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On an integrable differential-difference equation with a source

GEGENHASI [a,b] *and Xing-Biao HU* [a]

[a] *Institute of Computational Mathematics and Scientific Engineering Computing, AMSS, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, P.R. CHINA*

[b] *Graduate School of the Chinese Academy of Sciences, Beijing, P.R. CHINA*
E-mail: gegen@mail.amss.ac.cn, hxb@lsec.cc.ac.cn

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Abstract

We introduce an integrable differential-difference KdV equation with a source. A bilinear Bäcklund transformation and the associated nonlinear superposition formula are thereby obtained. And the multisoliton solution of the equation is also presented.

1 Introduction

Soliton equations with self consistent sources constitute an important class of integrable equations. Some of such type of equations have found physical applications. For example, the KdV equation with a source

$$u_t + 6uu_x + u_{xxx} = -\int_{-\infty}^{\infty} dk'\ \bar{\nu}(|\phi|^2)_x, \tag{1.1}$$

$$\phi_{xx} + (u + k'^2)\phi = 0 \tag{1.2}$$

describes the interaction of long and short capillary-gravity waves [1, 2], where $u = u(x,t)$ and $\phi = \phi(x,t;k')$ are real and complex functions respectively, k' is a real parameter and $\bar{\nu} = \bar{\nu}(k',t)$ is a given real function. By the dependent variable transformation $u = 2(\ln f)_{xx}, \phi = \bar{\phi}_0 e^{ik'x} g/f$, (1.1) and (1.2) are transformed into the following bilinear equation [3, 4]

$$D_x(D_t + D_x^3) f \cdot f = -\int_{-\infty}^{\infty} dk'\ \bar{\nu}|\bar{\phi}_0|^2(|g|^2 - f^2), \tag{1.3}$$

$$(D_x^2 + 2ik'D_x) g \cdot f = 0, \tag{1.4}$$

where $\bar{\phi}_0 = \bar{\phi}_0(k',t)$ is a given function.

In recent years, there has been active research on soliton equations with self consistent sources, see, e.g. [5]-[22]. A variety of methods have been proposed to deal with these soliton equations with sources, such as via IST method , $\bar{\partial}$-method, Gauge transformations, Darboux transformations, Wronskian technique, Hirota's bilinear method etc. However, most results have been achieved just in *continuous* case. Comparatively less work has been done in *discrete* case.

In view of this unsatisfactory situation, it would be of interest to produce new discrete soliton equations with self consistent sources. The purpose of this paper is to give a differential-difference version to the KdV equation with a source.

We now propose the following bilinear differential-difference equation with a source:

$$(D_t^2 e^{D_n} - \frac{1}{2} D_t e^{D_n}) f(n) \cdot f(n) = \\ - \int_{-\infty}^{\infty} dk \; \nu |\phi_0|^2 \cosh(D_n)[g(n) \cdot g^*(n) - f(n) \cdot f(n)], \tag{1.5}$$

$$(D_t^2 + 2ikD_t) g(n) \cdot f(n) = 0, \tag{1.6}$$

where $f(n) = f(n,t)$ and $g(n) = g(n,t;k)$ are real and complex functions respectively, $\nu = \nu(k)$ and $\phi_0 = \phi_0(k)$ are real and complex functions of k respectively, and the bilinear operators D_t and $\exp(\delta D_n)$ [23, 24] are defined by

$$D_t^m a \cdot b \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m a(t) b(t') \Big|_{t'=t},$$
$$\exp(\delta D_n) \; a \cdot b \equiv a(n+\delta) b(n-\delta),$$

respectively. We can show that under some condition the continuous analogue of (1.5) and (1.6) is the KdV equation with a source (1.3) and (1.4). In fact, setting $D_t = \epsilon D_X, D_n = 2\epsilon D_X - \frac{8}{3}\epsilon^3 D_T$ and $k = \epsilon k', \nu(\epsilon k')|\phi_0(\epsilon k')|^2 = \frac{4}{3}\epsilon^3 \bar{\nu}(k')|\bar{\phi}_0(k')|^2 + O(\epsilon^4)$ in (1.5) and (1.6) and letting $\epsilon \longrightarrow 0$, we obtain the KdV equation with a source (1.3) and (1.4) under the condition $\bar{\nu} = \bar{\nu}(k'), \bar{\phi}_0 = \bar{\phi}_0(k')$.

By making dependent variable transformation $u = (\ln f)_t, v = e^{ikt}\phi_0(k) g/f$, the equations (1.5-1.6)are transformed into the following nonlinear system:

$$u_{tt}(n+1) + u_{tt}(n-1) + [2(u(n+1) - u(n-1)) - \frac{1}{2}](u_t(n+1) - u_t(n-1)) \\ = -\frac{1}{2} \int_{-\infty}^{\infty} dk \nu(k) [v_t(n+1,k) v^*(n-1,k) + v(n+1,k) v_t^*(n-1,k) \\ + v_t(n-1,k) v^*(n+1,k) + v(n-1,k) v_t^*(n+1,k)], \tag{1.7}$$

$$v_{tt}(n,k) + (k^2 + 2u_t(n)) v(n,k) = 0, \tag{1.8}$$

or equivalently, under the transformation $U = (\ln f)_{tt}, v = e^{ikt}\phi_0(k) g/f$, the equations (1.5-1.6) become:

$$U_t(n+1) + U_t(n-1) + \\ + \; [2 \int_{-\infty}^{t} U(n+1,\xi) d\xi - 2 \int_{-\infty}^{t} U(n-1,\xi) d\xi - \frac{1}{2}](U(n+1) - U(n-1)) \\ = -\frac{1}{2} \int_{-\infty}^{\infty} dk \nu(k) [v_t(n+1,k) v^*(n-1,k) + v(n+1,k) v_t^*(n-1,k) \\ + v_t(n-1,k) v^*(n+1,k) + v(n-1,k) v_t^*(n+1,k)], \tag{1.9}$$

$$v_{tt}(n,k) + (k^2 + 2U(n)) v(n,k) = 0, \tag{1.10}$$

2 Bilinear Bäcklund transformation and the nonlinear superposition formula

In this section, we devote to deriving the bilinear Bäcklund transformation and the associated nonlinear superposition formula for the equations (1.5-1.6). The multisoliton solution of the equation is also given.

Proposition 1. *The bilinear equations (1.5) and (1.6) have a Bäcklund transformation*

$$D_t g \cdot f' = -(\lambda + ik)(g' f + g f'), \tag{2.1}$$
$$D_t g' \cdot f = (\lambda - ik)(g' f + g f'), \tag{2.2}$$
$$(D_t^2 - 2\lambda D_t) f' \cdot f = 0, \tag{2.3}$$
$$D_t e^{-D_n} f \cdot f' + [\mu e^{D_n} + (\frac{1}{4} + \lambda) e^{-D_n}] f \cdot f' = \int_{-\infty}^{\infty} dk\ \nu |\phi_0|^2 \{ -\frac{1}{4} \frac{1}{\lambda + ik} e^{-D_n} g \cdot g'^* - \frac{1}{4} \frac{1}{\lambda - ik} e^{D_n} g' \cdot g^* \}, \tag{2.4}$$

where λ and μ are arbitrary real constants.

Proof. Let $(f(n), g(n))$ be a solution of equation (1.5) and (1.6). If we can show that $(f'(n), g'(n))$ given by equations (2.1)-(2.4) satisfies the relation

$$P_1 \equiv (D_t^2 e^{D_n} - \frac{1}{2} D_t e^{D_n}) f'(n) \cdot f'(n) + \int_{-\infty}^{\infty} dk \nu |\phi_0|^2 \cosh(D_n)[g'(n) \cdot g'^*(n) - f'(n) \cdot f'(n)] = 0,$$
$$P_2 \equiv (D_t^2 + 2ikD_t) g'(n) \cdot f'(n) = 0.$$

then equations (2.1)-(2.4) form a BT. In fact, similar to the proof in [3, 4], we know that $P_2 = 0$ can be proved by using (2.1)-(2.2). Thus it suffices to show that $P_1 = 0$. For this, by making use of (A1)-(A5) and (2.1)-(2.4), we have

$$
\begin{aligned}
&-[e^{D_n}f(n)\cdot f(n)]P_1\\
&=[(D_t^2e^{D_n}-\frac{1}{2}D_te^{D_n})f(n)\cdot f(n)+\int_{-\infty}^{\infty}dk\ \nu|\phi_0|^2\cosh(D_n)g(n)\cdot g^*(n)]\\
&\quad[e^{D_n}f'(n)\cdot f'(n)]-[e^{D_n}f(n)\cdot f(n)][(D_t^2e^{D_n}-\frac{1}{2}D_te^{D_n})f'(n)\cdot f'(n)\\
&\qquad+\int_{-\infty}^{\infty}dk\ \nu|\phi_0|^2\cosh(D_n)g'(n)\cdot g'^*(n)]\\
&=2D_t(D_te^{-D_n}f\cdot f')\cdot(e^{D_n}f\cdot f')+2\sinh(D_n)(D_t^2f\cdot f')\cdot ff'\\
&\quad-\sinh(D_n)(D_tf\cdot f')\cdot ff'-\int_{-\infty}^{\infty}dk\ \nu|\phi_0|^2\{\frac{1}{2}e^{-D_n}g'f\cdot fg'^*\\
&\qquad+\frac{1}{2}e^{D_n}g'f\cdot fg'^*-\frac{1}{2}e^{-D_n}gf'\cdot f'g^*-\frac{1}{2}e^{D_n}gf'\cdot f'g^*\}\\
&=2D_t(D_te^{-D_n}f\cdot f')\cdot(e^{D_n}f\cdot f')+2\sinh(D_n)(D_t^2f\cdot f')\cdot ff'-\sinh(D_n)(D_tf\cdot f')\cdot ff'\\
&\quad-\int_{-\infty}^{\infty}dk\ \nu|\phi_0|^2\{-\frac{1}{2}\frac{1}{\lambda+ik}e^{-D_n}[(D_tg\cdot f')\cdot fg'^*-gf'\cdot(D_tf\cdot g'^*)]\\
&\qquad+\frac{1}{2}\frac{1}{\lambda-ik}e^{D_n}[g'f\cdot(D_tf'\cdot g^*)-(D_tg'\cdot f)\cdot f'g^*]\}\\
&=2D_t(D_te^{-D_n}f\cdot f')\cdot(e^{D_n}f\cdot f')+2\sinh(D_n)(D_t^2f\cdot f')\cdot ff'-\sinh(D_n)(D_tf\cdot f')\cdot ff'\\
&\quad-\int_{-\infty}^{\infty}dk\ \nu|\phi_0|^2\{-\frac{1}{2}\frac{1}{\lambda+ik}D_t(e^{-D_n}g\cdot g'^*)\cdot(e^{D_n}f\cdot f')\\
&\qquad-\frac{1}{2}\frac{1}{\lambda-ik}D_t(e^{D_n}g'\cdot g^*)\cdot(e^{-D_n}f'\cdot f)\}\\
&=2\sinh(D_n)[(D_t^2-\frac{1}{2}D_t)f\cdot f']\cdot ff'-2D_t(\lambda+\frac{1}{4})(e^{-D_n}f\cdot f')\cdot(e^{D_n}f\cdot f')\\
&=2\sinh(D_n)[(D_t^2-\frac{1}{2}D_t)f\cdot f']\cdot ff'+4(\lambda+\frac{1}{4})\sinh(D_n)(D_tf\cdot f')\cdot ff'\\
&=0.
\end{aligned}
$$

Thus we have completed the proof of proposition 1. ■

Proposition 2. *Let (f_0,g_0) be a solution of (1.5-1.6) and suppose that (f_1,g_1) and (f_2,g_2) are solutions of (1.5-1.6) given by the Bäcklund transformation 2.1-2.4 with starting solution $(f,g)=(f_0,g_0)$ and Bäcklund parameters $(\lambda,\mu)=(\lambda_1,\mu_1)$ and $(\lambda,\mu)=(\lambda_2,\mu_2)$, respectively. i.e., $(f_0,g_0)\overset{(\lambda_i,\mu_i)}{\longrightarrow}(f_i,g_i)$ $(i=1,2)$, $\lambda_1\lambda_2\neq 0, f_j\neq 0$ $(j=0,1,2)$. Then (f_{12},g_{12}) defined by*

$$f_0f_{12}=c[D_t-(\lambda_1-\lambda_2)]f_1\cdot f_2, \tag{2.5}$$
$$g_0g_{12}=c[D_t-(\lambda_1-\lambda_2)]g_1\cdot g_2, \tag{2.6}$$

is a new solution to (1.5) and (1.6). Here c is a nonzero real constant.

Proof. Similar to the deduction in [4], we can show that

$$D_t f_0 \cdot f_{12} = -c(\lambda_1+\lambda_2) D_t f_1 \cdot f_2, \tag{2.7}$$
$$c(\lambda_2^2-\lambda_1^2) f_1 f_2 = [D_t + (\lambda_1+\lambda_2)] f_2 \cdot f_{12}, \tag{2.8}$$
$$(D_t^2 - 2\lambda_2 D_t) f_{12} \cdot f_1 = 0, \tag{2.9}$$
$$(D_t^2 - 2\lambda_1 D_t) f_{12} \cdot f_2 = 0, \tag{2.10}$$
$$[D_t + (\lambda_1+ik)] g_2 \cdot f_{12} + (\lambda_1+ik) g_{12} f_2 = 0, \tag{2.11}$$
$$[D_t + (\lambda_2+ik)] g_1 \cdot f_{12} + (\lambda_2+ik) g_{12} f_1 = 0, \tag{2.12}$$
$$[D_t - (\lambda_1-ik)] g_{12} \cdot f_2 - (\lambda_1-ik) g_2 f_{12} = 0, \tag{2.13}$$
$$[D_t - (\lambda_2-ik)] g_{12} \cdot f_1 - (\lambda_2-ik) g_1 f_{12} = 0. \tag{2.14}$$

From (2.11) , (2.12) or (2.13) and (2.14), we know that (f_{12}, g_{12}) is a solution of (1.6). Besides, we have

$$\begin{aligned}
0 &= \{[(D_t + (\lambda_1+ik)) g_0 \cdot f_1 + (\lambda_1+ik) g_1 f_0] g_2 f_{12} \\
&\quad - g_0 f_1 [(D_t + (\lambda_1+ik)) g_2 \cdot f_{12} + (\lambda_1+ik) g_{12} f_2] \\
&= D_t g_0 f_{12} \cdot g_2 f_1 + (\lambda_1+ik) g_1 g_2 f_0 f_{12} - (\lambda_1+ik) g_0 g_{12} f_1 f_2 \\
&= D_t g_0 f_{12} \cdot f_1 g_2 + c(\lambda_1+ik)[g_1 g_2 D_t f_1 \cdot f_2 - f_1 f_2 D_t g_1 \cdot g_2] \\
&= D_t g_0 f_{12} \cdot f_1 g_2 + c(\lambda_1+ik) D_t f_1 g_2 \cdot f_2 g_1 \\
&= D_t [g_0 f_{12} - c(\lambda_1+ik) f_2 g_1] \cdot f_1 g_2
\end{aligned}$$

which implies that

$$g_0 f_{12} = c(\lambda_1+ik) f_2 g_1 + \bar{c} f_1 g_2, \tag{2.15}$$

with $\bar{c}$ being some constant. Similarly, we have

$$g_0 f_{12} = -c(\lambda_2+ik) f_1 g_2 + \tilde{c} f_2 g_1, \tag{2.16}$$

where $\tilde{c}$ is some constant. From (2.15) and (2.16), we deduce

$$g_0 f_{12} = c(\lambda_1+ik) f_2 g_1 - c(\lambda_2+ik) f_1 g_2. \tag{2.17}$$

Furthermore, in a similar way, we may obtain

$$f_0 g_{12} = -c(\lambda_2-ik) f_2 g_1 + c(\lambda_1-ik) f_1 g_2. \tag{2.18}$$

In the following we will show that (f_{12}, g_{12}) is a solution of (1.5). In fact, since (f_1, g_1)

and (f_2, g_2) are solutions of (1.5-1.6), by using (A1)-(A5), (2.5)-(2.18), we have:

$$\begin{aligned}
0 = \quad & [e^{D_n} f_2 \cdot f_2]\{(D_t^2 e^{D_n} - \frac{1}{2} D_t e^{D_n}) f_1 \cdot f_1 + \int_{-\infty}^{\infty} dk\ \nu |\phi_0|^2 \cosh(D_n)[g_1 \cdot g_1^* - f_1 \cdot f_1]\} \\
& - [e^{D_n} f_1 \cdot f_1]\{(D_t^2 e^{D_n} - \frac{1}{2} D_t e^{D_n}) f_2 \cdot f_2 \\
& + \int_{-\infty}^{\infty} dk\ \nu |\phi_0|^2 \cosh(D_n)[g_2 \cdot g_2^* - f_2 \cdot f_2]\} \\
= \quad & 2D_t \cosh(D_n)(D_t f_1 \cdot f_2) \cdot f_1 f_2 - \sinh(D_n)(D_t f_1 \cdot f_2) \cdot f_1 f_2 \\
& - \int_{-\infty}^{\infty} dk\ \nu |\phi_0|^2 \cosh(D_n)(g_2 f_1 \cdot f_1 g_2^* - g_1 f_2 \cdot f_2 g_1^*) \\
= \quad & -\frac{1}{c^2(\lambda_2^2 - \lambda_1^2)}\{2D_t \cosh(D_n)(D_t f_0 \cdot f_{12}) \cdot f_0 f_{12} - \sinh(D_n)(D_t f_0 \cdot f_{12}) \cdot f_0 f_{12} \\
& - \int_{-\infty}^{\infty} dk\ \nu |\phi_0|^2 \cosh(D_n)(g_{12} f_0 \cdot f_0 g_{12}^* - g_0 f_{12} \cdot f_{12} g_0^*) \\
= \quad & \frac{1}{c^2(\lambda_2^2 - \lambda_1^2)}[e^{D_n} f_0 \cdot f_0]\{(D_t^2 e^{D_n} - \frac{1}{2} D_t e^{D_n}) f_{12} \cdot f_{12} \\
& + \int_{-\infty}^{\infty} dk\ \nu |\phi_0|^2 \cosh(D_n)[g_{12} \cdot g_{12}^* - f_{12} \cdot f_{12}]\}
\end{aligned}$$

which means that (f_{12}, g_{12}) is a solution of (1.5). Thus we have completed the proof of proposition 2. ■

As an application of the propositions 1 and 2, we may obtain the soliton solutions of equations (1.5) and (1.6). For example, using BT (2.1-2.4) with $\mu(\lambda) = A - \lambda - \frac{1}{4}$, where λ is arbitrary real constant and $A = \frac{1}{2}\lambda \int_{-\infty}^{+\infty} \frac{\nu|\phi_0|^2}{\lambda^2+k^2} dk$, we have, from the starting solution $f = 1, g = -1$,

$$\begin{aligned}
f' &= 1 + e^{\eta} \\
g' &= 1 + e^{\eta + i\alpha},
\end{aligned}$$

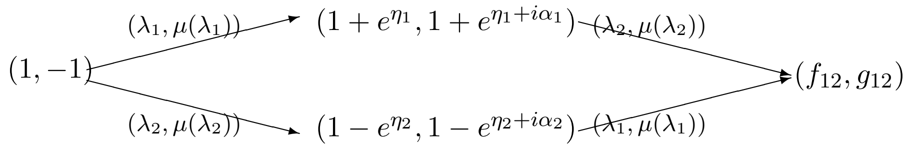
or

$$\begin{aligned}
f' &= 1 - e^{\eta} \\
g' &= 1 - e^{\eta + i\alpha},
\end{aligned}$$

where

$$\begin{aligned}
\eta &= 2\lambda t + pn + \delta \\
e^{i\alpha} &= \frac{k + i\lambda}{k - i\lambda} \\
p &= \frac{1}{2} \ln \frac{A - (\lambda + \frac{1}{4})}{(\lambda - \frac{1}{4}) - A}
\end{aligned}$$

and δ is a real phase constant. Furthermore, using the nonlinear superposition formula (2.5-2.6) with $c = \frac{1}{\lambda_2 - \lambda_1}$, we have



with

$$f_{12} = 1 + \frac{\lambda_1+\lambda_2}{\lambda_2-\lambda_1}e^{\eta_1} + \frac{\lambda_1+\lambda_2}{\lambda_2-\lambda_1}e^{\eta_2} + e^{\eta_1+\eta_2}, \tag{2.19}$$

$$g_{12} = -(1 + \frac{\lambda_1+\lambda_2}{\lambda_2-\lambda_1}e^{\eta_1+i\alpha_1} + \frac{\lambda_1+\lambda_2}{\lambda_2-\lambda_1}e^{\eta_2+i\alpha_2} + e^{\eta_1+\eta_2+i\alpha_1+i\alpha_2}), \tag{2.20}$$

where

$$\eta_j = 2\lambda_j t + p_j n + \delta_j, \tag{2.21}$$

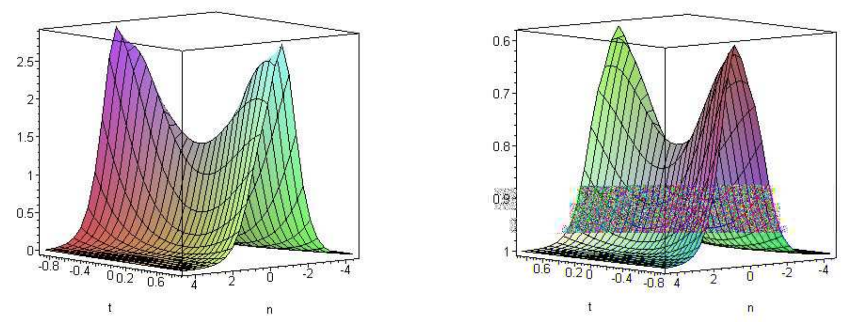
$$e^{i\alpha_j} = \frac{k+i\lambda_j}{k-i\lambda_j}, \tag{2.22}$$

$$p_j = \frac{1}{2}\ln\frac{A-(\lambda_j+\frac{1}{4})}{(\lambda_j-\frac{1}{4})-A}, \qquad j=1,2 \tag{2.23}$$

and λ_j, δ_j are real constants. We notice that (f_{12}, g_{12}) given by (2.19-2.20) with (2.21-2.23) is a 2-soliton solution. If we take

$$\lambda_1 = 1.26, \quad \lambda_2 = 1.74, \quad p_1 = -\ln(7), \quad p_2 = \ln(7), \quad A = 1.5$$

in (2.21-2.23), we can show the behaviors of $U = (\ln f_{12})_{tt}$ and $|v|^2 = |e^{ikt}\phi_0(k)g_{12}/f_{12}|^2 = |g_{12}/f_{12}|^2$ (here we assume $|\phi_0(k)| = 1$) in equations (1.9-1.10) graphically:



(a) (b)

Figure 1. The 2-soliton solution: (a) $U-$field , (b) behavior of $|v|^2$ with $k = 2$.

A general determinantal representation of N-soliton solution can be derived using BT (2.1-2.4) and nonlinear superposition formula (2.5) :

$$F_N = \frac{c}{f_0^{N-1}} \begin{vmatrix} f_1 & (-\frac{d}{dt}+\lambda_1)f_1 & \cdots & (-\frac{d}{dt}+\lambda_1)^{N-1}f_1 \\ f_2 & (-\frac{d}{dt}+\lambda_2)f_2 & \cdots & (-\frac{d}{dt}+\lambda_2)^{N-1}f_2 \\ \vdots & \vdots & \vdots & \vdots \\ f_N & (-\frac{d}{dt}+\lambda_N)f_N & \cdots & (-\frac{d}{dt}+\lambda_N)^{N-1}f_N \end{vmatrix},$$

where $f_i(i=1,2,\cdots,N)$ is obtained from starting solution f_0 using BT, i.e. $f_0 \overset{(\lambda_i,\mu_i)}{\longrightarrow} f_i$. Similar to the proof in [25], this result can be proved by induction. For example, when $N=2$, we have

$$\begin{aligned} F_2 &= \frac{c}{f_0^{N-1}} \begin{vmatrix} f_1 & (-\frac{d}{dt}+\lambda_1)f_1 \\ f_2 & (-\frac{d}{dt}+\lambda_2)f_2 \end{vmatrix} = \frac{c}{f_0} \begin{vmatrix} f_1 & (-\frac{d}{dt}+\lambda_1)f_1 \\ f_2 & (-\frac{d}{dt}+\lambda_2)f_2 \end{vmatrix} \\ &= \frac{c}{f_0}[D_t - (\lambda 1 - \lambda 2)]f_1 \cdot f_2]. \end{aligned}$$

From the nonlinear superposition formula, we know that F_2 is a two-solition solution.

Particularly, if we take

$$f_j = 1 \pm e^{\eta_j}, \qquad g_j = 1 \pm e^{\eta_j + i\alpha_j} \qquad (j=1,2,\cdots,N), \qquad c = \frac{1}{\lambda_2 - \lambda_1},$$

where η_j and $e^{i\alpha_j}$ are given in (2.21-2.23), then the N-soliton solution of the equations (1.5-1.6) can be expressed as:

$$F_N = \frac{1}{\lambda_2 - \lambda_1} \begin{vmatrix} f_1 & (-\frac{d}{dt}+\lambda_1)f_1 & \cdots & (-\frac{d}{dt}+\lambda_1)^{N-1}f_1 \\ f_2 & (-\frac{d}{dt}+\lambda_2)f_2 & \cdots & (-\frac{d}{dt}+\lambda_2)^{N-1}f_2 \\ \vdots & \vdots & \vdots & \vdots \\ f_N & (-\frac{d}{dt}+\lambda_N)f_N & \cdots & (-\frac{d}{dt}+\lambda_N)^{N-1}f_N \end{vmatrix},$$

and

$$G_N = \frac{(-1)^{N-1}}{\lambda_2 - \lambda_1} \begin{vmatrix} g_1 & (-\frac{d}{dt}+\lambda_1)g_1 & \cdots & (-\frac{d}{dt}+\lambda_1)^{N-1}g_1 \\ g_2 & (-\frac{d}{dt}+\lambda_2)g_2 & \cdots & (-\frac{d}{dt}+\lambda_2)^{N-1}g_2 \\ \vdots & \vdots & \vdots & \vdots \\ g_N & (-\frac{d}{dt}+\lambda_N)g_N & \cdots & (-\frac{d}{dt}+\lambda_N)^{N-1}g_N \end{vmatrix}.$$

3 Conclusion

In this paper, we proposed a differential-difference version of the kdv equation with a source (1.5-1.6) and presented a bilinear Bäcklund transformation as well as a nonlinear superposition formula for it. As an application of the obtained results, N-soliton solution is derived.

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Appendix

A Hirota's bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions a, b, c and d.

$$(D_t e^{D_n} a \cdot a)(e^{D_n} b \cdot b) - (e^{D_n} a \cdot a)(D_t e^{D_n} b \cdot b) = 2\sinh(D_n)(D_t a \cdot b) \cdot ab. \quad (A.1)$$

$$\begin{aligned}(D_t^2 e^{D_n} a \cdot a)(e^{D_n} b \cdot b) &- (e^{D_n} a \cdot a)(D_t^2 e^{D_n} b \cdot b)\\ &= 2D_t(D_t e^{-D_n} a \cdot b) \cdot (e^{D_n} a \cdot b) + 2\sinh(D_n)(D_t^2 a \cdot b) \cdot ab\\ &= 2D_t \cosh(D_n)(D_t a \cdot b) \cdot ab. \end{aligned} \quad (A.2)$$

$$e^{-D_n}[(D_t a \cdot b) \cdot cd - ab \cdot (D_t c \cdot d)] = D_t(e^{-D_n} a \cdot d) \cdot (e^{D_n} c \cdot b). \quad (A.3)$$

$$e^{D_n}[(D_t a \cdot b) \cdot cd - ab \cdot (D_t c \cdot d)] = D_t(e^{D_n} a \cdot d) \cdot (e^{-D_n} c \cdot b). \quad (A.4)$$

$$2\sinh(D_n)(D_t a \cdot b) \cdot ab = D_t(e^{D_n} a \cdot b) \cdot (e^{-D_n} a \cdot b). \quad (A.5)$$

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