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Extensions of 1-Dimensional Polytropic Gas Dynamics

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Abstract

1-dimensional polytropic gas dynamics is integrable for trivial reasons, having $2 < 3$ components. It is realized as a subsystem of two different integrable systems: an infinite-component hydrodynamic chain of Lax type, and a 3-component system not of Lax type.

1 Introduction

The polytropic gas dynamics in $1 + 1$ dimensions has the form

$$u_t = uu_x + const\ \rho^\Gamma \rho_x, \qquad (1.1a)$$
$$\rho_t = (\rho u)_x, \qquad (1.1b)$$

where x is the space coordinate, t is the (minus physical) time coordinate, subscripts t and x denote partial derivatives, u is the velocity, ρ is the density,

$$\Gamma = \gamma - 2, \qquad (1.2)$$

and γ is the polytropic exponent. The constant "const" entering equation (1.1) can be removed by a rescaling of ρ.

Being a *two-component* system, the polytropic gas dynamics (1.1) is integrable by the general Tsarev theory [9, 10]. Indeed, in the Riemann invariants

$$r_{1,2} = u \pm \frac{2}{\Gamma + 1}\rho^{(\Gamma+1)/2}, \quad \Gamma \neq -1, \qquad (1.3)$$

the system (1.1) can be re-written as

$$r_{1,t} = [(\Gamma + \frac{3}{2})r_1 - (\Gamma + \frac{1}{2})r_2]r_{1,x}, \qquad (1.4a)$$
$$r_{2,t} = [-(\Gamma + \frac{1}{2})r_2 + (\Gamma + \frac{3}{2})r_2]r_{2,x}. \qquad (1.4b)$$

The question is: can the polytropic gas dynamics be regularized, i.e., embedded into an N-component integrable system with $N \geq 3$? Two such regularizations are described below. (Integrability here is understood in the sense of Tsarev's theory.)

First we make the system (1.1) into a quadratic one, by introducing the variables [1, 2]

$$u = u; \; v = \rho^{\Gamma+1}\theta, \quad \theta = const, \quad \Gamma \neq -1. \tag{1.5}$$

In these variables, the system (1.1) becomes:

$$u_t = uu_x + v_x, \tag{1.6a}$$
$$v_t = (\Gamma+1)vu_x + uv_x. \tag{1.6b}$$

Now consider the hydrodynamic chain

$$A_{n,t} = A_{n+1,x} + (an+b)A_n A_{0,x} + \bar{c}A_0 A_{n,x}, \quad n \in \mathbf{Z}_{\geq 0}, \tag{1.7}$$

where

$$a, b, \bar{c} \quad \text{are constants.} \tag{1.8}$$

This hydrodynamic chain is integrable for any $a, b, \bar{c}$ [5, 6]. Take

$$a = \Gamma + 1, \quad b = 0, \quad \bar{c} = 1. \tag{1.9}$$

The hydrodynamic chain (1.7) becomes:

$$A_{n,t} = A_{n+1,x} + (\Gamma+1)nA_n A_{0,x} + A_0 A_{n,x}, \quad n \in \mathbf{Z}_{\geq 0}. \tag{1.10}$$

In particular, when

$$\{A_n = 0, \quad n \geq 2\}, \tag{1.11}$$

we get:

$$A_{0,t} = A_{1,x} + A_0 A_{0,x}, \tag{1.12a}$$
$$A_{1,t} = (\Gamma+1)A_1 A_{0,x} + A_0 A_{1,x}, \tag{1.12b}$$

which is the polytropic gas dynamics (1.6) under identification

$$u = A_0, \quad v = A_1. \tag{1.13}$$

Thus, the infinite chain (1.10) provides a regularization of the polytropic gas dynamics. It was shown by Brunelli and Das [1] that the system (1.12) has a Lax representation. That representation applies also to the full infinite chain (1.10). This is shown in the next Section.

The *minimal* regularization of the polytropic gas dynamics in the above form results when the infinite chain (1.10) is restricted onto the submanifold

$$\{A_n = 0, \quad n \geq 3\}. \tag{1.14}$$

In the notation

$$u = A_0, \quad v = A_1, \quad w = A_2, \tag{1.15}$$

we find:

$$u_t = v_x + uu_x, \tag{1.16a}$$
$$v_t = w_x + (\Gamma + 1)vu_x + uv_x, \tag{1.16b}$$
$$w_t = 2(\Gamma + 1)wu_x + uw_x. \tag{1.16c}$$

If one twists the RHS of the equation (1.16c into

$$u_t = v_x + uu_x, \tag{1.17a}$$
$$v_t = w_x + (\Gamma + 1)vu_x + uv_x, \tag{1.17b}$$
$$w_t = (\Gamma + 1)wu_x + \frac{1 - \Gamma}{2}uw_x, \tag{1.17c}$$

the resulting system is no longer of Lax form and it is no longer Galilean invariant. Nevertheless, for reasons unknown, the twisted system (1.17) is still integrable. This is proven in Section 3,4.

All this was under assumption

$$\Gamma \neq -1. \tag{1.18}$$

when

$$\Gamma = -1, \tag{1.19}$$

formula (1.5) is replaced by

$$\bar{v} = \ell n\rho, \tag{1.20}$$

and the system (1.6) becomes

$$u_t = uu_x + \bar{v}_x, \tag{1.21a}$$
$$v_t = u_x + u\bar{v}_x. \tag{1.21b}$$

This system can be gotten from the system (1.6) via the shift

$$v = \bar{v} + \frac{1}{\Gamma + 1} \tag{1.22}$$

and *then* by letting $\Gamma = -1$. The corresponding regularizations (1.16, 17) *coincide and* take the form:

$$u_t = \bar{v}_x + uu_x, \tag{1.23a}$$
$$\bar{v}_t = (w + u)_x + u\bar{v}_x, \tag{1.23b}$$
$$w_t = uw_x. \tag{1.23c}$$

2 The Lax Representation

Set

$$L = p^{\mu} + u + \frac{1}{\mu}\, vp^{-\mu}, \tag{2.1}$$

$$\mu = \gamma - 1 = \Gamma + 1. \tag{2.2}$$

Brunelli and Das [1] found that the Lax equation

$$L_t = \frac{\mu}{\mu+1}\{(L^{(\mu+1)/\mu})_{\geq 1}, L\} \tag{2.3}$$

reproduces the polytropic equations (1.6). Here

$$\{A, B\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} \tag{2.4}$$

is the symplectic Poisson bracket, and

$$\left(\sum_n f_n p^{1+\mu n}\right)_{\geq 1} = \sum_{n\geq 1} f_n p^{1+\mu n} \tag{2.5}$$

is understood as the projection in the space of power series.

When μ is a *positive integer*, the above picture is the quasiclassical limit of the $k = 1$ nonstandard systems from [4]. Since in our case μ is arbitrary, the notion $(\cdot)_{\geq 1}$ (2.5) requires a proper definition.

We can argue as follows. Let K be a differential ring with a derivative $\partial : K \to K$ (such as $C^{\infty}(\mathbf{R}^1)$.) Let

$$\tilde{K} = K((z^{-1})) = \{\sum_{i=-\infty}^{<\infty} f_i z^i | f_i \in K\} \tag{2.6}$$

be the ring of the Laurent series in z with coefficients in K. This ring is again a differential algebra, with the derivation $\partial : K \to K$ extended as

$$\partial(z) = 0. \tag{2.7}$$

Consider now the objects

$$C_{\alpha} = p^{\alpha} K((z^{-1})), \quad \alpha \in \mathcal{A}, \tag{2.8}$$

with a new formal parameter p. C_{α} is not itself a ring, but if $\mathcal{A}$ is an additive space containing $\mathbf{Z}$ then all C_{α}'s together form a new ring $C_{\mathcal{A}}$:

$$(p^{\alpha} f)(p^{\beta} g) = p^{\alpha+\beta} fg, \quad f, g \in K((z^{-1})). \tag{2.9}$$

$C_{\mathcal{A}}$ is obviously again a differential ring with respect to ∂:

$$\partial(p^{\alpha}) = 0, \tag{2.10}$$

but we can introduce another derivation into $C_{\mathcal{A}}$,

$$\partial_p = \frac{\partial}{\partial p}, \tag{2.11}$$

acting by the rule

$$\partial_p(f) = 0, \quad f \in K, \tag{2.12a}$$
$$\partial_p(p^\alpha) = \alpha p^{\alpha-1}, \tag{2.12b}$$
$$\partial_p(z) = \mu p^{-1} z, \tag{2.12c}$$

where μ is a fixed formal parameter. Since the derivations ∂ and ∂_p obviously commute, we can define the Poisson bracket

$$\{,\} : C_\alpha \times C_\beta \to C_{\alpha+\beta-1} \tag{2.13}$$

by the rule

$$\{p^\alpha f, p^\beta g\} \; = \partial_p(p^\alpha f)\partial(p^\beta g) - \partial(p^\alpha f)\partial_p(p^\beta g), \;\; f, g \in \tilde{K}. \tag{2.14}$$

The projections $(\cdot\cdot)_{\geq(\cdot)}$, can now be defined for *each individual* C_α as

$$(p^\alpha \sum_n f_n z^n)_{\geq \alpha+N} = p^\alpha \sum_{n\geq N} f_n z^n, \quad N \in \mathbf{Z}, \;\alpha \in \mathcal{A}. \tag{2.15}$$

Now set

$$L = p^\mu (1 + \sum_{i=0}^{\infty} A_i z^{-i-1}), \tag{2.16}$$

so that

$$L \in C_\mu. \tag{2.17}$$

Hence

$$L^{1/\mu} = p(1 + \sum_{i=0}^{\infty} A_i z^{-i-1})^{1/\mu} \in C_1, \tag{2.18}$$
$$L^{n+1/\mu} \in C_{1+n\mu}, \;\; n \in \mathbf{Z}. \tag{2.19}$$

Thus, the objects

$$(L^{n+1/\mu})_{\geq 1}, \;\; n \in \mathbf{Z}, \tag{2.20}$$

are well-defined, and we can consider the equation

$$\frac{\partial L}{\partial t_n} = p^\mu \sum_{i=0}^{\infty} \frac{\partial A_i}{\partial t_n} z^{-1-i} = \; const_n \; \{(L^{n+1/\mu})_{\geq 1}, \; L\} = \tag{2.21a}$$
$$= const_n \; \{-(L^{n+1/\mu})_{<1}, L\}, \;\; n \in \mathbf{Z}_{\geq 0}. \tag{2.21b}$$

The LHS of this equation, $\partial L/\partial t_n$, belongs to C_μ, while the RHS $\{,\}$ belongs to $C_{(n+1)\mu}$. Thus, this equation makes sense if we impose the constrain

$$p^\mu = z. \tag{2.22}$$

Formulae (2.12b) and (2.12c) show that this constrain is compatible with the derivation ∂_p. Thus, both sides of the equation (2.21) belong to C_0, with the LHS in

$$\{\sum_{i\geq 0} f_i z^{-i} | f_i \in K\}, \tag{2.23}$$

and the RHS, by formula (2.21b), in the same subspace. Moreover, formula (2.21a) shows that the ideals

$$\{A_i = 0,\ i \geq N\}, \quad N \in \mathbf{Z}_{\geq 0}, \tag{2.24}$$

are invariant with respect to the dynamics. The usual arguments show that flows (2.21) commute between themselves and have an infinite set of common conserved densities

$$Res(L^{1+n-1/\mu}), \quad n \in \mathbf{Z}_{\geq 0}, \tag{2.25}$$

where Res singles out the coefficient in front of

$$p^{-1} z^0. \tag{2.26}$$

Alternatively, we can use the identification $p^\mu = z$ to set

$$L = z + \sum_{i=0}^{\infty} A_i z^{-i}, \tag{2.27a}$$

$$L^{1/\mu} = p(1 + \sum_{i>0} A_i z^{-i-1})^{1/\mu}, \tag{2.27b}$$

$$L^{n+1/\mu} = pz^n(1 + \sum_{i\geq 0} A_i z^{-i-1})^{n+1/\mu}, \tag{2.27c}$$

$$(L^{n+1/\mu})_{\geq 1} = p(z^n(1 + \sum_{i\geq 0} A_i z^{-i-1})^{n+1/\mu})_{\geq 0}, \tag{2.27d}$$

$$(L^{n+1/\mu})_{<1} = p(z^n(1 + \sum_{i\geq 0} A_i z^{-i-1})^{n+1/\mu})_{<0}, \tag{2.27e}$$

$$Res(L^{1+n-1/\mu}) = Res(L^{1+n}(L^{1/\mu})^{-1}). \tag{2.27f}$$

Let us consider the first two flows.
For $n = 0$,

$$(L^{1/\mu})_{\geq 1} = (p(1 + \sum A_i z^{-i-1})^{1/\mu})_{\geq 1} = p, \tag{2.28}$$

and with

$$const_0 = 1 \tag{2.29}$$

the equation (2.21) yields

$$\frac{\partial A_i}{\partial t_0} = \frac{\partial A_i}{\partial x}, \quad i \in \mathbf{Z}_{\geq 0}. \tag{2.30}$$

For $n = 1$,

$$\begin{aligned}
&(L^{1+1/\mu})_{\geq 1} = (pz(1 + \sum A_i z^{-i-1})^{1+1/\mu})_{\geq 1} = \\
&= pz(1 + \frac{\mu+1}{\mu} A_0 z^{-1}) = pz + \frac{\mu+1}{\mu} p A_0 \Rightarrow
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\frac{\partial L}{\partial t_1} = \sum_{i \geq 0} \frac{\partial A_i}{\partial t_i} z^{-i} = \ const_1 \{pz + \frac{\mu+1}{\mu} p A_0, z + \sum_{j \geq 0} A_j z^{-j}\} = \\
&= const_1 \{[(\mu+1)z + \frac{\mu+1}{\mu} A_0] \sum_{j \geq 0} \frac{\partial A}{\partial x} z^{-j} - \frac{\mu+1}{\mu} p \frac{\partial A_0}{\partial x} \mu p^{-1} z [1 - \sum_{j \geq 0} j A_j z^{-j-1}]\} = \\
&= const_1 (\mu+1) \{\sum_{j \geq 0} (\frac{\partial A_{j+1}}{\partial x} + \mu^{-1} A_0 \frac{\partial A_j}{\partial x} + j A_j \frac{\partial A_0}{\partial x}) z^{-j}\} \Rightarrow
\end{aligned} \tag{2.32}$$

$$\frac{\partial A_i}{\partial t_1} = A_{i+1,x} + i A_i A_{0,x} + \mu^{-1} A_0 A_{i,x}, \quad i \in \mathbf{Z}_{\geq 0}, \tag{2.33}$$

with

$$const_1 = (\mu + 1)^{-1}. \tag{2.34}$$

This is essentially the system (1.10) after rescaling

$$A_i \to \mu A_i, \quad i \in \mathbf{Z}_{\geq 0}. \tag{2.35}$$

Recall that $\mu = \Gamma + 1$ by formula (2.2).

3 The Twisted System

Let us consider the general quadratic system

$$u_t = \alpha u u_x + v_x, \tag{3.1a}$$

$$v_t = \beta v u_x + c u v_x + w_x, \tag{3.1b}$$

$$w_t = \gamma w u_x + \delta u w_x, \tag{3.1c}$$

where

$$\alpha, \beta, \gamma, c, \delta \tag{3.2}$$

are unspecified constants. To determine when the system (3.1) is integrable, we first notice that it is conservative: it has conserved densities

$$H_0 = u, \tag{3.3a}$$

$$H_2 = v + \frac{\beta - c}{2} u^2, \tag{3.3b}$$

$$H_3 = uv + \frac{1}{\gamma - \delta} w + (\frac{\alpha + \beta}{2} - c) \frac{u^3}{3}, \quad \gamma \neq \delta, \tag{3.3c}$$

$$H_3 = w, \quad \gamma = \delta. \tag{3.3d}$$

Our second step is to calculate the Haantjes tensor [7] for the system (3.1) and equate it to zero ([3,8].)

Writing the system (3.1) in the hydrodynamic form

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = A \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x , \tag{3.4}$$

$$A = \begin{pmatrix} \alpha u & 1 & 0 \\ \beta v & cu & 1 \\ \gamma w & 0 & \delta u \end{pmatrix}, \tag{3.5}$$

we first calculate the Nijenhuis tensor of the matrix A:

$$N_A(X, Y) = N(X, Y) = A^2[X, Y] - A([X, AY] - [Y, AX]) + [AX, AY]. \tag{3.6}$$

Here

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \ Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \tag{3.7}$$

and the commutators are understood as between vector fields.

Set

$$U = U_{X,Y} = X_1 Y_2 - Y_1 X_2, \tag{3.8a}$$
$$V = V_{X,Y} = X_1 Y_3 - Y_1 X_3, \tag{3.8b}$$
$$W = W_{X,Y} = X_2 Y_3 - Y_2 X_3. \tag{3.8c}$$

Since the Nijenhuis tensor is a *tensor*, we can calculate it for the case when X and Y are constant (u, v, w-independent) vectors, so that

$$N(X, Y) = [AX, AY] - A(\widehat{X}(AY) - \widehat{Y}(AX)), \tag{3.9}$$

where

$$\widehat{X} = X_1 \frac{\partial}{\partial u} + X_2 \frac{\partial}{\partial v} + X_3 \frac{\partial}{\partial w}. \tag{3.10}$$

Thus,

$$AX = \begin{pmatrix} \alpha u X_1 + X_2 \\ \beta v X_1 + cu X_2 + X_3 \\ \gamma w X_1 \qquad + \delta u X_3 \end{pmatrix}, \quad AY = \begin{pmatrix} \alpha u Y_1 + Y_2 \\ \beta v Y_1 + cu Y_2 + Y_3 \\ \gamma w Y_1 \qquad + \delta u Y_3 \end{pmatrix} \Rightarrow \tag{3.11}$$

$$[AX, AY] = \begin{pmatrix} -\alpha U \\ c(\alpha - \beta) u U - \beta V \\ \delta(\alpha - \gamma) u V + \delta W \end{pmatrix}, \tag{3.12}$$

$$\widehat{X}(AY) - \widehat{Y}(AX) = \begin{pmatrix} 0 \\ (c-\beta)U \\ (\delta-\gamma)V \end{pmatrix}, \tag{3.13}$$

$$A(\widehat{X}(AY) - \widehat{Y}(AX)) = \begin{pmatrix} (c-\beta)U \\ c(c-\beta)uU + (\delta-\gamma)V \\ \delta(\delta-\gamma)uV \end{pmatrix} \Rightarrow \tag{3.14}$$

$$N(X,Y) = \begin{pmatrix} \beta-\alpha-c & 0 & 0 \\ c(\alpha-c)u & \gamma-\delta-\beta & 0 \\ 0 & \delta(\alpha-\delta)u & \delta \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}. \tag{3.15}$$

Next comes the Haantjes tensor:

$$\begin{aligned} &H_A(X,Y) = H(X,Y) = \\ &= A^2N(X,Y) - A(N(X,AY) - N(Y,AX)) + N(AX,AY). \end{aligned} \tag{3.16}$$

We calculate it in the form

$$H_A(X,Y) = \widehat{H} \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{3.17}$$

where the matrix $\widehat{H}$ is

$$\widehat{H} = \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3, \tag{3.18}$$

corresponding to the three summands in the RHS of formula (3.16). Denote by $\widehat{N}$ the matrix in the RHS of (3.15):

$$N(X,Y) = \widehat{N} \begin{pmatrix} U \\ V \\ W \end{pmatrix}. \tag{3.19}$$

Then

$$\widehat{H}_1 = A^2\widehat{N} = \begin{pmatrix} \alpha^2u^2 + \beta v & (\alpha+c)u & 1 \\ \beta(\alpha+c)u + \gamma w & c^2u^2 + \beta v & (c+\delta)u \\ \gamma wu(\alpha+\delta) & \gamma w & \delta^2u^2 \end{pmatrix} \widehat{N} =$$

$$= \begin{pmatrix} u^2[\alpha^2(\beta-\alpha) - c^3] + \beta v(\beta-\alpha-c) \\ u^3c^3(\alpha-c) + uv\beta[(\alpha+c)(\beta-\alpha-c) + c(\alpha-c)] + \gamma w(\beta-\alpha-c) \\ \gamma wu[(\alpha+\delta)(\beta-\alpha-c)] + c(\alpha-c) \end{pmatrix} \oplus \tag{3.20.1}$$

$$\oplus \begin{pmatrix} u[\delta(\alpha-\delta)+(\alpha+c)(\gamma-\beta-\delta)] \\ u^2[c^2(\gamma-\beta-\delta)+\delta(c+\delta)(\alpha-\delta)]+\beta v(\gamma-\beta-\delta) \\ \delta^3 u^3(\alpha-\delta)+\gamma w(\gamma-\beta-\delta) \end{pmatrix} \oplus \tag{3.20.2}$$

$$\oplus \begin{pmatrix} \delta \\ \delta u(c+\delta) \\ \delta^3 u^2 \end{pmatrix}, \tag{3.20.3}$$

where formula (3.20.i) gives column #i of the matrix $\widehat{H}_1$, $i = 1, 2, 3$.
Next,

$$\begin{aligned} &U_{X,AY} - U_{Y,AX} = u(c+\alpha)U + V, \\ &V_{X,AY} - V_{Y,AX} = u(\delta+\alpha)V + W, \\ &W_{X,AY} - W_{Y,AX} = -\gamma w U + \beta v V + u(\delta+c)W \Rightarrow \\ &-\widehat{H}_2 = A\widehat{N} \begin{pmatrix} u(c+\alpha) & 1 & 0 \\ 0 & u(\delta+\alpha) & 1 \\ -\gamma w & \beta v & u(\delta+c) \end{pmatrix} = \end{aligned} \tag{3.21}$$

$$A \begin{pmatrix} u(\beta-\alpha-c)(c+\alpha) & \beta-\alpha-\rho & 0 \\ u^2 c(\alpha^2-c^2) & u[c(\alpha-c)+(\gamma-\beta-\delta)(\delta+\alpha)] & \gamma-\beta-\delta \\ -\delta\gamma w & u^2\delta(\alpha^2-\delta^2)+\delta\beta v & \delta u(\alpha+c) \end{pmatrix} =$$

$$= \begin{pmatrix} u^2[\alpha(\beta-\alpha-c)(\alpha+c)+c(\alpha^2-c^2)] \\ u^3c^2(\alpha^2-c^2)+uv\beta(\beta-\alpha-c)(c+\alpha)-\delta\gamma w \\ \gamma uw[(\beta-\alpha-c)(c+\alpha)-\delta^2] \end{pmatrix} \oplus \tag{3.22.1}$$

$$\oplus \begin{pmatrix} u[\alpha(\beta-\alpha-c)+c(\alpha-c)+(\gamma-\beta-\delta)(\delta+\alpha)] \\ \beta v(\beta-\alpha-c+\delta)+u^2[\delta(\alpha^2-\delta^2)+c^2(\alpha-c)+c(\gamma-\beta-\delta)(\delta+\alpha)] \\ \gamma w(\beta-\alpha-c)+u^3\delta^2(\alpha^2-\delta^2)+uv\delta^2\beta \end{pmatrix} \oplus \tag{3.22.2}$$

$$\oplus \begin{pmatrix} \gamma-\beta-\delta \\ u[c(\gamma-\beta-\delta)+\delta(\alpha+c)] \\ u^2\delta^2(\alpha+c) \end{pmatrix}. \tag{3.22.3}$$

Finally,

$$\begin{aligned} &U_{AX,AY} = (\alpha c u^2 - \beta v)U + \alpha u V + W, \\ &V_{AX,AY} = -\gamma w V + \alpha\delta u^2 V + \delta u W, \\ &W_{AX,AY} = -c\gamma u w U + (\beta\delta u v - \gamma w)V + c\delta u^2 W \Rightarrow \end{aligned} \tag{3.23}$$

$$\widehat{H}_3 = \widehat{N}\begin{pmatrix} \alpha c u^2 - \beta v & \alpha u & 1 \\ -\gamma w & \alpha\delta u^2 & \delta u \\ -c\gamma u w & \beta\delta u v - \gamma w & c\delta u^2 \end{pmatrix} =$$

$$= \begin{pmatrix} (\beta - \alpha - c)(\alpha c u^2 - \beta v) \\ c(\alpha - c)u(\alpha c u^2 - \beta v) - \gamma w(\gamma - \beta - \delta) \\ -\delta\gamma u w(\alpha - \delta + c) \end{pmatrix} \oplus \tag{3.24.1}$$

$$\oplus \begin{pmatrix} \alpha u(\beta - \alpha - c) \\ \alpha u^2[c(\alpha - c) + \delta(\gamma - \beta - \delta)] \\ \delta[\alpha\delta(\alpha - \delta)u^3 + \beta\delta u v - \gamma w] \end{pmatrix} \oplus \tag{3.24.2}$$

$$\oplus \begin{pmatrix} \beta - \alpha - c \\ u[c(\alpha - c) + \delta(\gamma - \beta - \delta)] \\ \delta^2 u^2(\alpha - \delta + c) \end{pmatrix}. \tag{3.24.3}$$

Collecting together formulae (3.20*, 22*, 24*), we obtain:

$$\widehat{H} = \begin{pmatrix} 0 \\ \gamma w(2\delta + 2\beta - \alpha - c - \gamma) \\ \gamma w u(\delta - c)(2\delta + \beta - 2\alpha) \end{pmatrix} \oplus \tag{3.25.1}$$

$$\oplus \begin{pmatrix} u(\gamma + c - \alpha - \beta)(c - \delta) \\ u^2(c - \alpha)(c - \delta)(\gamma + c - \alpha - \beta) + \beta v(\gamma + \alpha + c - 2\beta - 2\delta) \\ \gamma w(\gamma + \alpha + c - 2\beta - 2\delta) \end{pmatrix} \oplus \tag{3.25.2}$$

$$\oplus \begin{pmatrix} 2\delta + 2\beta - \gamma - \alpha - c \\ u(\delta - c)(\gamma + c - \alpha - \beta) \\ 0 \end{pmatrix}. \tag{3.25.3}$$

The vanishing of the matrix $\widehat{H}$ (3.25*) amounts to a system of linear and quadratic relations on the coefficients $\alpha, \beta, \gamma, c, \delta$. We examine these relations in the next Section.

4 The Relations

The first entry of the vector (3.25.3) yields

$$2\beta + 2\delta - \gamma - \alpha - c = 0. \tag{4.1}$$

With this relation satisfied, the remaining entries of the matrix (3.25) yield:

$$\gamma(\delta - c)(2\delta + \beta - 2\alpha) = 0, \tag{4.2a}$$
$$(\delta - c)(\gamma + c - \alpha - \beta) = 0. \tag{4.2b}$$

If

$$\delta = c \tag{4.3}$$

then both equations (4.2) are satisfied. The relations (4.1,3) can be rewritten as

$$c = \delta, \tag{4.4a}$$
$$\gamma = \delta + 2\beta - \alpha. \tag{4.4b}$$

This is exactly the untwisted 3-component subsystem of the general $(a, b, \bar{c})$ chain (1.7), with

$$a = \beta + \delta - \alpha, \quad b = \alpha - \delta, \quad \bar{c} = \delta. \tag{4.5}$$

If the relation $\delta = c$ is not assumed, then the system (4.2) reduces to the system

$$\gamma(2\delta + \beta - 2\alpha) = 0, \tag{4.6a}$$
$$\gamma + c - \alpha - \beta = 0. \tag{4.6b}$$

Adding up equations (4.1) and (4.6b) we find:

$$2\delta + \beta - 2\alpha = 0, \tag{4.7}$$

which implies the relation (4.6a). Thus, we obtain the second, twisted solution, {(4.1) & (4.6b)}:

$$\gamma = \alpha + \beta - c, \tag{4.8a}$$
$$\delta = \alpha - \frac{1}{2}\beta. \tag{4.8b}$$

For

$$\alpha = 1, \quad \beta = \Gamma + 1, \quad c = 1, \tag{4.9}$$

formulae (4.8) give

$$\gamma = \Gamma + 1, \quad \delta = \frac{1 - \Gamma}{2}, \tag{4.10}$$

and we recover the mysterious system (1.17).

In terms of the hydrodynamic chain (1.7), its regular 3-component reduction

$$u_t = v_x + (b + \bar{c})uu_x, \tag{4.11a}$$
$$v_t = w_x + (a + b)vu_x + \bar{c}uv_x, \tag{4.11b}$$
$$w_t = (2a + b)wu_x + \bar{c}uw_x, \tag{4.11c}$$

has, by formulae (4.8), the twisted form

$$u_t = v_x + (b + \bar{c})uu_x, \tag{4.12a}$$

$$v_t = w_x + (a + b)vu_x + \bar{c}uv_x, \tag{4.12b}$$

$$w_t = (a + 2b)wu_x + (\bar{c} + \frac{b-a}{2})uw_x. \tag{4.12c}$$

For $a = b$ both forms coincide.

It seems likely that all the other finite-component reductions of the infinite hydrodynamic chain (1.7),

$$\{A_i = 0, \quad i > N\}, \quad N = 2, 3... \tag{4.13}$$

possess a twist of the $(N + 1)^{st}$ equation

$$A_{N,t} = (aN + b)A_N A_{0,x} + \bar{c}A_0 A_{N,x}. \tag{4.14}$$

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