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# Massless Pseudo-scalar Fields and Solution of the Federbush Model

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**Abstract**

The formal Heisenberg equations of the Federbush model are linearized and then are directly integrated applying the method of dynamical mappings. The fundamental role of two-dimensional free massless pseudo-scalar fields is revealed for this procedure together with their locality condition taken into account. Thus the better insight into solvability of this model is obtained together with the additional phase factor for its general solution, and the meaning of the Schwinger terms is elucidated.

## 1 Introduction

Generalizing the methods, giving solutions in sufficiently simple models one can hope to develop a technique to be effective for the comparatively non-trivial and physically important problems [29]. Examples of such partly or entirely solvable models are models with four-fermion interactions in two-dimensional space-time and related to them non-linear bosonic models of the sine-Gordon (SG) type. The relentless interest in these models is due to the fact that their non-Abelian four-dimensional analogs are more or less successfully used for the analysis and explanation of various non-perturbative effects in modern theory of strong interactions, such as: description of the processes of quark hadronization and the phenomenon of spontaneous symmetry breaking [8], [20]. They serve also as a "testing ground" for various non-perturbative methods [1], [2], [5], [6], [9]. As a rule an operator solution for these models is obtained with the help of proper operator ansatz [11], [13], [17], [18], [29] instead of consecutive integration the corresponding equations of motion.

Recently Faber, Ivanov in a series of papers (see e.g. [9], [10]), following Morchio et al. [18], re-examined some ambiguities [7] of the Thirring model and elucidated the importance of existence of two-dimensional free massless scalar fields for its solution.

One of the aims of the present work is to show a similar role of free massless *pseudo-scalar* fields for a direct step-by-step integration of the Heisenberg equations (HEqs) of the Federbush model [11], and to advocate as a general method the corresponding linearization procedure, successfully used to solve some non-relativistic and relativistic phenomenological models [16], [27]. It is interesting to note that unlike free massless scalar field, the free massless pseudo-scalar field has a well-defined generating functional [10] and appears in a definite sense as a gauge invariant object [2].

Another goal is to understand the role of Schwinger terms for this model. We show that, in spite of a long story of this sloppy question [23], [25], [28] the suggested linearization procedure allows naturally to adopt an idea of solution found in QED [24].

In order to linearize the HEqs we use the notion of physical fields [4], [14], [26]. Let us recall that the HEqs are formal relations between operators, as long as they are not defined on a corresponding vector space. This implies, that in order to give a physical meaning to the quantum-field-system description by means of the Heisenberg fields $\Psi(x)$, it is necessary to represent them in the space of physical states, that, in turn, necessitates to express them in terms of physical fields $\psi(x)$. The relationship between Heisenberg and physical fields is called dynamical mapping (DM), and may be formally expressed generally only as a weak equality: $\Psi(x) \stackrel{\mathrm{w}}{=} \Upsilon[\psi(x)]$. After quantization of the physical fields, satisfying some free field equations, one obtains the HEqs and DM in the normal-ordered form with respect to creation and annihilation operators of a chosen set of physical particles. This normal-ordered form in fact is equivalent in the sense of DM to their primary representations as a weak limit, but is, of course, representation dependent [4].

Since dynamical HEqs contain time derivatives it is necessary to choose initial conditions. There are two physically different choices leading respectively to two essentially different sets of physical fields: for asymptotic *in*-fields [3], [26], for $t \to -\infty$, one has $\lim\limits_{t\to-\infty} \Psi(x^1,t) \stackrel{\mathrm{w}}{=} \Upsilon[\psi_{in}(x^1,-\infty)]$; alternatively for Schrödinger fields [22] one has $\lim\limits_{t\to 0} \Psi(x^1,t) \stackrel{\mathrm{w}}{=} \Upsilon[\psi(x^1,0)]$ as $t \to 0$. The latter choice will be used here. It was advocated in our previous works [16], [27] as some generalization of the usual interaction representation [26] shown its usefulness on the bound-state problem, and in fact was already used for the construction of the solution of the Federbush model [11].

The work is organized as follows. First in section 2 we discuss a linearization procedure for HEqs. Then in section 3 we obtain their solutions in terms of Feynman functional integrals calculated in sections 4 and 5 by pure algebraic operator methods using various bosonization rules for the *free fermion currents only* and the properties of free massless pseudo-scalar fields only. It should be emphasized in order to avoid some confusions, that, as is shown below, the bosonization rules in fact could not be applied here to the local Heisenberg currents directly, and that in two-dimension space-time both the used in section 4 bosonization rules: (4.1) for the local free massive fermion current and (4.6) for the local free massless fermion current, are immediate well-known consequence of conservation of both the vector currents [13], [29]. These two rules introduce however two different pseudo-scalar fields: the free massless field $\phi(x)$ in (4.6) and the field $\Phi(x)$ in (4.1) which in fact is the sine-Gordon field, as is finally checked in subsection 4.2. These fields certainly serve as building blocks for the construction of solution, nevertheless, are necessary but insufficient ingredients to obtain the explicit operator solution of the model, as is shown in sections 4 and 5. The role of Schwinger terms is also discussed in the Summary.

## 2 Step 1. Linearization of Heisenberg equations

The model under consideration describes a parity non-conserving local interaction of two different two-component fermion fields $\Psi_\xi(x)$ in two-dimensional space-time with different masses $m_\xi$, $\xi = \pm 1$, and is defined by a Lagrangian density, bilinear in the two Nöther vector currents $J^\nu_{\xi(\Psi)}(x) = \overline{\Psi}_\xi(x)\gamma^\nu\Psi_\xi(x)$. Splitting the total Hamiltonian of the model into free and interaction parts as

$$H_{(\Psi)}(t) = H_{0(\Psi)}(t) + H_{I(\Psi)}(t) \tag{2.1}$$

$$\equiv \int\limits_{-\infty}^{\infty} dx^1 \sum_{\xi=\pm 1} \left\{ \left( \Psi^\dagger_\xi(x) E_\xi(P^1)\Psi_\xi(x) \right) + \xi\pi\lambda\epsilon_{\mu\nu} J^\mu_{\xi(\Psi)}(x) J^\nu_{-\xi(\Psi)}(x) \right\}, \tag{2.2}$$

and using the equal-time anticommutation relations for the Heisenberg operators[1]

$$\left\{ \Psi_{\xi,\alpha}(x)\,, \Psi_{\xi',\beta}(y) \right\}\Big|_{x_0=y_0} = 0, \quad \left\{ \Psi_{\xi,\alpha}(x)\,, \Psi^\dagger_{\xi',\beta}(y) \right\}\Big|_{x_0=y_0} = \delta_{\xi\xi'}\delta_{\alpha\beta}\,\delta(x^1 - y^1), \tag{2.3}$$

we get the formal HEqs for the fields $\Psi_\xi(x)$

$$i\partial_0\Psi_\xi(x) = \left[ \Psi_\xi(x)\,, H_{(\Psi)}(t) \right] = \left[ E_\xi(P^1) + V_{-\xi(\Psi)}(x) \right] \Psi_\xi(x), \tag{2.4}$$

$$\text{where } V_{-\xi(\Psi)}(x) = \xi 2\pi\lambda\epsilon_{\mu\nu}\gamma^0\gamma^\mu J^\nu_{-\xi(\Psi)}(x) = \xi 2\pi\lambda \left[ \gamma^5 J^0_{-\xi(\Psi)}(x) - I J^1_{-\xi(\Psi)}(x) \right], \tag{2.5}$$

and since $\left[ J^\nu_{-\xi(\Psi)}(x)\,, \Psi_\xi(x) \right] = 0$, we even need not to symmetrize or to pass the operator product in the r.h.s. of the HEqs (2.4) into the normal-ordered form, but the careful definition of the Heisenberg "currents" $V_{-\xi(\Psi)}(x)$ in Eq. (2.5) is still necessary. On the other hand, in order to linearize Eq. (2.4), we should first determine the dynamics of these "currents". For this purpose we rewrite them in a more convenient abstract form: $\nu = A$,

$$\gamma^0\gamma^\nu = O^A, \;\; \xi 2\pi\lambda\epsilon_{\mu\nu}\gamma^0\gamma^\mu = \tilde{O}^A_{-\xi}, \;\; V_{-\xi(\Psi)}(x) \equiv \sum_A \left( \Psi^\dagger_{-\xi}(x) O^A \Psi_{-\xi}(x) \right) \tilde{O}^A_{-\xi}, \tag{2.6}$$

and then by means of Eqs. (2.1)–(2.6), we find for these "currents" the formal HEqs

$$i\partial_0 V_{-\xi(\Psi)}(x) = \left[ V_{-\xi(\Psi)}(x)\,, H_{0(\Psi)}(t) \right] + \left[ V_{-\xi(\Psi)}(x)\,, H_{I(\Psi)}(t) \right] \tag{2.7a}$$

$$= \sum_A \Bigg\{ \left( \Psi^\dagger_{-\xi}(x) \left[ O^A E_{-\xi}(P^1) - E_{-\xi}(P^{1\dagger}) O^A \right] \Psi_{-\xi}(x) \right) \tag{2.7b}$$

$$+ \left( \Psi^\dagger_{-\xi}(x) \left[ O^A V_{\xi(\Psi)}(x) - V^\dagger_{\xi(\Psi)}(x) O^A \right] \Psi_{-\xi}(x) \right) \Bigg\} \left( \tilde{O}^A_{-\xi} \right), \tag{2.7c}$$

$$\text{which, since } O^A \propto \tilde{O}^B_\xi \propto I, \gamma^5, \quad \sum_A \sum_B \left( O^B \right) \left( \left[ O^A\,, \tilde{O}^B_\xi \right] \right) \left( \tilde{O}^A_{-\xi} \right) = 0, \tag{2.7d}$$

$$\text{read as } i\partial_0 V_{-\xi(\Psi)}(x) \equiv \left[ V_{-\xi(\Psi)}(x)\,, H_{(\Psi)}(t) \right] = \left[ V_{-\xi(\Psi)}(x)\,, H_{0(\Psi)}(t) \right]. \tag{2.7e}$$

Of course the conditions (2.7d) are satisfied by any set of the mutually commuting matrices $O^A, \tilde{O}^B$. But the final Eq. (2.7e) implies also the absence of possible contributions from

[1] Our conventions are: $x^\mu = (x^0, x^1)$, $x^0 = t$, $\hbar = c = 1$, $\partial_\mu = (\partial_0, \partial_1)$, metric tensor $g^{00} = -g^{11} = 1$, antisymmetric tensor $\epsilon^{01} = -\epsilon^{10} = 1$, Dirac conjugate field $\overline{\Psi}_\xi(x) = \Psi^\dagger_\xi(x)\gamma^0$, gamma matrices $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$, with $\sigma_i$ and $I$ as Pauli and unit matrices respectively, and operators $P^1 = -i\partial_1 \equiv -i\partial/\partial x^1$, $P^{1\dagger} = i\overleftarrow{\partial}_1$, $E_\xi(P^1) = \gamma^5 P^1 + \gamma^0 m_\xi$.

the Schwinger terms [23]. Only under the assumption, as discussed in the Summary, the evolution of operators $V_{-\xi(\Psi)}(x)$ is governed by the free one-particle Hamiltonian $H_{0(\Psi)}(t)$ only leaving just the line (2.7b). A simple main observation of our approach developed in [16], [27], is, that since the contribution responsible for the interaction vanishes, these Heisenberg operators (in the Heisenberg-field representation), as solutions of HEqs (2.7e) in a weak sense of DM are equivalent to itself in some free-trial-physical-field representation[2] which, according to the explicit form of (2.7a) given below by (6.1), (6.2), we should choose as free fields $\psi_{-\xi}(x)$ with the same mass $m_{-\xi}$ from Hamiltonian (2.2). In other words, from homogeneous HEqs (2.7e) with (2.4), (2.5) we conclude that the DM $\Upsilon[\psi(x)]$ leaves the form of these "currents" invariant up to possible multiplicative constant fixed by initial condition at $t = 0$ equal to unity. That is

$$V_{-\xi(\Psi)}(x) = e^{iH_{(\Psi)}t}V_{-\xi(\Psi)}(x^1,0)e^{-iH_{(\Psi)}t} = e^{iH_{0(\Psi)}t}V_{-\xi(\Psi)}(x^1,0)e^{-iH_{0(\Psi)}t} \longleftrightarrow \qquad (2.8)$$
$$\longleftrightarrow e^{iH_{0(\psi)}t}V_{-\xi(\psi)}(x^1,0)e^{-iH_{0(\psi)}t} = V_{-\xi(\psi)}(x), \quad \text{or} \quad V_{-\xi(\Upsilon[\psi])}(x) \stackrel{\mathrm{w}}{=} V_{-\xi(\psi)}(x),$$

where the last line takes place due to the fact, that *just these and only these* "currents" (2.5) determine the interaction in the HEqs (2.4), so that the latter get a *linearized* form

$$\partial_0\Psi_\xi(x) = -i\left[E_\xi(P^1) + V_{-\xi(\psi)}(x)\right]\Psi_\xi(x). \qquad (2.9)$$

Strictly speaking the operator product in the r.h.s. should now be symmetrized or normal-ordered. We show below, that the correct result however may be obtained proceeding from Eq. (2.9), and argue the only possible weak sense of relation (2.8).

From a dynamical point of view the suggested linearization condition (2.7e), as *a free* equation of motion for Heisenberg "currents", is a natural and straightforward extension of the usual vector current conservation condition. Indeed, supposing for a moment the unit matrices in Eq. (2.6) for all $A$, $O^A = I$, we immediately turn Eq. (2.7e) into the well-known divergence identity for the vector current, that is

$$i\partial_0 J^0_{-\xi(\Psi)}(x) = \left[J^0_{-\xi(\Psi)}(x)\,,H_{0(\Psi)}(t)\right], \quad \text{means} \quad \partial_\mu J^\mu_{-\xi(\Psi)}(x) = 0, \qquad (2.10)$$

$$\text{and} \quad \left[Q_{-\xi(\Psi)}(t)\,,H_{(\Psi)}(t)\right] = 0, \quad \text{where} \quad Q_{-\xi(\Psi)}(t) = \int\limits_{-\infty}^{\infty} dx^1 J^0_{-\xi(\Psi)}(x^1,t). \qquad (2.11)$$

Thus the conservation of the local Heisenberg vector current with corresponding fermionic charge $Q_{-\xi(\Psi)}(t)$ is equivalent to the fact, that the time evolution of its zero component $J^0_{-\xi(\Psi)}(x)$ is governed by the free part $H_{0(\Psi)}(t)$ of the total Hamiltonian $H_{(\Psi)}(t)$ only (a similar phenomenon in the SG model is described in Ref. [6]). However, since the component $J^0_{-\xi(\Psi)}(x)$ alone does not define entirely the r.h.s. of the equation of motion (2.4), then the conservation condition (2.10) alone does not enough for the second line in Eq. (2.8) to exist, whereas the condition (2.7e) in common with Eq. (2.4) is enough.

# 3 Step 2. Functional integral representation of the solution

The linearized HEqs (2.9) allow to write their formal solution in the chosen representation of the physical fields (2.8), as a time-ordered ($\mathcal{T}$-ordered) exponential of different sets of

[2] That in fact takes place for all known solvable fermionic models.

mutually non-commuting operators, such as $X^1 = x^1$ and $P^1 = -i\partial_1$, $\gamma^\nu$, $\psi_{-\xi}(x)$

$$\Psi_\xi(x^1, T) = \mathcal{T}\left[\exp\left\{-i\int\limits_0^T d\eta\left(E_\xi(P^1) + V_{-\xi(\psi)}(x)\right)\right\}\right]\Psi_\xi(x^1, 0)\,, \tag{3.1}$$

where the Schrödinger initial conditions at $t = 0$ are imposed. Following Feynman [12], the Heisenberg field (3.1) for $T > 0$ may be reconstructed from the initial field $\Psi_\xi(x^1, 0)$ using the matrix element of the evolution kernel $\mathcal{Y}_{(\psi_{-\xi})}$, which in turn is expressed as a functional integral over the phase space variables $x^1(\eta)$ and $p^1(\eta)$

$$\Psi_\xi(x^1, T) = \int\limits_{-\infty}^{\infty} dy^1 \mathcal{Y}_{(\psi_{-\xi})}(T, x^1|y^1, 0)\Psi_\xi(y^1, 0)\,, \tag{3.2a}$$

$$\mathcal{Y}_{(\psi_{-\xi})}(T, x^1|y^1, 0) = \langle x^1|\mathcal{T}\left[\exp\left\{-i\int\limits_0^T d\eta\left(E_\xi(P^1) + V_{-\xi(\psi)}(X^1, \eta)\right)\right\}\right]|y^1\rangle \tag{3.2b}$$

$$= \int\limits_{x^1(0)=y^1}^{x^1(T)=x^1} \mathcal{D}x^1(\eta)\int \mathcal{D}p^1(\eta)\exp\left(i\int\limits_0^T d\eta\, p^1(\eta)\,\dot{x}^1(\eta)\right)$$

$$\cdot\,\mathcal{T}_{\gamma,\psi_{-\xi}}\left[\exp\left\{-i\int\limits_0^T d\eta\left(E_\xi(p^1(\eta)) + V_{-\xi(\psi)}\left(x^1(\eta), \eta\right)\right)\right\}\right]. \tag{3.2c}$$

However, the integrand still contains a "mixture" of different structures, the $\gamma$-matrix and the field ones $\mathcal{T}_{\gamma,\psi_{-\xi}}[\ldots]$. Nevertheless, the explicit form (2.5) of the free "currents" $V_{-\xi(\psi)}(x) = \xi 2\pi\lambda\left[\gamma^5 J^0_{-\xi(\psi)}(x) - I\, J^1_{-\xi(\psi)}(x)\right]$ prompts the shift of functional variable $p^1(\eta) = \underline{p}^1(\eta) - \xi 2\pi\lambda J^0_{-\xi(\psi)}\left(x^1(\eta), \eta\right)$ to obtain for the matrix element, returning to the previous designation $\underline{p}^1(\eta) \mapsto p^1(\eta)$ with $\dot{x}^\mu(\eta) = (1, \dot{x}^1(\eta))$, the following expression

$$\mathcal{Y}_{(\psi_{-\xi})}(T, x^1|y^1, 0) = \int\limits_{x^1(0)=y^1}^{x^1(T)=x^1} \mathcal{D}x^1(\eta)\mathcal{T}_{\psi_{-\xi}}\left[\exp\left\{-i\xi 2\pi\lambda\int\limits_0^T d\eta\epsilon_{\mu\nu}\dot{x}^\mu(\eta)J^\nu_{-\xi(\psi)}\left(x^1(\eta), \eta\right)\right\}\right]$$

$$\cdot\int \mathcal{D}p^1(\eta)\mathcal{T}_\gamma\left[\exp\left\{-i\int\limits_0^T d\eta\left(\gamma^5 p^1(\eta) + \gamma^0 m_\xi - p^1(\eta)\dot{x}^1(\eta)\right)\right\}\right]. \tag{3.3}$$

The most important property of this expression is the splitting of the $\gamma$-matrix and the field operator orderings, that moreover, are integrated over different functional variables. It is a simple matter to see that up to this point the space-time dimension does not play any essential role: if the linearization conditions (2.7d), (2.7e) are satisfied in higher-dimensional cases as well the above transformations yield results [16], similar to Eq. (3.3).

# 4 Step 3. Bosonization and $\mathcal{T}$-ordering

## 4.1 Bosonization

The local vector current of the free massive Dirac field is defined by means of antisymmetrization or normal-ordering of a formal field product [3]. Due to the particular property of two-dimensional space-time, like $\gamma^5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$ and the conservation law similar to (2.10) these *free* massive currents bosonize to pseudo-scalar fields $\Phi_{-\xi}(x)$ [21], [29]

$$\epsilon_{\mu\nu} J^{\nu}_{-\xi(\psi)}(x) = \frac{\partial_\mu \Phi_{-\xi}(x)}{\sqrt{\pi}}, \quad \text{where} \quad J^{\mu}_{-\xi(\psi)}(x) =: \overline{\psi}_{-\xi}(x)\gamma^\mu\psi_{-\xi}(x):, \tag{4.1}$$

leading to the simple full-derivative representation

$$-\epsilon_{\mu\nu}\dot{x}^\mu(\eta) J^{\nu}_{-\xi(\psi)}\left(x^1(\eta),\eta\right) = -\frac{1}{\sqrt{\pi}}\frac{d}{d\eta}\,\Phi_{-\xi}\left(x^1(\eta),\eta\right), \tag{4.2}$$

that recasts the first of the time-ordered exponentials in Eq. (3.3) to the form

$$\mathcal{T}_{\psi_{-\xi}}\left[\ldots\right] = \mathcal{T}_{\Phi_{-\xi}}\left[\exp\left\{-ig\int\limits_0^T d\eta \frac{d}{d\eta}\,\Phi_{-\xi}\left(x^1(\eta),\eta\right)\right\}\right], \quad \text{with} \quad g = 2\xi\lambda\sqrt{\pi}. \tag{4.3}$$

Thus, the problem of $\mathcal{T}$-ordering of the exponential bilinear in the fermion field is transformed to the $\mathcal{T}$-ordering of an exponential linear in the pseudo-scalar field $\Phi_{-\xi}(x)$. This is a considerably more simple exercise at least for the free boson case, as is shown in Appendix. The necessary relation between the pseudo-scalar fields $\Phi_{-\xi}(x)$ and free pseudo-scalar fields $\phi_{-\xi}(x)$, turning the problem to the free-field case, is found in the next subsection.

## 4.2 Reduction to the free massless pseudo-scalar fields

A simplest way to relate the field $\Phi_{-\xi}(x)$ to the free field $\phi_{-\xi}(x)$ can be derived from the relation between the free massive $\psi_{-\xi}(x^1,t)$ and the free massless $\chi_{-\xi}(x^1,t)$ Dirac fields, with the formal unitary transformation operator $G_{-\xi}(t)$ [4], so that for arbitrary $t$, from

$$i\gamma^\mu\partial_\mu\psi_{-\xi}(x) = m_{-\xi}\psi(x), \qquad i\gamma^\mu\partial_\mu\chi_{-\xi}(x) = 0, \tag{4.4a}$$

$$\text{one has} \quad \psi_{-\xi}(x^1,t) = G^{-1}_{-\xi}(t)\chi_{-\xi}(x^1,t)G_{-\xi}(t), \tag{4.4b}$$

$$\text{as well as} \quad J^{\mu}_{-\xi(\psi)}(x^1,t) = G^{-1}_{-\xi}(t) j^{\mu}_{-\xi(\chi)}(x^1,t)G_{-\xi}(t), \tag{4.4c}$$

$$\text{where} \quad j^{\mu}_{-\xi(\chi)}(x) =: \overline{\chi}_{-\xi}(x)\gamma^\mu\chi_{-\xi}(x):. \tag{4.4d}$$

Now we temporarily omit for brevity the index $-\xi$, using $m_{-\xi} = m$ and so on, to obtain by means of these Dirac equations the following relations

$$[R(t),\chi(x^1,t)] = im\gamma^0\chi(x^1,t), \quad \text{for} \quad R(t) \equiv \{\partial_0 G(t)\}\, G^{-1}(t), \tag{4.5a}$$

$$\text{so, that} \;\; R(t) = -im\int\limits_{-\infty}^{\infty} dy^1\left(\overline{\chi}(y^1,t)\chi(y^1,t)\right) = -R^\dagger(t), \tag{4.5b}$$

$$G(t) = \mathcal{T}_\chi\left[\exp\left\{\int\limits_0^t d\eta R(\eta)\right\}\right] G(0), \quad G^\dagger(t) = G^{-1}(t), \tag{4.5c}$$

where the solution (4.5b), (4.5c) of Eq. (4.5a) in terms of operators $\chi(x^1,t)$ is obtained by means of the corresponding equal-time anti-commutation relations of the type (2.3).

The following comments are in order. First of all, we note, that an operator of the same type as $R(t)$ is present in the Hamiltonian (2.2) and should exist under appropriate regularization for example in normal-ordered form [9], [29]. The $G(t)$-transformation relates the time evolutions of two different free quantized fields, each of them is defined on a corresponding space of states. In fact, it is a Bogoliubov transformation diagonalizing the free massive fermionic Hamiltonian in terms of the free massless fermion fields for arbitrary time $t$ similarly to [4]. The operator $G(0)$ can be used to impose additional initial conditions. Because this operator does not change the final results of further calculations, we eliminate it further on choosing for simplicity $G(0) = I$.

For the *free massless* fermion current (4.4d) the above arguments, originative earlier the bosonization rules (4.1), introduce now the *free massless* pseudo-scalar field $\phi(x)$ [13]

$$\epsilon_{\mu\nu} j^{\nu}_{(\chi)}(x) = \frac{1}{\sqrt{\pi}}\,\partial_\mu \phi(x)\,, \qquad \overline{\chi}(x)\chi(x) = \upsilon \cos\left(\beta\phi(x)\right), \qquad \beta = 2\sqrt{\pi}, \tag{4.6}$$

connecting this current and the field by the equation similar to Eq. (4.2), so that both the equations amount to the relations between the pseudo-scalar fields only

$$\frac{d\Phi\left(x^1(\eta),\eta\right)}{d\eta} = G^{-1}(\eta)\left(\frac{d\phi\left(x^1(\eta),\eta\right)}{d\eta}\right)G(\eta), \quad \Phi(x^1,t) = G^{-1}(t)\phi(x^1,t)G(t), \tag{4.7}$$

$$\text{where } \; G(t) = \mathcal{T}_\phi\left[\exp\left\{\int\limits_0^t d\eta R(\eta)\right\}\right], \quad R(\eta) = -im\upsilon \int\limits_{-\infty}^{\infty} dy^1 \cos(\beta\phi(y^1,\eta)), \tag{4.8}$$

and the index $-\xi$ is assumed for all quantities. The value of $\upsilon$ depends on the renormalization prescription chosen for the normal-ordering procedures of boson and fermion fields [9], [10] assumed in both sides of Eqs. (4.6). These procedures are however irrelevant for our further operator transformations, so we do not specify further the value of $\upsilon$.

The next comment concerns the properties of the field $\Phi(x)$. These may be easily derived substituting into the free field equation for the field $\phi(x)$ its formal expression from Eq. (4.7). Namely, for $x^0 = t$

$$\partial_\mu\partial^\mu\phi(x) \equiv \left(\partial_0^2 - \partial_1^2\right)\phi(x^1,t) = 0, \quad \text{where} \quad \phi(x^1,t) = G(t)\Phi(x^1,t)G^{-1}(t). \tag{4.9}$$

Using here Eqs. (4.8) we arrive at the following form of the second-time-derivative

$$\partial_0^2\phi(x^1,t) = G(t)[\partial_0^2\Phi(x^1,t)]G^{-1}(t) - [\partial_0\phi(x^1,t), R(t)]. \tag{4.10}$$

The commutator is easy computed by means of (4.8) and the canonical commutation relation for the free pseudo-scalar field $\left[\partial_0\phi(x^1,t)\,,\phi(y^1,t))\right] = -i\delta(x^1-y^1)$ as follows

$$\left[\partial_0\phi(x^1,t), R(t)\right] = m\upsilon\beta\sin\left(\beta\phi(x^1,t)\right). \tag{4.11}$$

Thus the second relation (4.9) leads to the SG equation of motion for the field $\Phi(x)$

$$\partial_\mu\partial^\mu\phi(x^1,t) = G(t)\left[\partial_\mu\partial^\mu\Phi(x) - m\upsilon\beta\sin\left(\beta\Phi(x)\right)\right]G^{-1}(t) = 0, \tag{4.12}$$

which is reproduced for $\beta = 2\sqrt{\pi}$ by the well-known Lagrangian density [7], [15], [17], [21]

$$\mathcal{L}^{SG}(x) = \frac{1}{2}\partial_\mu\Phi(x)\,\partial^\mu\Phi(x) - \upsilon m\left[\cos\left(\beta\Phi(x)\right) - 1\right]. \tag{4.13}$$

### 4.3 Conversion of the $\mathcal{T}$-exponential into the unordered form

Now we are ready to get rid of the $\mathcal{T}$-ordering in Eq. (4.3). This problem is simplified by using the well-known operator formula for the $\mathcal{T}$-exponentials [19]

$$\mathcal{U}_{A+B}(T) = \mathcal{U}_A(T)\,\mathcal{U}_C(T), \quad \text{where} \quad C(\eta) = \Big(\mathcal{U}_A(\eta)\Big)^{-1} B(\eta)\,\mathcal{U}_A(\eta), \tag{4.14}$$

$$\text{with} \quad \mathcal{U}_{A+B}(T) = \mathcal{T}\left[\exp\left\{\int_0^T d\eta\Big(A(\eta)+B(\eta)\Big)\right\}\right], \quad \text{and so on.}$$

Identifying here $A(\eta) = R(\eta)$, $B(\eta) = -ig\left(d\phi(x^1(\eta),\eta)/d\eta\right)$, by means of the first Eq. (4.7) and Eq. (4.8), we transform the expression on the r.h.s. of Eq. (4.3) as follows

$$G^{-1}(T)\mathcal{T}_\phi\left[\exp\left\{\int_0^T d\eta\left(R(\eta) - ig\,\frac{d\phi(x^1(\eta),\eta)}{d\eta}\right)\right\}\right], \quad \text{with} \;\; g = \xi\lambda\beta. \tag{4.15}$$

According to Eqs. (A.10), (A.11) of Appendix, the $\mathcal{T}$-exponential (4.3) for the case of a *free* (pseudo-) scalar field is independent of the trajectory $x^1(\eta)$ and depends only on its end points

$$W(T) \equiv \mathcal{T}_\phi\left[\exp\left\{-ig\int_0^T d\eta\frac{d\phi(x^1(\eta),\eta)}{d\eta}\right\}\right] = U(T)U^{-1}(0)\,, \tag{4.16}$$

$$\text{where} \quad U(T) = \exp\left\{-i\frac{g^2}{4}\theta\left[1-\left(\dot{x}^1(T)\right)^2\right]\right\}\exp\left[-ig\phi(x^1(T),T)\right]\,, \tag{4.17}$$

that allows to apply again the relation (4.14), written now from left-to-right, to the last $\mathcal{T}_\phi-$ exponential in Eq. (4.15), just interchanging the roles of operators only: $B(\eta) = R(\eta)$, $A(\eta) = -ig\left(d\phi(x^1(\eta),\eta)/d\eta\right)$, recasting Eq. (4.15) into the form

$$G^{-1}(T)U(T)\mathcal{T}_\phi\left[\exp\left\{\int_0^T d\eta\, U^{-1}(\eta)R(\eta)U(\eta)\right\}\right]U^{-1}(0)\,. \tag{4.18}$$

From the Eqs. (4.8), (4.17), with the locality condition for the field $\phi(x)$, originated in Eqs. (A.1), (A.2) of Appendix, it is a simple matter to see that, when $x = x(\eta) = (\eta, x^1(\eta))$, $y = (\eta, y^1)$, then $x(\eta) - y = (0, x^1(\eta) - y^1)$,

$$\left[\phi\left(x(\eta)\right),\phi(y)\right] = -i\,D_0\left(x(\eta)-y\right) = 0, \quad U^{-1}(\eta)R(\eta)U(\eta) = R(\eta), \tag{4.19}$$

and the integrand in Eq. (4.18) is reduced to $R(\eta)$. Therefore, by means of (4.7), we arrive at the final expression for the $\mathcal{T}$-exponential of Eqs. (4.3) = (4.18), as

$$G^{-1}(T)U(T)G(T)U^{-1}(0) = \Omega(T,0)\exp\left[-ig\Phi(x(T))\right]\exp\left[ig\Phi(x(0))\right], \tag{4.20}$$

$$\text{where} \quad \Omega(T,0) = \exp\left\{-i\frac{g^2}{4}\left(\theta\left[1-\left(\dot{x}^1(T)\right)^2\right] - \theta\left[1-\left(\dot{x}^1(0)\right)^2\right]\right)\right\}, \tag{4.21}$$

is a c-number phase factor. It different from unity, for example, if absolute value of the velocity $|\dot{x}^1(t)|$ during the evolution has crossed the speed of light an odd number of times.

## 5 Step 4. Initial and Heisenberg fields

Now we are finally able to obtain the solution of the Federbush model [5], [29]. First of all we notice that the above obtained expression (4.20) for the $\mathcal{T}$-exponential is fully independent of the trajectory $x^1(\eta)$ and factorizes out from the path integral in Eq. (3.3) which is then immediately calculated by definition (3.2a), (3.2c) as free evolution kernel $Y_\xi^{(0)}$, expressing the time evolution of another free fermion field with the mass $m_\xi$

$$\mathcal{Y}_{(\psi_{-\xi})}(T,x^1|y^1,0) = \Omega(T,0)\exp\left[-ig\Phi_{-\xi}(x(T))\right]\exp\left[ig\Phi_{-\xi}(x(0))\right] Y_\xi^{(0)}(T,x^1|y^1,0)\,, \tag{5.1a}$$

$$\text{where}\quad Y_\xi^{(0)}(T,x^1|y^1,0) = \theta(T)\exp\left[-iTE_\xi\left(P^1\right)\right]\,\delta(x^1-y^1) \tag{5.1b}$$

$$= \int\limits_{x^1(0)=y^1}^{x^1(T)=x^1} \mathcal{D}x^1(\eta)\int \mathcal{D}p^1(\eta)\mathcal{T}_\gamma\left[\exp\left\{-i\int\limits_0^T d\eta\left(\gamma^5 p^1(\eta)+\gamma^0 m_\xi - p^1(\eta)\dot{x}^1(\eta)\right)\right\}\right],$$

$$\text{so, that}\quad \int\limits_{-\infty}^{\infty} dy^1\, Y_\xi^{(0)}(T,x^1|y^1,0)\;\psi_\xi(y^1,0) = \psi_\xi(x^1,T)\,. \tag{5.1c}$$

In order to check Eq. (5.1b) we need only the following relation [20]

$$\int\limits_{x^1(0)=y^1}^{x^1(T)=x^1} \mathcal{D}x^1(\eta)\,[\ldots] = \int \mathcal{D}\dot{x}^1(\eta)\delta\left(x^1-y^1-\int\limits_0^T d\eta\,\dot{x}^1(\eta)\right)\,[\ldots],$$

and the Fourier-representation for the delta-function. From the Eqs. (5.1) above it is a simple matter to see, that in order to obtain the correct solution for $x^1(T)=x^1$, the initial field for $x^1(0)=y^1$ should be chosen as follows

$$\Psi_\xi(y^1,0) = \exp\left[-\,ig\Phi_{-\xi}(y^1,0)\right]\psi_\xi(y^1,0), \qquad g = 2\xi\lambda\sqrt{\pi}, \tag{5.2}$$

$$\text{so, that}\quad \Psi_\xi(x^1,T) \overset{\text{w}}{=} \Upsilon[\Phi_{-\xi},\psi_\xi] = \Omega(T,0)\exp\left[-\,ig\Phi_{-\xi}(x^1,T)\right]\psi_\xi(x^1,T). \tag{5.3}$$

It should be stressed that, as a solution of the *linear homogeneous equation* (2.9), this expression may be easily renormalized following Wightman [21], [29] by dividing the r.h.s. of Eq. (5.3) on the vacuum expectation value $\langle 0|\exp\left[-\,ig\Phi_{-\xi}(x)\right]|0\rangle$. The factor $\Omega(T,0)$ (4.21) is generated in Eq. (A.9) of Appendix by non-vanishing contribution of zero modes into the dimensionless Pauli-Jordan commutator function (A.2) of the two-dimensional free massless (pseudo-) scalar fields. As was shown by Faber et al. [9], [10], these collective modes are not affected by the dynamics and their consecutive removal makes meaningful the theory of the free massless two-dimensional (pseudo-) scalar field. With this interpretation the value $|\dot{x}^1(t)|$ is the constant velocity of the "center of mass system" of these zero modes decoupled from everything, and $\Omega(T,0)\equiv 1$ for all $T$.

## 6 Summary

We have shown, how the solution of the Federbush model may be obtained by direct integration of the linearized formal HEqs using the notion of physical fields. These fields are conventionally taken as the Schrödinger ones, because the latter as well as Heisenberg fields form a complete set of fields itself, unlike the asymptotic *in (out)* fields [26], what is also useful for construction of explicit solutions of the bound-state problem [16], [27]. We have revealed the fundamental role of the free massless pseudo-scalar field for this integration procedure. We have found, that the solvability of the Federbush model, besides the properties of the linearization (2.8) and the bosonization (4.1), hides in the full-derivative expression (4.2) for the interaction term with an arbitrary trajectory in the time-ordered exponential of the path integral representation for the evolution kernel (3.3) of the Heisenberg field. It is worth to note that the method suggested here may be applied to another known solvable two-dimension models like the Thirring and derivative coupling models [13], the model in Ref. [1], as well as to some four-dimensional ones [16].

Our last comments concern a possible contribution to Eq. (2.7a) from the Schwinger terms. For the massless Thirring model it may be shown that their invariant contribution vanishes independently of the value of the constant in the current commutator. On the one hand, it is well-known [9], [28], that this constant directly depends on the chosen physical field representation – the chosen space of states. On the other hand, it was shown [23], that for the Federbush model its value strongly depends on the chosen point-splitting prescription for the *Heisenberg vector current* in the HEqs (2.4), (2.5), (2.7). That is why in this work we prefer to deal with local currents only.

Indeed, for these currents, because of $\gamma^1 = \gamma^0\gamma^5$, the "non-interacting" and "interacting" parts of the HEqs (2.7a)–(2.7c) may be transcribed straightforwardly as follows

$$i\partial_0 V_{-\xi(\Psi)}(x) - \left[V_{-\xi(\Psi)}(x)\,,H_{0(\Psi)}(t)\right] = \gamma^5\, g\sqrt{\pi}\Big\{ i\partial_\nu\left(\overline{\Psi}_{-\xi}(x)\gamma^\nu\Psi_{-\xi}(x)\right)\Big\} \tag{6.1}$$

$$+I\, g\sqrt{\pi}\Big\{ i\partial_\nu\left(\overline{\Psi}_{-\xi}(x)\gamma^5\gamma^\nu\Psi_{-\xi}(x)\right) - 2m_{-\xi}\left(\overline{\Psi}_{-\xi}(x)\gamma^5\Psi_{-\xi}(x)\right)\Big\},$$

$$\left[V_{-\xi(\Psi)}(x)\,,H_{I(\Psi)}(t)\right] = \gamma^5\, g\sqrt{\pi}\left[J^0_{-\xi(\Psi)}(x)\,,H_{I(\Psi)}(t)\right] \tag{6.2}$$

$$-I\, g\sqrt{\pi}\left[J^1_{-\xi(\Psi)}(x)\,,H_{I(\Psi)}(t)\right].$$

Matching the independent matrix structures $I$ and $\gamma^5$ of these two equations we easily recognize both vector current and axial current divergence identities if the "interacting" part (6.2) vanishes identically. Therefore the conservation of the Heisenberg vector current (2.10) preserves here the absence of the full contribution (6.2) from the Schwinger terms in the r.h.s. of Eqs. (2.7a)–(2.7c), because for the given interaction $H_{I(\Psi)}(t)$ (2.1), (2.2) both commutators in the r.h.s. of Eq. (6.2) reduce to the one and the same Schwinger term in the one and the same commutator $\left[J^1_{-\xi(\Psi)}(x)\,,J^0_{-\xi(\Psi)}(y)\right]\Big|_{x_0=y_0}$, and are vanishing or not only both together. It means that the bosonization rules like (4.1), (4.6) cannot be applied here to the local Heisenberg vector current $J^\nu_{-\xi(\Psi)}(x)$ directly, unlike [7], [9], [13].

The observed contradiction may be overcomed following Ref. [24] in QED, if we impose bosonization rules to the *free* fermion currents (4.1), (4.6) *only*, and if *only* weak correspondence (2.8) between the Heisenberg and free currents take place. Thus the Schwinger term does not contribute into the HEqs (2.7a), because it becomes meaningful *only after* the identification transformation (2.8) determining the representation space. These arguments, besides of all, demonstrate a self-consistence of the suggested linearization procedure (2.4), (2.5) $\mapsto$ (2.7e) $\mapsto$ (2.8), (2.9) with obtained solution (5.3) or (6.3), (6.4).

Thus we come to the conclusion: the transformation of the "currents" (2.8), generated for the Federbush model by the DM (5.3) with the pseudo-scalar SG field $\Phi_{-\xi}(x^1,t)$ and normal-ordered with respect to the Schrödinger free-fermion-physical fields $\psi_{\pm\xi}(x)$

$$\Psi_\xi(x^1,t) \overset{\text{w}}{=} \Upsilon[\psi_{\pm\xi}] =: \exp\left[-2i\xi\lambda\sqrt{\pi}\,\Phi_{-\xi}(x^1,t)\right]\psi_\xi(x^1,t) :, \tag{6.3}$$

$$\text{where} \quad \Phi_{-\xi}(x^1,t) = \frac{\sqrt{\pi}}{2}\left(\int\limits_{-\infty}^{x^1} dy^1 - \int\limits_{x^1}^{\infty} dy^1\right) J^0_{-\xi(\psi)}(y^1,t), \tag{6.4}$$

(see (4.1)) leaving invariant the local form of the both components of vector current $J^\nu_{-\xi(\Psi)}(x) \overset{\text{w}}{=} J^\nu_{-\xi(\psi)}(x)$, as well as conservation law (2.10), changes their algebra, extending it by the Schwinger term, whereas for the conserving charges (2.11)

$$Q_{-\xi(\Psi)}(t) \overset{\text{w}}{=} Q_{-\xi(\psi)}(t) = \frac{\Phi_{-\xi}(\infty,t) - \Phi_{-\xi}(-\infty,t)}{\sqrt{\pi}} \equiv \frac{2}{\sqrt{\pi}}\,\Phi_{-\xi}(\infty,t), \tag{6.5}$$

the algebra remains unchanged.

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# Appendix

## A

Let us consider more general time-ordered exponential of the total derivative of the free *massive* (pseudo-) scalar field $\varphi(\eta) = g\phi(x^1(\eta),\eta)$, taken on an arbitrary trajectory $x^1(\eta)$, where its commutator function $\Delta_m(t|\eta)$ is expressed via the Bessel function $\mathcal{J}_0(z)$ as

$$[\varphi(t),\varphi(\eta)] = \Delta_m(t|\eta) = -ig^2 D_m\left(x(t) - x(\eta)\right), \tag{A.1}$$

$$D_m(x) = 2\pi i \int \frac{d^2k}{(2\pi)^2}\,\varepsilon(k^0)\delta\left(k^2 - m^2\right)e^{-i(kx)} = \frac{\varepsilon(x_0)}{2}\,\theta\left(x^2\right)\mathcal{J}_0\left(m\sqrt{x^2}\right). \tag{A.2}$$

Since this commutator is a c-number, by repeated application of the well known formula: $\exp(A)\exp(B) = \exp(A+B)\exp(\{1/2\}[A,B])$, one can obtain for the $\mathcal{T}-$ exponential

$$\mathcal{T}_\varphi\left[\exp\left\{-i\int\limits_0^T dt\frac{d\varphi(t)}{dt}\right\}\right] = \exp\left\{-i\varphi(T)\right\}\exp\left\{i\varphi(0)\right\}\exp\left\{-\frac{1}{2}\Delta_m(T|0)\right\} \tag{A.3}$$
$$\cdot\exp\left[-\frac{1}{2}\int\limits_0^T d\eta\int\limits_0^\eta d\tau\left\{\frac{d}{d\eta}\frac{d}{d\tau}\Delta_m(\eta|\tau)\right\}\right],$$

and transcribe it, by means of the locality condition (4.19) and the next equality

$$\int\limits_0^T d\eta\int\limits_0^\eta d\tau\left\{\frac{d}{d\eta}\frac{d}{d\tau}\Delta_m(\eta|\tau)\right\} = \Delta_m(T|T) - \Delta_m(T|0) - \int\limits_0^T d\eta\left[\frac{d}{d\tau}\Delta_m(\eta|\tau)\Big|_{\tau=\eta}\right],$$

as

$$\mathcal{T}_\varphi\Big[\dots\Big] = \exp\{-i\varphi(T)\}\exp\{i\varphi(0)\}\exp\left\{\frac{1}{2}\int_0^T d\eta\left[\frac{d}{d\tau}\Delta_m(\eta|\tau)\Big|_{\tau=\eta}\right]\right\}. \tag{A.4}$$

The function $\Delta_m(\eta|\tau)$ is discontinuous on the light-cone, at $\eta = \tau$, and its derivative has a singularity there. Therefore the expression for the integral in the last exponential is indefinite and should be redefined as the following limit with $\sigma \to +0$ of

$$\int_0^T d\eta\left[\frac{d}{d\tau}\Delta_m(\eta|\tau)\Big|_{\tau=\eta-\sigma}\right] = \frac{g^2}{i}\int_0^T d\eta\, 2\pi i\int\frac{d^2k}{(2\pi)^2}\varepsilon(k^0)\delta(k^2-m^2) \tag{A.5}$$
$$\cdot\left[\frac{d}{d\tau}e^{-i(k\cdot(x(\eta)-x(\tau)))}\Big|_{\tau=\eta-\sigma}\right].$$

For the latter derivative, using the Taylor expansion of the trajectory $x^1(\eta-\sigma)$ with respect to $\sigma > 0$, and changing the momentum variables as $q^\mu = \sigma k^\mu$, we obtain

$$\frac{d}{d\tau}e^{-i\left(k^0(\eta-\tau)-k^1(x^1(\eta)-x^1(\tau))\right)}\Big|_{\tau=\eta-\sigma} = \left[\frac{d}{d\eta}-\frac{1}{\sigma}\frac{\partial}{\partial\gamma}\right]e^{-i\gamma\left(q^0-q^1\dot{x}^1(\eta)\right)}\Big|_{\gamma=1}. \tag{A.6}$$

After these changes the integral (A.5) becomes

$$\frac{g^2}{i}\int_0^T d\eta\left[\frac{d}{d\eta}-\frac{1}{\sigma}\frac{\partial}{\partial\gamma}\right]2\pi i\int\frac{d^2q}{(2\pi)^2}\varepsilon(q^0)\delta(q^2-\sigma^2m^2)e^{-i\gamma\left(q^0-q^1\dot{x}^1(\eta)\right)}\Big|_{\gamma=1} \tag{A.7}$$
$$= \frac{g^2}{2i}\int_0^T d\eta\left[\frac{d}{d\eta}-\frac{1}{\sigma}\frac{\partial}{\partial\gamma}\right]\varepsilon(\gamma)\theta\left(\gamma^2\left[1-\left(\dot{x}^1(\eta)\right)^2\right]\right)\mathcal{J}_0\left(\sigma m\gamma\sqrt{1-\left(\dot{x}^1(\eta)\right)^2}\right)\Big|_{\gamma=1}.$$

Because $\mathcal{J}_0(z) = 1 + O(z^2)$, it is easy to see, that the mass dependence of (A.2) leads here just to terms of order $\sigma$. So, as $\sigma \to +0$, the direct substitution $\delta(q^2-\sigma^2m^2) \mapsto \delta(q^2)$, gives for the integral (A.7)

$$\frac{g^2}{2i}\int_0^T d\eta\left[\frac{d}{d\eta}-\frac{1}{\sigma}\frac{\partial}{\partial\gamma}\right]\varepsilon(\gamma)\theta\left(\gamma^2\left[1-\left(\dot{x}^1(\eta)\right)^2\right]\right)\Big|_{\gamma=1}. \tag{A.8}$$

The integrand in (A.8) does not depend on $\gamma$ at all near the point $\gamma = 1$, and its corresponding derivative vanishes, eliminating the contribution singular at $\sigma \to +0$. The remaining integral on $d\eta$ is taken immediately, leading to the expression for the limit

$$\lim_{\sigma\to+0}\int_0^T d\eta\left[\frac{d}{d\tau}\Delta_m(\eta|\tau)\Big|_{\tau=\eta-\sigma}\right] = \frac{g^2}{2i}\left\{\theta\left[1-\left(\dot{x}^1(T)\right)^2\right]-\theta\left[1-\left(\dot{x}^1(0)\right)^2\right]\right\}. \tag{A.9}$$

As a result, the time-ordered exponential (A.3) takes the following form

$$\mathcal{T}_{\varphi}\left[\exp\left\{-i\int\limits_{0}^{T}dt\frac{d\varphi(t)}{dt}\right\}\right]=U(T)U^{-1}(0), \tag{A.10}$$

$$\text{where}\quad U(T)=\exp\left\{-i\frac{g^2}{4}\theta\left[1-\left(\dot{x}^1(T)\right)^2\right]\right\}\exp\left\{-i\varphi(T)\right\}. \tag{A.11}$$

This result is of fundamental importance for all computations, because it depends on the values of an unknown trajectory $x^1(\eta)$ and velocity $\dot{x}^1(\eta)$ only at the initial and final points, $t=0$ and $t=T$. Moreover, the phase factor from (A.4), (A.9) in (A.10) becomes the same for both the massive and massless cases, that, on the one hand, corresponds to mass independence of the light-cone behavior, but on the other hand, due to the dimensionless nature of the commutator function (A.2), represents by the construction (A.5)–(A.8) the non-trivial infrared properties of the free massless (pseudo-) scalar field, such as the existence and the non-vanishing contribution of its zero modes [9], [10].

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