first place from the right we can put the Bianchi identity as follows

$$d\,F_\omega + (\omega_{kl}\hat{\xi}_k\partial_l + 2\,ix^k\,\partial_k)(-x^2 + i\,\nabla_\omega x^k\,\hat{\xi}_k + \frac{1}{4}\Theta^\omega_{kl}\hat{\xi}_k\hat{\xi}_l) + \mathcal{R} = 0, \tag{3.15}$$

where $\mathcal{R}$ denotes the sum of all terms containing a partial derivative on the first place from the right. Acting by the operator standing in the left-hand side of the above relation (3.15) on the constant section $e$ and taking into account that $\mathcal{R}$ vanishes in this case we obtain

$$(d + \omega_{kl}\hat{\xi}_k\partial_l + 2\,ix^k\,\partial_k)(F_\omega(e)) = 0. \tag{3.16}$$

Thus the Bianchi identity gives us the operator which vanishes on $F_\omega(e) = \Phi_\omega$ and this operator is

$$\mathcal{Q}_\omega = d + \omega_{kl}\hat{\xi}_k\partial_l + 2\,ix^k\,\partial_k. \tag{3.17}$$

The first two terms in the operator $\mathcal{Q}_\omega$ can be replaced by the covariant differential $\nabla_\omega$. Indeed since a connection $\omega$ is compatible with the metric of the trivial bundle $P \times V$ we have $dx^2 =< \nabla_\omega x, x > + < x, \nabla_\omega x >$. Therefore in the next section we will use the operator $\mathcal{Q}_\omega$ in the equivalent form

$$\mathcal{Q}_\omega = \nabla_\omega + 2\,ix^k\partial_k. \tag{3.18}$$

# 4 Supersymmetry in finite dimensional case

The aim of this section is to show that making use of the operator $\mathcal{Q}_\omega$ obtained from the Bianchi identity in the previous section one can get the supersymmetry transformations of a finite dimensional model of a topological quantum field theory.

Let $P$ be a Riemannian manifold of dimension $l = 2m + r$, $G$ be a $r$-dimensional compact connected Lie group, $\underline{G}$ be the Lie algebra of the group $G$, $V$ be a real vector space of dimension $p = 2m$ with an inner product denoted by $<,>$, and $\rho$ be an orthogonal representation of $G$ on $V$. We also suppose that $G$ acts freely on $P$ by isometries. In this case $P$ is a principal fiber bundle over $2m$-dimensional base manifold $M$. Let $E = P \times_G W$ be the associated vector bundle, where $W = \underline{G} \oplus V$, and $G$ acts on $\underline{G}$ by adjoint representation. Obviously $E = \mathrm{Ad}(P) \oplus \tilde{E}$, where $\mathrm{Ad}(P) = P \times_G \underline{G}$ and $\tilde{E} = P \times_G V$.

Let $(x^1, x^2, ..., x^{2m}) = (x^\mu)$ be the coordinates of $V$, $(\lambda^1, \lambda^2, ..., \lambda^d) = (\lambda^i)$ be the coordinates of the Lie algebra $\underline{G}$ with respect to a basis $\{t_1, t_2, ..., t_d\}$. Then any element $\lambda$ of $\underline{G}$ is the linear combinition $\lambda = \lambda^i t_i$. Let $(\xi_1, \xi_2, ..., \xi_l) = (\chi_1, ..., \chi_{2m}, \zeta_1, ..., \zeta_r) = (\chi_\mu, \zeta_i)$ be the generators of the Grassmann algebra $\mathfrak{G}_l$. If $\omega$ is a connection 1-form on the principal bundle $P$ then the covariant differential $\nabla_\omega$ on the vector bundle $E$ can be decomposed into the sum

$$\nabla_\omega = \nabla'_\omega + \nabla''_\omega, \tag{4.1}$$

where $\nabla'_\omega = d + \mathrm{ad}(\omega)$ is the covariant differential on $\mathrm{Ad}(P)$ and $\nabla''_\omega = d + \rho'(\omega)$ is the covariant differential on $\tilde{E}$. The matrix of the curvature of the bundle $E$ splits into the following blocks

$$\begin{pmatrix} \mathrm{ad}(\Theta^\omega) & 0 \\ 0 & \rho'(\Theta^\omega) \end{pmatrix}, \tag{4.2}$$

where $\Theta^\omega$ is the curvature 2-form of a connection $\omega$. The 2-form $F_\omega(e)$ obtained in the previous section in the coordinates of the vector bundle $E$ has the form

$$\begin{aligned} F_\omega(e) = {} & \frac{1}{4}\mathrm{ad}(\Theta^\omega)_{ij}\zeta_i\zeta_j + \frac{1}{4}\rho'(\Theta^\omega)_{\mu\nu}\chi_\mu\chi_\nu \\ & + i\nabla'_\omega\lambda^i\zeta_i + i\nabla''_\omega x^\mu\chi_\mu - < x, x > - \mathrm{Tr}(\lambda, \lambda), \end{aligned} \tag{4.3}$$

where Tr is the Killing form on the algebra $\underline{G}$. The operator $\mathcal{Q}_\omega$ in the coordinates of $E$ has the form

$$\mathcal{Q}_\omega = \nabla'_\omega + \nabla''_\omega + 2i\lambda^i\partial_i + 2ix^\mu\partial_\mu, \tag{4.4}$$

where $\partial_i$ is a partial derivative with respect to $\zeta_i$ and $\partial_\mu$ is a partial derivative with respect to $\chi_m u$. From the previous section it follows that

$$\mathcal{Q}_\omega(F_\omega(e)) = 0. \tag{4.5}$$

The fibre bundle $E$, the coordinates of this bundle and the forms of connection and curvature may be viewed as a finite dimensional model of a field theory. From this point of view the multiplet of "bosonic fields" of a theory consists of $\phi = F_\omega$ (curvature), $x^\mu$ (the coordinates of $V$) and $\lambda^i$ (the coordinates of $\underline{G}$). The multiplet of the "fermionic fields" consists of $(\zeta_i, \chi_\mu)$ (the generators of the Grassmann algebra $\mathfrak{G}_l$), $\eta^i$ (the 1-forms $\nabla'_\omega \lambda^i$), $\varphi^\mu$ (the 1-forms $\nabla''_\omega x^\mu$). The form (4.3) may be viewed as the Lagrangian of a field theory

$$\mathcal{L} = \frac{1}{4}\mathrm{ad}(\phi)_{ij}\zeta_i\zeta_j + \frac{1}{4}\rho'(\phi)_{\mu\nu}\chi_\mu\chi_\nu + i\eta^i\zeta_i + i\varphi^\mu\chi_\mu - < x, x > -\mathrm{Tr}(\lambda,\lambda). \tag{4.6}$$

Acting by the operator (4.4) on each field from the mentioned above multiplets we obtain the supersymmetry transformations of our theory

$$\begin{aligned} \mathcal{Q}_\omega(\phi) &= 0, \quad \mathcal{Q}_\omega(\chi) = 2ix, \quad \mathcal{Q}_\omega(\varphi) = \rho'(\phi)x, \quad \mathcal{Q}_\omega(\lambda) = \eta. \\ \mathcal{Q}_\omega(x) &= \varphi, \quad \mathcal{Q}_\omega(\zeta) = 2i\lambda, \quad \mathcal{Q}_\omega(\eta) = \mathrm{ad}(\phi)\lambda, \end{aligned} \tag{4.7}$$

The supersymmetry transformations (4.7) are not nilpotent. Indeed if we apply twice the supersymmetry $\mathcal{Q}_\omega$ then the transformations of the fields $x, \lambda, \eta, \varphi$ take on the form

$$\mathcal{Q}^2_\omega(\Psi) = R(\phi)\Psi, \tag{4.8}$$

where $R$ is either $r$ or $\rho'$. The twice applied supersymmetry $\mathcal{Q}_\omega$ in the case of the fields $\zeta, \chi$ gives

$$\mathcal{Q}^2_\omega(\zeta) = 2i\eta, \qquad \mathcal{Q}^2_\omega(\chi) = 2i\varphi. \tag{4.9}$$

These transformations are on-shell nilpotent since the equations for the fields $\eta$ and $\chi$ have the form

$$\zeta^i = 0, \qquad \varphi^\mu = 0. \tag{4.10}$$

In order to obtain the supersymmetry transformations of a topological field theory on a four dimensional manifold (within the framework of its finite dimensional model) we shall use a fibre bundle $P$ instead of the associated vector bundle $E$. Let us suppose that there is a smooth section of the bundle $\tilde{E} = P \times_G V$ induced by a $G$-invariant section of the trivial bundle $P \times V$. This $G$-equvariant section $p \longrightarrow (p, x)$, where $p \in P, x \in V$, determines on $P$ the $V$-valued function $\Sigma : p \longrightarrow x = \Sigma(p)$ which satisfies $\Sigma(p\,g) = g^{-1}\Sigma(p)$ for any $g \in G$. The Lagrangian (4.6) becomes the $G$-equivariant form on the principal fibre bundle $P$ if we replace $x$ in the expression (4.6) for the Lagrangian by this $V$-valued function $\Sigma$.

Let $a = (a^1, a^2, ..., a^l) = (a^\alpha)$ be the local coordinates on a principal fibre bundle $P$, and $(\psi^1, \psi^2, ..., \psi^l)$ be their differentials. Then $x = \Sigma(a)$ and the Lagrangian (4.6) depends on the local coordinates of $P$ which may be viewed as the new fields of our theory. Let $TP$ be the tangent bundle of a principal fibre bundle $P$, and let $\psi = \psi^\alpha X_\alpha$ be a $TP$-valued 1-form on $P$, where $\{X_1, X_2, ..., X_l\}$ is a local basis of $T_pP$ dual to $(\psi^1, \psi^2, ..., \psi^l)$. We remind that the field $\varphi$ may be viewed as a 1-form on the vector bundle $P \times V$. Consequently the pull-back of this form by $\Sigma$ is the 1-form $\Sigma^*(\varphi)$ on $P$ which can be expressed by means of $a, \psi$. Thus the multiplet of bosonic fields of the theory becomes now the set $\phi, a, \lambda$ and the multiplet of fermion fields is $\psi, \chi, \zeta, \eta$.

It is shown in [3] that the horizontality condition is very important in obtaining the representative of a Thom class of a vector bundle. We also use the horizontality condition in deriving the supersymmetry transformations of the fields. Therefore we can identify the field $\varphi$ with exterior differential $dx$ instead of $\nabla''_\omega x$ keeping in mind the necessity of taking the horizontal part. Thus the pull-back by $\Sigma$ of the form $\varphi$ is the 1-form

$(\Sigma^*\varphi)(\psi) = D\Sigma(\psi)$, where $D\Sigma : TP \longrightarrow V$ is the differential of the section $\Sigma$. We have changed the structure of the field $\varphi$ ($\nabla''_\omega x \longrightarrow dx$ ) and this will change it transformation in the supersymmetry transformations (4.7). This in turn will break the invariance of the Lagrangian $\mathcal{L}$. Since the operator (4.4) generating the supersymmetry transformations does not depend on the new fields $a, \psi$ we shall use this degree of freedom to choose transformations for $a, \psi$ in such a way that the transformation for $\varphi$ would be the same as in (4.7). In this way we shall retain the invariance of the Lagrangian with respect to the previously found supersymmetry. We obtained the transformation for $\varphi$ by applying twice the covariant differential $\nabla''_\omega$ to $x$. Since $\nabla''_\omega x$ is replaced by $dx$ we have to evaluate the horizontal part of $D\Sigma(\psi)$ then to differentiate it and finally to evaluate the horizontal part of the resulting 2-form. Obviously this procedure is equivalent to applying twice the covariant differential. The horizontal part $\psi_H$ of $\psi$ can be evaluated by means of the formula

$$\psi_H = \psi - \Lambda\omega, \tag{4.11}$$

where $\Lambda : \underline{G} \longrightarrow TP$ is the infinitesimal action of $G$ on $P$. Taking the horizontal part and differentiating as it is explained above we get

$$\begin{aligned} \mathcal{Q}_\omega(D\Sigma(\psi)) = (d\,D\Sigma(\psi_H))_H &= D^2\Sigma(\mathcal{Q}_\omega(a)_H, \psi_H) + D\Sigma(d(\psi_H)_H) \\ &= D^2\Sigma(\mathcal{Q}_\omega(a)_H, \psi_H) - D\Sigma((d(\Lambda\omega)_H)) \\ &= D^2\Sigma(\mathcal{Q}_\omega(a)_H, \psi_H) - D\Sigma(\Lambda(d\omega)_H) \\ &= D^2\Sigma(\mathcal{Q}_\omega(a)_H, \psi_H) - D\Sigma(\Lambda\Theta^\omega) \\ &= D^2\Sigma(\mathcal{Q}_\omega(a)_H, \psi_H) - D\Sigma(\Lambda\phi). \end{aligned} \tag{4.12}$$

On the other hand

$$\mathcal{Q}_\omega(D\Sigma(\psi)) = D^2\Sigma(\mathcal{Q}_\omega(a), \psi) + D\Sigma(\mathcal{Q}_\omega, \psi). \tag{4.13}$$

We define the supersymmetry transformation of the field $a$ by the formula

$$\mathcal{Q}_\omega(a) = \psi. \tag{4.14}$$

As a consequence of this definition the second order differential $D^2\Sigma$ vanishes in (4.12) and (4.13). Comparing (4.12) and (4.13)we obtain the supersymmetry transformation for the field $\psi$

$$\mathcal{Q}_\omega(\psi) = -\Lambda\phi. \tag{4.15}$$

Finally the supersymmetry $\mathcal{Q}_\omega$ in terms of the fields $\phi, a, \lambda, \psi, \chi, \zeta, \eta$ take on the form

$$\mathcal{Q}_\omega(\phi) = 0, \qquad \mathcal{Q}_\omega(a) = \psi, \qquad \mathcal{Q}_\omega(\lambda) = \eta, \tag{4.16}$$

$$\mathcal{Q}_\omega(\psi) = -\Lambda\phi, \quad \mathcal{Q}_\omega(\chi) = 2i\Sigma, \quad \mathcal{Q}_\omega(\eta) = \mathrm{ad}(\phi)\lambda, \quad \mathcal{Q}_\omega(\zeta) = 2i\lambda. \tag{4.17}$$

## 5 Supersymmetry in infinite dimensional case

Our aim in this section is to show that the supersymmetry transformations obtained from the Bianchi identity in a finite dimensional case give the supersymmetry transformations of a topological field theory on a four dimensional manifold [9].

Let us remind that in the previous section $P$ is a finite dimensional Riemannian manifold and $G$ is a finite dimensional compact connected Lie group acting on $P$ by isometries. In this section let $P$ be the infinite dimensional affine space of all irreducible connections of a finite dimensional principal fibre bundle, and $G$ be the infinite dimensional group of gauge transformations or the group of all automorphisms of a same finite dimensional principal fiber bundle. We assume that a base manifold of a finite dimensional principal fibre bundle has dimension four. It is well known that one can define a norm on the infinite dimensional space $P$ in such a way that $P$ becomes an infinite dimensional Banach manifold. Similarly one can define a norm on the infinite dimensional group $G$ of gauge transformations in such a way that $G$ becomes an infinite dimensional Banach group, and $G$ acts on $P$ by isometries.

Since a base manifold of our finite dimensional principal fibre bundle is a four dimensional Riemannian manifold there is the Hodge operator on a base manifold and we can decompose the space of all 2-forms on this manifold with values in the associated adjoint vector bundle into the direct sum of self-dual 2-forms and the space of anti-self-dual 2-forms. Let $V$ be the infinite dimensional space of anti-self-dual 2-forms. Then the action of $G$ on $V$, denoted in the previous section by $\rho$, is defined by $g \cdot \theta = g^{-1}\,\theta\, g$, where $g \in G$ is a gauge transformation and $\theta$ is a anti-self-dual 2-form. Let $A$ be a point of the infinite dimensional principal fibre bundle $P$, i.e. $A$ is an irreducible connection on a finite dimensional principal fibre bundle. Then the infinitesimal action of $G$ on $P$ is the covariant differential $\nabla_A$. We define the $G$-equivariant section of the trivial bundle $P \times V$ denoted in the previous section by $\Sigma$ with the help of the formula

$$\Sigma(A) = F_A^-, \tag{5.1}$$

where $F_A^-$ is the anti-self-dual part of the curvature $F_A$ of a connection $A$.

The supersymmetry transformations (4.16) - (4.17) of the previous section expressed in the terms of this section take on the form

$$\mathcal{Q}_\omega(\phi) = 0, \qquad \mathcal{Q}_\omega(\psi) = -\nabla_A \phi, \tag{5.2}$$

$$\mathcal{Q}_\omega(A) = \psi, \qquad \mathcal{Q}_\omega, \chi\} = 2i\, F_A^-, \tag{5.3}$$

$$\mathcal{Q}_\omega(\lambda) = \eta, \qquad \mathcal{Q}_\omega(\eta) = [\phi, \lambda], \tag{5.4}$$

where $\omega$ is a connection on the infinite dimensional principal fibre bundle $P$. The supersymmetry transformations (5.2) - (5.4) are the supersymmetry transformations of a topological quantum field theory on a four dimensional smooth manifold [9].

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