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Symmetries and invariants for the 2D-Ricci flow model

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Abstract

The paper investigates some special Lie type symmetries and associated invariant quantities which appear in the case of the 2D Ricci flow equation in conformal gauge. Starting from the invariants some simple classes of solutions will be determined.

1 Introduction

One of the most fruitful models used in study of the black holes and in the attempt of obtaining a quantum theory of gravity is connected with the Ricci flow equations [2]. Because of the difficulties which appear when a quantum field theory is formulated, various models in less dimensions were intensively studied. These are the so called "mechanical models" and the most known examples are given by the classical model of the Yang-Mills gauge field [9], as well as some 3-dimensional models of dynamical systems as the Hénon-Heiles one [5].

We will investigate a 2D model for the Ricci flow equation, a nonlinear parabolic equation obtained when the components of the metric tensor $g_{\alpha\beta}$ are deformed following the equation:

$$\frac{\partial}{\partial t} g_{\alpha\beta} = -R_{\alpha\beta} \tag{1.1}$$

where $R_{\alpha\beta}$ is the Ricci tensor for the n-dimensional Riemann space. The main interest will be connected with the integrability of this equation. There are many possible ways to solve this problem, as for example the reduction of the differential equation to an algebraic one by finding a sufficient number of Lie symmetries or the application of the Painlevé test [6]. In this paper we start from the standard form of the Lie operators [10] and we look for the existence of a sufficient number of Lie symmetries [7]. As is well known, the Lie symmetries of nonlinear equations may be used to construct exact solutions and conservation laws [4]. The algorithm we will apply is the same with that from [11]. In a special case of linearization, we will point out some particular classes of invariants and of solutions. Some special classes of solutions for the same equations are considered in [3].

The paper is organized as follows: after this introductive section, the 2D model of the Ricci flow equation is presented in the second section. It will lead to a non-linear differential equation of second order. This equation could be seen as describing a non-autonomous dynamical system and, by its consequences, a specific technique could be used in order to recover the symmetry operators and the invariant quantities which could be attached to them. We will discuss all these symmetry matters in the third section of the paper and we will obtain there special classes of invariants. The paper will end with some concluding remarks, the most important one regarding the algebra satisfied by the symmetry operators of the considered model.

2 The mechanical model for the Ricci flow

Let us consider the case of the Ricci flow equation of the type (1.1). The metric tensor of the space $g_{\alpha\beta}$ will be connected with the Riemann metric in the conformal gauge:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{2}\exp\{\Phi(X,Y,t)\}(dX^2 + dY^2) \tag{2.1}$$

The "potential" $\Phi(X,Y,t)$ satisfies the equation:

$$\frac{\partial}{\partial t}e^\Phi = \triangle\Phi \tag{2.2}$$

It has been noticed [1] that the equation (2.2) is pretty similar with the Toda equation describing the integrable interaction of a collection of two dimensional fields $\{\Phi_i, i = 1,2\}$ coupled by a Cartan matrix (K_{ij}):

$$\sum_j K_{ij}e^{\Phi_j(X,Y)} = \triangle\Phi_i(X,Y) \tag{2.3}$$

Introducing the field $v(x,y,t)$ given by

$$v(x,y,t) = e^\Phi \tag{2.4}$$

the equation (2.2) takes the form:

$$v_t = (\ln v)_{xy} \tag{2.5}$$

An equivalent form for the previous equation, which will be used in the next sections of the paper, is:

$$v^2 v_t + v_y v_x - v v_{xy} = 0 \tag{2.6}$$

It is a well-known equation which has been studied as a continuum limit of the Toda-type equation. Among the main results concerning (2.5) we mention: *(i)* it could be obtained as a particular case of the 3D Ricci flow equation which accepts a Killing vector; *(ii)* by linearization, it presents various classes of solutions depending on the "sector" where it is defined [3]. Up to our knowledge, no effective studies on the Lie symmetries of this equation were performed. This will be the main objective of the following section of our paper.

3 Symmetries and invariants

3.1 The general form of the Lie symmetry operators

The Lie symmetry operator for differential equations with independent variables x^i, $i = \overline{1,p}$ and dependent variables v^α, $\alpha = \overline{1,q}$ has the form [10]:

$$U = \sum_{i=1}^{p} \xi^i(x^i, v^q)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x^i, v^q)\frac{\partial}{\partial v^\alpha} \tag{3.1}$$

The n-th extension of (3.1) is the operator:

$$U^{(n)} = U + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x^i, v^{(n)})\frac{\partial}{\partial v_J^\alpha} \tag{3.2}$$

where $v^{(n)}$denotes the set of variables which includes v and the partial derivatives of v up to n-th order and

$$v_J^\alpha = \frac{\partial^m v^\alpha}{\partial x^{j_1}\partial x^{j_2}..\partial x^{j_m}} \tag{3.3}$$

Also, in (3.1) the second summation refers to the all multi-indices $J = (j_1, ...j_m)$,with $1 \leq j_m \leq\ p, 1 \leq m \leq\ n$.The coefficient functions ϕ_α^J of (3.2) are given by the following formula:

$$\phi_\alpha^J(x^i, v^{(n)}) = D_J[\phi_\alpha - \sum_{i=1}^{p} \xi^i v_i^\alpha] + \sum_{i=1}^{p} \xi^i v_{J,i}^\alpha \tag{3.4}$$

where

$$v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i},\ i = \overline{1,p} \tag{3.5}$$

$$v_{J,i}^\alpha = \frac{\partial v_J^\alpha}{\partial x^i} = \frac{\partial^{m+1} v^\alpha}{\partial x^i \partial x^{j_1}\partial x^{j_2}..\partial x^{j_m}} \tag{3.6}$$

$$D_J = D_{j_1}D_{j_2}...D_{j_m}, D_j = \frac{d^{\ m}}{dx^{j_1}dx^{j_2}..dx^{j_m}} \tag{3.7}$$

A differential equation of n-th order should remain invariant in respect with a point transformation if the action of n-th order extended operator (3.2) upon the equation would vanish.

3.2 Lie operators for 2D Ricci flow

Let us come back to the equation (2.6). Because it is a partial differential equation of second order, which has three independent variables x, y, t and one dependent variable v, the Lie operator which leaves (2.6) invariant, has the form:

$$U(x,y,t,v) = \xi^1(x,y,t,v)\frac{\partial}{\partial x} + \xi^2(x,y,t,v)\frac{\partial}{\partial y} + \xi^3(x,y,t,v)\frac{\partial}{\partial t} + \phi(x,y,t,v)\frac{\partial}{\partial v} \tag{3.8}$$

Following the general formula (3.2), the second extension of (3.8) has the form:

$$\begin{aligned}U^{(2)} &= U+\phi^{x}\frac{\partial}{\partial v_x}+\phi^{y}\frac{\partial}{\partial v_y}+\phi^{t}\frac{\partial}{\partial v_t}+\phi^{2x}\frac{\partial}{\partial v_{2x}}+\phi^{xy}\frac{\partial}{\partial v_{xy}}+\phi^{xt}\frac{\partial}{\partial v_{xt}}+\\&\quad+\phi^{2y}\frac{\partial}{\partial v_{2y}}+\phi^{yt}\frac{\partial}{\partial v_{yt}}+\phi^{2t}\frac{\partial}{\partial v_{2t}}\end{aligned}\tag{3.9}$$

We impose the invariance condition for the Ricci flow equation. In other words, we ask for the vanishing of the action of operator (3.9) upon the equation (2.6). We obtain the condition:

$$\phi[2vv_t-v_{xy}]+\phi^{x}v_y+\phi^{y}v_x+\phi^{t}v^2-\phi^{xy}v=0\tag{3.10}$$

Using (3.4), the function ϕ^x from (3.10) has the form:

$$\phi^{x}=\phi_x+[\phi_v-\xi^1_x]v_x-\xi^1_v v_x^2-\xi^2_x v_y-\xi^3_x v_t-\xi^2_v v_x v_y-\xi^3_v v_x v_t\tag{3.11}$$

If we should substitute v_t from (2.6) into (3.11), we would obtain:

$$\begin{aligned}\phi^{x} &= \phi_x+[\phi_v-\xi^1_x]v_x-\xi^1_v v_x^2-\xi^2_x v_y+[v^{-2}\xi^3_x-\xi^2_v]v_x v_y-\\&\quad-v^{-1}\xi^3_v v_y v_x^2-v^{-1}\xi^3_v v_x v_{xy}\end{aligned}\tag{3.12}$$

Similarly, after the substitution of v_t, the function ϕ^y has the expression:

$$\begin{aligned}\phi^{y} &= \phi_y+[\phi_v-\xi^2_y]v_y-\xi^1_y v_x+[-\xi^1_v+v^{-2}\xi^3_y]v_x v_y-\xi^2_v v_y^2-\\&\quad-v^{-1}\xi^3_y v_{xy}+v^{-2}\xi^3_v v_x v_y^2-v^{-1}\xi^3_v v_y v_{xy}\end{aligned}\tag{3.13}$$

Following the same procedure, we obtain for the functions ϕ^t and ϕ^{xy} the forms:

$$\begin{aligned}\phi^{t} &= \phi^{t}-v^{-2}[\phi_v-\xi^3_t]v_x v_y+v^{-1}[\phi_v-\xi^3_t]v_{xy}-\xi^1_t v_x+v^{-2}\xi^1_v v_y v_x^2-\\&\quad-v^{-1}\xi^1_v v_x v_{xy}-\xi^2_t v_y+v^{-2}\xi^2_v v_x v_y^2-v^{-1}\xi^2_v v_y v_{xy}-v^{-4}\xi^3_v v_x^2 v_y^2-\\&\quad-v^{-2}\xi^3_v v_{xy}^2+2v^{-3}\xi^3_v v_x v_y v_{xy}\end{aligned}\tag{3.14}$$

$$\begin{aligned}\phi^{xy} &= \phi_{xy}+[\phi_{xv}-\xi^2_{xv}]v_y+[\phi_{yv}-\xi^1_{xy}]v_x+\\&\quad+[\phi_{2v}-\xi^1_{vx}-\xi^2_{yv}+\xi^3_{xy}v^{-2}]v_y v_x+[\phi_v-\xi^1_x-\xi^2_y-v^{-1}\xi^3_{xy}]v_{xy}-\\&\quad--\xi^1_{vy}v_x^2+[-\xi^1_{2v}+v^{-2}\xi^3_{vy}-2v^{-3}\xi^3_y]v_y v_x^2+[-2\xi^1_v-v^{-1}\xi^3_{vy}+2v^{-2}\xi^3_y]v_x v_{xy}-\\&\quad-\xi^1_y v_{2x}+[-\xi^1_v+v^{-2}\xi^3_y]v_y v_{2x}-\xi^2_{xv}v_y^2+[-\xi^2_{2v}+v^{-2}\xi^3_{xv}-2v^{-3}\xi^3_x]v_x v_y^2-\\&\quad-\xi^2_x v_{2y}+[-\xi^2_v+v^{-2}\xi^3_x]v_{2y}v_x+[-2\xi^2_v-v^{-1}\xi^3_{xv}+2v^{-2}\xi^3_x]v_y v_{xy}+\\&\quad+[v^{-2}\xi^3_{2v}-4v^{-3}\xi^3_v]v_x^2 v_y^2+[-v^{-1}\xi^3_{2v}+5v^{-2}\xi^3_v]v_y v_x v_{xy}+v^{-2}\xi^3_v v_x^2 v_{2y}-\\&\quad-v^{-1}\xi^3_x v_{xyy}-v^{-1}\xi^3_v v_x v_{xyy}-v^{-1}\xi^3_y v_{xxy}-v^{-1}\xi^3_v v_y v_{xxy}+v^{-2}\xi^3_v v_y^2 v_{2x}-v^{-1}\xi^3_v v_{xy}^2\end{aligned}\tag{3.15}$$

Substituting the relations (3.12)-(3.15) into (3.10) and equalizing the coefficients of the various monomials in the first and second order partial derivatives of v,we find a system with 18 equations which can be reduced to the following:

$$
\begin{aligned}
&-v\phi_{2v} + \xi^3_t + \phi_v - \xi^2_y - \xi^1_x - 2v^{-1}\phi = 0 \\
&-v\phi_{yv} - \xi^1_t v^2 + \phi_y = 0 \\
&-v\phi_{xv} - \xi^2_t v^2 + \phi_x = 0 \\
&\phi - v\xi^3_t + v\xi^1_x + v\xi^2_y = 0 \\
&\phi_t v - \phi_{xy} = 0
\end{aligned} \tag{3.16}
$$

By solving (3.16), we conclude that the most general symmetry generator of the Ricci flow equation, has coefficient functions of the form:

$$
\begin{aligned}
& & \xi^1(x),\ \xi^2(y),\ \text{arbitrary functions;} \\
\xi^3(x) &=& c_1 t + c_2; c_1,\ c_2 \text{ arbitrary constants;} \\
\phi(x,y,t,v) &=& v[\xi^3_t(t) - \xi^1_x(x) - \xi^2_y(y)]
\end{aligned} \tag{3.17}
$$

Thereby, the Lie symmetry operator for (2.6) has the final form:

$$
U = \xi^1(x)\frac{\partial}{\partial x} + \xi^2(y)\frac{\partial}{\partial y} + [c_1 t + c_2]\frac{\partial}{\partial t} + v[c_1 - \xi^1_x(x) - \xi^2_y(y)]\frac{\partial}{\partial v} \tag{3.18}
$$

As U contains coefficients in the form of 2 arbitrary functions $\{\xi^i, i = 1,2\}$, we deal with an infinite number of symmetry operators. The action of U can be split in various "sectors", depending on the concrete form we might choose for the functions $\{\xi^i, i = 1,2\}$.

3.3 The linear sector of invariance

One of the simplest cases we could consider for the action of the symmetry operator U given by (3.18) is the case when

$$
\Phi \equiv v[c_1 - \xi^1_x(x) - \xi^2_y(y)] = k\, v \tag{3.19}
$$

The condition $k = const.$ imposes in fact that ξ^1 and ξ^2 should be linear in their arguments:

$$
\xi^1(x) = c_5 x + c_3;\ \xi^2(y) = c_6 y + c_4 \tag{3.20}
$$

This case corresponds to the situation when the variables are not "melt" in the action of the symmetry U. On the sector given by the curves (3.20), the 2D Ricci flow equation (2.6) admits a 6-parameters family of Lie operators. It creates 6 independent operators of the form:

$$
\begin{aligned}
U_1 &=& t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v};\ U_2 = \frac{\partial}{\partial t}; U_3 = \frac{\partial}{\partial x};\ U_4 = \frac{\partial}{\partial y} \\
U_5 &=& x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v};\ U_6 = y\frac{\partial}{\partial y} - v\frac{\partial}{\partial v}
\end{aligned} \tag{3.21}
$$

The forms of the operators U_i, $i = \overline{1,6}$ suggest their significations: U_1 represents a dilatation, U_2 describes the symmetry of time translation and U_3, U_4 generate the symmetry of space translations, U_5, U_6 are associated with the scaling transformation [8].

When the Lie algebra of these operators is computed, the only non-vanishing relations are obtained in the form:

$$[U_2, U_1] = U_2;\ [U_3, U_5] = U_3; [U_4, U_6] = U_4 \tag{3.22}$$

It is interesting to remark the coupling of the operators in three independent pairs with similar action on the equation. This means that the whole algebra splits in a direct sum of 3 subalgebras. Each of them is 2-dimensional and, following Ado's theorem, is generated by the only one non-commutative 2-dimensional algebra of matrices generated by

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.23}$$

The associated Lie group has the form: $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

Let us pass now to the problem of the invariant quantities associated with the symmetry operators $\{U_i, i = 1, ..., 6\}$. They will be solutions of the system of equations having the form:

$$U_j [I_k] = 0; j = \overline{1, 6}, k = 1, 2, ... \tag{3.24}$$

We will consider by turns the expressions (3.21) of $U_i, i = 1, 6$.

- The invariants associated to the symmetry operator U_1, are obtained by integrating the characteristic equations:

$$\frac{dy}{0} = \frac{dx}{0} = \frac{dt}{t} = \frac{dv}{v} \tag{3.25}$$

and have the forms x, y, $\frac{v}{t}$. Taking into account the last invariant, we assume a similarity solution of the form:

$$v = f(x, y)t \tag{3.26}$$

and we substitute it into (2.6) to determine the form of the function $f(x, y)$. We obtain that $f(x, y)$ is a solution for the following differential equation:

$$f^3 + f_x f_y - f f_{xy} = 0 \tag{3.27}$$

- The invariants induced by U_5 are obtained in a similar way from the equalities:

$$\frac{dy}{0} = \frac{dt}{0} = \frac{dx}{x} = \frac{dv}{-v} \tag{3.28}$$

They are t, y, vx. By direct computation, when we impose the similarity condition $vx = g(t, y)$,we see that, in fact, $g(y, t)$ has to depend on y only, that is to say it is an arbitrary function of the form $g(y)$. The 2D Ricci flow equation (2.5) admits in this case the stationary solution:

$$v = \frac{g(y)}{x} \tag{3.29}$$

- Because (2.6) is symmetric in x and y, it has a similarity solution of the form:

$$v = \frac{g_1(x)}{y} \tag{3.30}$$

- By similar arguments, the invariants generated by the operators $U_i, i = 2, 3, 4$ are respectively the arbitrary functions $h(x, y, v)$, $k(t, y, v); p(x, t, v)$.

4 Conclusions

The aim of this paper was to study the Lie symmetries and the associated invariants of the two dimensional model for the Ricci flow equation. Apart from its intrinsic importance, this model allows the application of an interesting algorithm for analyzing the Lie symmetries of the non-autonomous dynamical systems. It starts from the general form of the symmetry operators (3.8), continues with the computation of its extensions till the order equal to the one of the differential equation and concludes by imposing the invariance condition on the evolution equation. By applying this approach, we obtained some interesting results which can be synthesized as follows: *(i)* the Lie operator has the general form given by (3.18). It depends on the 2 constants and 2 arbitrary functions $\xi^1(x), \xi^2(y)$; *(ii)* choosing linear expressions for $\xi^1(x), \xi^2(y)$, a set of 6 symmetry operators can be generated. They have the expressions (3.21) and satisfy the algebra (3.22). It can be split in a direct sum of 3 independent subalgebras admitting an interesting matriceal representation; *(iii)* the Lie operators generate interesting forms of invariant quantities, from the simplest (the coordinates themselves) to arbitrary functions. Using these invariants and imposing the similarity condition, we were able to obtain very simple solutions of the 2D Ricci flow equation, as for example stationary solutions (3.29) and (3.30) or solutions that propagate linearly in time, as (3.26).

As an alternative investigation, the invariants could be obtained by the extension of the evolutionary operator in the form

$$U_Q = \sum_{\alpha=1}^{q} Q_\alpha \frac{\partial}{\partial v^\alpha}, \tag{4.1}$$

where $Q_\alpha(x^i, v, v_{x^i}) = (Q_1, ..., Q_q)$, $i = \overline{1, p}$ are referred to as the characteristics of the operator (4.1) and have the expression:

$$Q_\alpha = \phi_\alpha(x^i, v) - \sum_{i=1}^{p} \xi^i \left(x^i, v\right) v_i^\alpha, \;\; \alpha = \overline{1, q} \tag{4.2}$$

This approach will be tackled with in a next to come paper.

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