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Total Differentiation Under Jet Composition

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Abstract

Total differentiation operators as linear vector fields, their flows, invariants and symmetries form the geometry of jet space. In the jet space the dragging of tensor fields obeys the exponential law.

The composition of smooth maps induces a composition of jets in corresponding jet spaces. The prolonged total differentiation operators generalize the differentiation of composite function. The relations between Cartan forms under the jet composition are described.

Key words: jet, jet space, total differentiation operator, Cartan form.
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Introduction

When speaking about *Group Analysis*, *Symmetry Analysis* or *Secondary Calculus*, when prolonging differential equations and group operators or classifying singularities of smooth maps, we are always dealing with jets of maps, see [1]–[4],[7].

In the present paper we consider infinite jets of smooth maps $\mathbb{R}^n \to \mathbb{R}^m$. The set of infinite jets $\mathfrak{J}_{n,m}$ is a bundle space with n-dimensional base and infinite-dimensional fibers. In $\mathfrak{J}_{n,m}$ total differentiation operators (TDOs) D and Cartan forms ω are defined. At the same time the additive group $\mathbb{R}^n$ acts in $\mathfrak{J}_{n,m}$ along orbits that are integral manifolds of distribution spanned by D. In $\mathfrak{J}_{n,m}$ the operators D are linear vector fields and their flows are determined by the exponential law. The same law determines invariants and infinitesimal symmetries of D.

In the first section we introduce an algebraic scheme for calculation of invariants and symmetries of D in $\mathfrak{J}_{1,1}$. This scheme is universal in the following sense. There is a triple (D, t, U) defined in $\mathfrak{J}_{1,1}$, where D is a TDO, t is its parameter and U is the set of fiber coordinates. Let X be a vector field with canonical parameter s on a manifold M, and let f be a smooth function differentiable with respect to X. Then we associate a triple (X, s, F), where F is the set consisting of f and all its derivatives with respect to X, by a certain map $\varphi : M \to \mathcal{J}_{1,1}$ with the triple (D, t, U). Namely, suppose $t \circ \varphi = s$, $U \circ \varphi = F$ and, as a result, the vector field X is φ-related to D. All implications for (D, t, U) are carried over from $\mathcal{J}_{1,1}$ onto M and give us useful information about (X, s, F). This scheme can be also generalized for the case $\mathcal{J}_{n,m}$ using multi-indices.

In the second section we consider the rule of total differentiation under the jet composition. It is shown how the corresponding TDOs are prolonged and how the Cartan forms are related under the jet composition. The convenient recurrent formulas are derived.

1 Jet Calculus

1.1 Jet space $\mathcal{J}_{1,1}$

A *pure jet* as an element of $\mathcal{J}_{1,1}$ is an infinite sequence of symbols

$$t, u, u', u'', \dots \tag{1.1}$$

In general, these symbols are not connected with each other. $\mathcal{J}_{1,1}$ is a trivial bundle $\mathbb{R}\times\mathbb{R}^\infty$ with time axis $\mathbb{R}$ as its base and infinite dimensional fibers $\mathbb{R}^\infty$. The symbols (1.1) are considered as coordinate functions in $\mathcal{J}_{1,1}$.

If the symbols $u^{(k)}$, $k = 0, 1, 2, \dots$, are replaced by a smooth function u of argument t and its derivatives, then the sequence (1.1) becomes a jet of the given function, a section of the bundle $\mathbb{R} \times \mathbb{R}^\infty$. Any relation between the quantities in (1.1) can be interpreted as an ODE. After prolongation this ODE can be considered as a surface in $\mathcal{J}_{1,1}$. For instance, being prolonged, an ODE of order n

$$u^{(n)} = \mathcal{F}(t, u, u', u'', \dots, u^{(n-1)})$$

allows to express $u^{(n)}$, $u^{(n+1)}, \dots$ by means of $n+1$ quantities $t, u, u', u'', \dots, u^{(n-1)}$. One can speak about parametric equations of a $(n+1)$-dimensional surface in $\mathcal{J}_{1,1}$. If the jet of the function $u(t)$ which can be considered as the section of the bundle $\mathbb{R}\times\mathbb{R}^\infty$ lies entirely on this surface, then the function $u(t)$ is called a solution of the ODE.

Note that in the present paper we study pure jets (1.1) which are not connected with a certain map.

1.2 Basic implications

Let us define in $\mathcal{J}_{1,1}$ the following infinite matrices:

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \qquad e^{Ct} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \dots \\ 0 & 1 & t & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$U = \begin{pmatrix} u \\ u' \\ u'' \\ \vdots \end{pmatrix}, \quad U' = \begin{pmatrix} u' \\ u'' \\ u''' \\ \vdots \end{pmatrix}, \quad U_t = \begin{pmatrix} u_t \\ u'_t \\ u''_t \\ \vdots \end{pmatrix}, \quad I = \begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ \vdots \end{pmatrix}, \quad \omega = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \vdots \end{pmatrix},$$

where E is the unit matrix, C is the shift matrix, e^{Ct} is the exponential of Ct and U, U', U_t, I, ω are column-matrices. Define also the infinite row-matrix

$$\frac{\partial}{\partial U} = \begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial u'} & \frac{\partial}{\partial u''} & \dots \end{pmatrix}.$$

In the bundle $\mathbb{R} \times \mathbb{R}^\infty$ there is defined the *adapted basis* (see [5],[6])

$$\begin{pmatrix} D & \frac{\partial}{\partial U} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial U} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ U' & E \end{pmatrix}, \tag{1.2}$$

$$\begin{pmatrix} dt \\ \omega \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -U' & E \end{pmatrix} \cdot \begin{pmatrix} dt \\ dU \end{pmatrix}. \tag{1.3}$$

This basis contains the *total differentiation operator* D and the *Cartan forms* ω:

$$D = \frac{\partial}{\partial t} + \frac{\partial}{\partial U} U', \qquad \omega = dU - U'dt. \tag{1.4}$$

In order to denote differentiation with respect to D (Lie differentiation), we use ordinary primes.

Let us start with a summary of the main implications.

First, the following double implication takes place:

$$U' = CU \quad \Rightarrow \quad U_t = e^{Ct}U \quad \Rightarrow \quad I = e^{-Ct}U. \tag{1.5}$$

Since D is a linear vector field on each fibre, the dynamical system $U' = CU$ determines its *flow* $U_t = e^{Ct}U$ in $J_{1,1}$. Any point U moves along its trajectory (t, U_t). Changing the sign of t, one obtains the set of invariants $I = e^{-Ct}U$ of D. Indeed, being dragged along the flow of D, the quantities I do not change: $I' = e^{-Ct}(U' - CU) = 0$. The formulas $U_t = e^{Ct}U$ and $I = e^{-Ct}U$ both can be written as follows:

$$u_t^{(k)} = \sum_{l=0}^{\infty} u^{(k+l)} \frac{t^l}{l!}, \quad k = 0, 1, 2, \dots, \tag{1.6}$$

$$i_k = \sum_{l=0}^{\infty} u^{(k+l)} \frac{(-t)^l}{l!}, \quad k = 0, 1, 2, \dots \tag{1.7}$$

Second, for the Cartan forms ω we have the double implication

$$\omega' = C\omega \quad \Rightarrow \quad \omega_t = e^{Ct}\omega \quad \Rightarrow \quad dI = e^{-Ct}\omega. \tag{1.8}$$

The leftmost equation $\omega' = C\omega$ means that each next form in the sequence $\omega_0, \omega_1, \omega_2, \dots$ is the Lie derivative of the previous one with respect to D. The middle equation $\omega_t = e^{Ct}\omega$ describes the dragging of ω along the flow of D. The rightmost equation $dI = e^{-Ct}\omega$ implies that the exponential e^{-Ct} is an *integrating matrix* for the system ω. Thus the Cartan forms become the exact differentials dI. The replacement of fiber coordinates U with invariants I is accompanied by transformations in the basis:

$$U \rightsquigarrow I \quad \Rightarrow \quad \omega \rightsquigarrow dI, \quad \frac{\partial}{\partial U} \rightsquigarrow \frac{\partial}{\partial I}.$$

Third, for the vertical frame we have the following double implication

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U}C \quad \Rightarrow \quad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U}e^{-Ct} \quad \Rightarrow \quad \frac{\partial}{\partial I} = \frac{\partial}{\partial U}e^{Ct}. \tag{1.9}$$

The leftmost and the middle equations describe the dragging of the frame $\frac{\partial}{\partial U}$ along the flow of D. The rightmost equation determines the invariant frame $\frac{\partial}{\partial I}$ consisting of operators

$$\begin{aligned}
&\frac{\partial}{\partial i_0} = \frac{\partial}{\partial u},\\
&\frac{\partial}{\partial i_1} = t\frac{\partial}{\partial u} + \frac{\partial}{\partial u'},\\
&\frac{\partial}{\partial i_2} = \frac{t^2}{2}\frac{\partial}{\partial u} + t\frac{\partial}{\partial u'} + \frac{\partial}{\partial u''},\\
&\qquad\vdots
\end{aligned}$$

These operators form a basis for *Lie vector fields* that are infinitesimal symmetries of D.

Forth, for the components of a Lie vector field we have the following double implication:

$$\mu' = C\mu \quad \Rightarrow \quad \mu_t = e^{Ct}\mu \quad \Rightarrow \quad \nu = e^{-Ct}\mu. \tag{1.10}$$

Let P be a vertical vector field in $\mathfrak{J}_{1,1}$. Denote by $\mu = PU$ and $\nu = PI$ its components in the natural frame $\frac{\partial}{\partial U}$ and the invariant frame $\frac{\partial}{\partial I}$, respectively, and calculate the Lie derivative of P with respect to D:

$$P = \frac{\partial}{\partial U}\mu = \frac{\partial}{\partial I}\nu \quad \Rightarrow \quad P' = \frac{\partial}{\partial U}(\mu' - C\mu) = \frac{\partial}{\partial I}\nu\,'.$$

This implies that P commutes with D, i.e. P is an infinitesimal symmetry of D if and only if either $\mu' = C\mu$ or $\nu\,' = 0$. These two conditions are obviously equivalent since ν and μ are related: $\nu = e^{-Ct}\mu$. The equality $\mu' = C\mu$ means that each element of the column μ is the derivative of the previous one, i.e. $\mu_k = \mu'_{k-1}$, $k = 1, 2, \ldots$, or, that is the same,

$$\mu_k = \mu_0^{(k)}, \quad k = 1, 2, \ldots$$

The first function μ_0 is said to be a *generating function* for P. For instance, the invariant frame $\frac{\partial}{\partial I}$ consists of Lie vertical vector fields with generating functions $1, t, \frac{t^2}{2}, \ldots$, respectively.

The leftmost and the middle equations in (1.10) show how the elements of μ are dragged along the flow of D. The rightmost equation in (1.10) shows that the components ν and μ are related in such a way that the invariants I are related to the fiber coordinates U in (1.5).

Now suppose that P is not necessarily vertical vector field. Such a vector field can have both vertical and horizontal components. The natural question arises: under what conditions P is an infinitesimal symmetry of D, or when the condition $P' \parallel D$ holds. In the later case P is obviously an infinitesimal symmetry of D. The sign $\parallel$ means the equality of operators up to a coefficient of proportionality. Let us present P in the natural, the adapted and the invariant frames respectively:

$$P = \xi\frac{\partial}{\partial t} + \frac{\partial}{\partial U}\lambda = \xi D + \frac{\partial}{\partial U}\mu = \xi\frac{\partial}{\partial t} + \frac{\partial}{\partial I}\nu.$$

Here the components ξ, λ, μ, ν have the following meaning:

$$\xi = Pt, \quad \lambda = PU = \mu + U'\xi, \quad \mu = \omega(P), \quad \nu = PI = e^{-tC}\mu.$$

Let us calculate the Lie derivative of P with respect to D in these frames:

$$P' = \xi' D + \frac{\partial}{\partial U}(\lambda' - C\lambda - \xi' U') = \xi' D + \frac{\partial}{\partial U}(\mu' - C\mu) = \xi' D + \frac{\partial}{\partial I}\nu'.$$

One can see that the next conditions are equivalent:

$$P' \parallel D \quad \Leftrightarrow \quad \lambda' - C\lambda - \xi' U' = 0 \quad \Leftrightarrow \quad \mu' = C\mu \quad \Leftrightarrow \quad \nu' = 0.$$

Now we can formulate a *rule* for prolongation of vector fields in the natural frame. Let a vector field $\overline{P} = \xi\frac{\partial}{\partial t} + \lambda_0\frac{\partial}{\partial u}$ be given on the tu plane. Our purpose is to construct a prolongation of $\overline{P}$ to a Lie vector field in $\mathfrak{J}_{1,1}$. First, using components ξ and λ_0 define the generating function $\mu_0 = \lambda_0 - \xi u'$. Second, calculate the derivatives $\mu' = \lambda' - \xi' U' - \xi U''$ and then according to the formula $C\lambda = \mu' + \xi U''$ write the column λ for $P = \xi\frac{\partial}{\partial t} + \frac{\partial}{\partial U}\lambda$.

Fifth and finally, for invariant (with respect to D) 1-form $\Psi = \psi\omega$ with coefficients taken in the form of row-matrix $\psi = \begin{pmatrix}\psi_0 & \psi_1 & \psi_2 & \cdots\end{pmatrix}$, we have the following double implication:

$$\psi' = -\psi C \quad \Leftrightarrow \quad \psi_t = \psi e^{-Ct} \quad \Leftrightarrow \quad \chi = \psi e^{Ct}. \tag{1.11}$$

¿From the derivative $\Psi' = (\psi' + \psi C)\omega$ there follows

$$\Psi' = 0 \quad \Leftrightarrow \quad \psi' = -\psi C.$$

So the 1-form Ψ is invariant with respect to D if and only if its components satisfy the condition $\psi' = -\psi C$. This condition means that the first element ψ_0 in the infinite row ψ is invariant with respect to D: $\psi_0' = 0$, and each next element is the antiderivative of the previous one with the opposite sign: $\psi_k' = -\psi_{k-1}$, or, that is the same, the term $(-1)^k\psi_k$ is the k-th antiderivative of ψ_0, $k = 1, 2, \ldots$

The equality $\psi' = -\psi C$ can be considered as an ODE. Its solution presented by the middle formula in (1.11) describes the dragging of ψ along the flow of D. The system of invariants $\chi = \psi e^{Ct}$ is obtained by changing the sign of t in the previous formula. This is the infinite row of elements

$$\begin{aligned}
\chi_0 &= \psi_0,\\
\chi_1 &= t\psi_0 + \psi_1,\\
\chi_2 &= \frac{t^2}{2}\psi_0 + t\psi_1 + \psi_2,\\
&\vdots
\end{aligned}$$

the components of the 1-form Ψ in the invariant coframe, $\Psi = \chi dI$. Such a form is called a *Lie form*, analogously to a Lie vector field.

2 Jet composition

2.1 Statement of problem

Each composition of smooth maps

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

is induced by a composition of the corresponding jets. As it was mentioned above we use only a composition of pure jets that are not connected with a certain map. Let $\mathcal{J}(\mathcal{A},\mathcal{B})$, $\mathcal{J}(\mathcal{B},\mathcal{C})$ and $\mathcal{J}(\mathcal{A},\mathcal{C})$ be spaces of jets from $\mathcal{A}$ to $\mathcal{B}$, from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{A}$ to $\mathcal{C}$, respectively. The question is how TDOs and Cartan forms from the spaces $\mathcal{J}(\mathcal{A},\mathcal{B})$ and $\mathcal{J}(\mathcal{B},\mathcal{C})$ are related to TDOs and Cartan forms from $\mathcal{J}(\mathcal{A},\mathcal{C})$ under a jet composition.

Let the spaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be of dimensions n_1, n_2, n_3, and provide them with coordinates (t^i), (u^α), (v^λ), respectively. Assume that the indices
i, j run over $1, 2, \dots, n_1$;
α, β run over $n_1+1, n_1+2, \dots, n_1+n_2$;
λ, μ run over $n_1+n_2+1, n_1+n_2+2, \dots, n_1+n_2+n_3$.
For the sake of brevity we use also multi-indices

$$(j) = j_1 j_2 \dots j_p, \quad (\beta) = \beta_1 \beta_2 \dots \beta_p, \quad p = 0, 1, 2, \dots$$

In each mentioned jet space there are defined coordinates, TDOs and Cartan forms as follows:

$$\begin{aligned}
&\mathcal{J}(\mathcal{A},\mathcal{B}): && (t^i, u^\alpha_{(j)}), && X_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial u^\alpha_{(j)}} u^\alpha_{i(j)}, && \omega^\alpha_{(j)} = du^\alpha_{(j)} - u^\alpha_{i(j)} dt^i,\\
&\mathcal{J}(\mathcal{B},\mathcal{C}): && (u^\alpha, v^\lambda_{(\beta)}), && Y_\alpha = \frac{\partial}{\partial u^\alpha} + \frac{\partial}{\partial v^\lambda_{(\beta)}} v^\lambda_{\alpha(\beta)}, && \theta^\lambda_{(\beta)} = dv^\lambda_{(\beta)} - v^\lambda_{\alpha(\beta)} du^\alpha,\\
&\mathcal{J}(\mathcal{A},\mathcal{C}): && (t^i, v^\lambda_{(j)}), && Y_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial v^\lambda_{(j)}} v^\lambda_{i(j)}, && \theta^\lambda_{(j)} = dv^\lambda_{(j)} - v^\lambda_{i(j)} dt^i.
\end{aligned}$$

Besides this we need also to define an intermediate space $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ with coordinates and TDOs $\overline{X}_i$ as follows:

$$\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C}): \quad (t^i, u^\alpha_{(j)}, v^\lambda_{(\beta)}), \quad \overline{X}_i = \frac{\partial}{\partial t^i} + \frac{\partial}{\partial u^\alpha_{(j)}} u^\alpha_{i(j)} + \frac{\partial}{\partial v^\lambda_{(\beta)}} v^\lambda_{\alpha(\beta)} u^\alpha_i,$$

It means that the space $\mathcal{J}(\mathcal{A},\mathcal{B})$ is extended to $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$ and the operators X_i are prolonged to the operators $\overline{X}_i$,

$$\mathcal{J}(\mathcal{A},\mathcal{B}) \rightsquigarrow \mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C}), \quad X_i \rightsquigarrow \overline{X}_i.$$

2.2 Composition formula

Note that when we apply X_i to a function of the form $F(t^i, u^\alpha_{(j)})$ in $\mathcal{J}(\mathcal{A},\mathcal{B})$ we extend the differentiation rule for composite function on pure jets. Indeed, assuming that u^α depend on t^i, the derivative of F with respect to t^i equals to $X_i F$. Analogously, if we apply $\overline{X}_i$ to a function of the form $F(t^i, u^\alpha_{(j)}, v^\lambda_{(\beta)})$ in $\mathcal{J}(\mathcal{A},\mathcal{B},\mathcal{C})$, then the differentiation rule for composite function is naturally generalized. Here we assume that v^λ depend on u^α that in their turn depend on t^i.

Theorem 2.1. *The jet composition is determined as the map*

$$\varphi : \mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \to \mathcal{J}(\mathcal{A}, \mathcal{C}) \tag{2.1}$$

by relation $t^i \circ \varphi = t^i$ *and recurrent formula*

$$(Y_{(i)} v^\lambda) \circ \varphi = \overline{X}_{(i)} v^\lambda, \tag{2.2}$$

where

$$Y_{(i)} = Y_{i_1} \dots Y_{i_p}, \quad \overline{X}_{(i)} = \overline{X}_{i_1} \dots \overline{X}_{i_p}, \quad (i) = (i_1 i_2 \dots i_p), \quad p = 0, 1, 2, \dots$$

Proof. Two first relations $t^i \circ \varphi = t^i$ and $v^\lambda \circ \varphi = v^\lambda$ are true since the jets from $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $\mathcal{J}(\mathcal{A}, \mathcal{C})$ have the common source space $\mathcal{A}$ and the common end space $\mathcal{C}$. For $p = 1, 2 \dots$ (2.2) can be rewritten in the form

$$\begin{aligned}
v_i^\lambda \circ \varphi &= v_\alpha^\lambda u_i^\alpha, \\
v_{ij}^\lambda \circ \varphi &= v_{\alpha\beta}^\lambda u_i^\alpha u_j^\beta + v_\alpha^\lambda u_{ij}^\alpha, \\
&\vdots
\end{aligned}$$

which corresponds to a composition of maps. ■

Remark. For comparison, let us present these formulas in the case of the composition of given maps $f : M_1 \to M_2$ and $g : M_2 \to M_3$, i.e. $h = g \circ f$ (see [5], p. 116):

$$\begin{aligned}
h^\lambda &= g^\lambda \circ f, \\
h_i^\lambda &= (g_\alpha^\lambda \circ f) f_i^\alpha, \\
h_{ij}^\lambda &= (g_{\alpha\beta}^\lambda \circ f) f_i^\alpha f_j^\beta + (g_\alpha^\lambda \circ f) f_{ij}^\alpha, \\
&\vdots
\end{aligned}$$

In contrast to (2.2) that hold in $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$, these formulas hold on a neighborhood of M_1.

2.3 Coupling of Cartan forms

Theorem 2.2. *Operators* $\overline{X}_i$ *and* Y_i *are* φ*-related.*

Proof. It is known that two vector fields are said to be φ-related for some map φ if the derivatives of any φ-related functions with respect to these vector fields are φ-related. In particular, vector fields are said to be φ-related if and only if the derivatives of coordinate functions v^λ and of φ-related with them functions $v^\lambda \circ \varphi$ are φ-related. We have exactly this situation for $\overline{X}_i$ and Y_i, see (2.2). Hence, $\overline{X}_i$ and Y_i are φ-related. ■

Theorem 2.3. *The Cartan forms* $\theta_{(i)}^\lambda$ *are mapped by (2.1) from* $\mathcal{J}(\mathcal{A}, \mathcal{C})$ *to* $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ *according to the following recurrent formula:*

$$(\mathcal{L}_{Y_{(i)}} \theta^\lambda) \circ T\varphi = \mathcal{L}_{\overline{X}_{(i)}} (\theta^\lambda + v_\alpha^\lambda \omega^\alpha), \tag{2.3}$$

where

$$Y_{(i)} = \mathcal{L}_{Y_{i_1}} \ldots \mathcal{L}_{Y_{i_p}}, \quad \overline{X}_{(i)} = \mathcal{L}_{\overline{X}_{i_1}} \ldots \mathcal{L}_{\overline{X}_{i_p}}, \quad (i) = (i_1 i_2 \ldots i_p), \quad p = 0, 1, 2, \ldots,$$

and (2.3) can be written as follows:

$$\theta^\lambda \circ T\varphi = \theta^\lambda + v^\lambda_\alpha \omega^\alpha,$$

$$\theta^\lambda_i \circ T\varphi = \theta^\lambda_\alpha u^\alpha_i + v^\lambda_\alpha \omega^\alpha_i + v^\lambda_{\alpha\beta} u^\beta_i \omega^\alpha,$$

$$\vdots$$

Proof. For $p = 0$ (2.3) gives obviously the first relation

$$\theta^\lambda \circ T\varphi = \theta^\lambda + v^\lambda_\alpha \omega^\alpha.$$

Note that $\theta^\lambda = dv^\lambda - v^\lambda_i dt^i$ are defined in $\mathcal{J}(\mathcal{A}, \mathcal{C})$, $\theta^\lambda = dv^\lambda - v^\lambda_\alpha du^\alpha$ and $\omega^\alpha = du^\alpha - u^\alpha_i dt^i$ are defined in $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Taking into account that $t^i \circ \varphi = t^i$, $v^\lambda \circ \varphi = v^\lambda$ and $v^\lambda_i \circ \varphi = v^\lambda_\alpha u^\alpha_i$, θ^λ are transported from $\mathcal{J}(\mathcal{A}, \mathcal{C})$ to $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as follows:

$$\theta^\lambda \circ T\varphi = (dv^\lambda - v^\lambda_i dt^i) \circ T\varphi = dv^\lambda - v^\lambda_\alpha u^\alpha_i dt^i = \theta^\lambda + v^\lambda_\alpha \omega^\alpha.$$

Hence, the first relation holds. In other words, $\theta^\lambda + v^\lambda_\alpha \omega^\alpha$ in $\mathcal{J}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are φ-related with θ^λ in $\mathcal{J}(\mathcal{A}, \mathcal{C})$. It is well-known that the Lie derivatives of φ-related tensor fields with respect to φ-related vector fields are φ-related. Differentiating φ-related forms with respect to φ-related vector fields $\overline{X}_i$ (from the right) and Y_i (from the left), for $p = 1, 2, \ldots$ we obtain φ-related Lie derivatives. Thus, the formula (2.3) is proved. ■

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