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A fuzzy difference equation of a rational form

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Abstract

In this paper, we prove some effects concerning a Fuzzy Difference Equation of a rational form.

1 INTRODUCTION

A Fuzzy Difference Equation is a Difference Equation where the constants and the initial values are fuzzy numbers (see preliminaries) and it's solutions are sequences of fuzzy numbers. In order to study the behavior of a parametric fuzzy Difference Equation, we prove results for the behavior of a related family of systems of parametric ordinary Difference Equations and using the fuzzy analog of concepts known by the theory of ordinary Difference Equations, we extend these results to the Fuzzy Difference Equation. We note that the behavior of the parametric fuzzy Difference Equation is not always the same as the behavior of the corresponding parametric ordinary Difference Equation. (Some results concerning the study of Fuzzy Difference Equations are included in the papers [2], [3], [8], [10], [11], [12], [15], [16]).

In paper [15], we give some results concerning the asymptotic behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1} = \frac{A + \sum_{i=0}^{k} a_i x_{n-i}}{B + \sum_{i=0}^{k} b_i x_{n-i}}, \tag{1.1}$$

where $k \in \{1, 2, ...\}$, A, B, a_i, b_i, $i \in \{0, 1, ..., k\}$, x_i, $i \in \{-k, -k+1, ..., 0\}$ are positive fuzzy numbers.

In addition, in paper [16], we studied the trichotomy character, the stability and the oscillatory behavior of the positive solutions of the fuzzy difference equation

$$x_{n+1} = A + \frac{\sum_{i=1}^{k} c_i x_{n-p_i}}{\sum_{j=1}^{m} d_j x_{n-q_j}},$$

where $k, m \in \{1, 2, ...\}$, A, c_i, d_j, $i \in \{1, 2..., k\}$, $j \in \{1, 2..., m\}$ are positive fuzzy numbers, p_i, $i \in \{1, 2..., k\}$, q_j, $j \in \{1, 2..., m\}$ are positive integers such that $p_1 < p_2 < ... < p_k$, $q_1 < q_2 < ... < q_m$ and the initial values x_i, $i \in \{-\pi, -\pi+1, ..., 0\}$, $\pi = max\{p_k, q_m\}$ are positive fuzzy numbers.

In this paper, we study the asymptotic behavior, the stability, the oscillation and the periodicity of the positive solutions of the following fuzzy difference equation

$$x_{n+1} = \frac{A + \sum_{i=1}^{k} a_i x_{n-p_i}}{B + \sum_{j=1}^{m} b_j x_{n-q_j}}, \tag{1.2}$$

where $k, m \in \{1, 2, ...\}$, A, B, a_i, b_j, $i \in \{1, 2..., k\}$, $j \in \{1, 2..., m\}$ are positive fuzzy numbers, p_i, $i \in \{1, 2..., k\}$, q_j, $j \in \{1, 2..., m\}$ are positive integers such that $p_1 < p_2 < ... < p_k$, $q_1 < q_2 < ... < q_m$ and the initial values x_i, $i \in \{-\pi, -\pi+1, ..., 0\}$, where

$$\pi = max\{p_k, q_m\}$$

are positive fuzzy numbers.

Obviously, equation (1.1) is a special case of equation (1.2).

2 PRELIMINARIES

We need the following definitions:

• If A is a function from $I\!R^+ = (0, \infty)$ into the interval $[0, 1]$, then A is called a fuzzy set.

• A is convex, if for every $t \in [0, 1]$ and $x_1, x_2 \in I\!R^+$ we have

$$A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}.$$

• A is normalized, if there exists an $x \in I\!R^+$ such that $A(x) = 1$.

• If A is a fuzzy set by a-cuts, $a \in [0, 1]$ we mean the sets

$$[A]_a = \left\{x \in I\!R^+ : A(x) \geq a\right\}.$$

It is known that the a-cuts determine the fuzzy set A.

• For a set B we denote by $\overline{B}$ the closure of B. We say that A is a fuzzy number if the following conditions hold:

(i) A is normal,

(ii) A is a convex fuzzy set,

(iii) A is upper semicontinuous,

(iv) The support of A, supp$A = \overline{\bigcup_{a \in (0,1]} [A]_a} = \overline{\{x : A(x) > 0\}}$ is compact.

Then from Theorems 3.1.5 and 3.1.8 of [9] the a-cuts of A are closed intervals.

• We say that a fuzzy number A is positive if $\mathrm{supp}A \subset (0,\infty)$.

• The fuzzy analog of the boundedness and persistence (see [4]) is given as follows: We say that a sequence of positive fuzzy numbers x_n persists (resp. is bounded) if there exists a positive number M (resp. N) such that

$$\mathrm{supp}x_n \subset [M,\infty), \quad (\text{resp. } \mathrm{supp}x_n \subset (0,N]), \quad n = 1,2,...$$

In addition, we say that x_n is bounded and persists if there exist numbers $M, N \in (0,\infty)$ such that

$$\mathrm{supp}x_n \subset [M,N], \quad n = 1,2,...$$

• We say x_n is a positive solution of (1.2), if x_n is a sequence of positive fuzzy numbers which satisfies (1.2).

• We say that a positive fuzzy number x is a positive equilibrium for (1.2) if

$$x = \frac{A + \sum_{i=1}^{k} a_i x}{B + \sum_{j=1}^{m} b_j x}.$$

• Let A, B be fuzzy numbers with

$$[A]_a = [A_{l,a}, A_{r,a}], \quad [B]_a = [B_{l,a}, B_{r,a}], \quad a \in (0,1]. \tag{2.1}$$

Then we define the following metric (see [5], [14], [17])

$$D(A,B) = \sup \max\{|A_{l,a} - B_{l,a}|, |A_{r,a} - B_{r,a}|\}$$

where sup is taken for all $a \in (0,1]$.

• Let x_n be a sequence of positive fuzzy numbers and x is a positive fuzzy number. Suppose that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \quad a \in (0,1], \quad n = 0,1,... \tag{2.2}$$

and

$$[x]_a = [L_a, R_a], \quad a \in (0,1]. \tag{2.3}$$

We say that the sequence x_n converges to x with respect to D as $n \to \infty$ if

$$\lim D(x_n, x) = 0, \quad \text{as} \quad n \to \infty.$$

In addition, we say that x_n nearly converges to x with respect to D as $n \to \infty$ if for every $\delta > 0$ there exists a measurable set T, $T \subset (0,1]$ of measure less than δ such that

$$\lim D_T(x_n, x) = 0, \quad \text{as} \quad n \to \infty$$

where

$$D_T(x_n, x) = \sup_{a\in(0,1]-T} \left\{ \max\{|L_{n,a} - L_a|, |R_{n,a} - R_a|\} \right\}$$

• Let E be the set of positive fuzzy numbers. Suppose that A, B belong to E satisfying (2.1). From Theorem 2.1 of [17] we have that, $A_{l,a}$, $B_{l,a}$ (resp. $A_{r,a}$, $B_{r,a}$) are increasing (resp. decreasing) functions on $(0, 1]$. Therefore, using condition (iv) of the definition of fuzzy number, there exist the Lebesque integrals

$$\int_J |A_{l,a} - B_{l,a}|da, \quad \int_J |A_{r,a} - B_{r,a}|da$$

where $J = (0, 1]$. We define the function $D_1 : E \times E \to R^+$ such that

$$D_1(A, B) = \max\Big\{\int_J |A_{l,a} - B_{l,a}|da, \quad \int_J |A_{r,a} - B_{r,a}|da\Big\}.$$

If $D_1(A, B) = 0$ we have that, there exists a measurable set T of measure zero such that

$$A_{l,a} = B_{l,a} \quad A_{r,a} = B_{r,a} \quad \text{for all} \quad a \in (0, 1] - T. \tag{2.4}$$

We consider however, two fuzzy numbers A, B to be equivalent, if there exists a measurable set T of measure zero such that (2.4) hold and if we do not distinguish between equivalent of fuzzy numbers then, E becomes a metric space with metric D_1.

• We say that a sequence of positive fuzzy numbers x_n converges to a positive fuzzy number x with respect to D_1 as $n \to \infty$ if

$$\lim D_1(x_n, x) = 0, \quad \text{as} \quad n \to \infty.$$

• Suppose that (1.2) has a unique positive equilibrium x. We say that the positive equilibrium x of (1.2) is stable if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that for every positive solution x_n of (1.2), which satisfies $D(x_{-i}, x) \le \delta$, $i = 0, 1, ..., \pi$ we have $D(x_n, x) \le \epsilon$ for all $n \ge 0$.

Moreover, we say that the positive equilibrium x of (1.2) is nearly asymptotically stable, if it is stable and every positive solution of (1.2) nearly tends to the positive equilibrium of (1.2) with respect to D as $n \to \infty$.

• We give the fuzzy analog of the concept of oscillation (see [1],[7]). Let x_n be a sequence of positive fuzzy numbers and let x be a positive fuzzy number. We say that x_n oscillates about x, if for every $n_0 \in I\!N$ there exist $s, m \in I\!N$, $s, m \ge n_0$ such that

$$MIN\,\{x_m, x\} = x_m \quad \textit{and} \quad MIN\,\{x_s, x\} = x \quad \textit{or}$$

$$MIN\,\{x_m, x\} = x \quad \textit{and} \quad MIN\,\{x_s, x\} = x_s.$$

• We define the fuzzy analog for periodicity (see [7]) as follows.

A sequence $\{x_n\}$ of positive fuzzy numbers x_n is said to be periodic of period p, if

$$D(x_{n+p}, x_n) = 0, \quad n = 0, 1, ... \tag{2.5}$$

3 MAIN RESULTS

Arguing as in [15] we can easily prove the following proposition.

Proposition 1. *Consider equation (1.2) where* $k, m \in \{1, 2, ...\}$, A, B, a_i, b_j, $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., m\}$ *are positive fuzzy numbers,* p_i, $i \in \{1, 2..., k\}$, q_j, $j \in \{1, 2..., m\}$ *are positive integers such that* $p_1 < p_2 < ... < p_k$, $q_1 < q_2 < ... < q_m$. *Then for any positive fuzzy numbers* $x_{-\pi}, x_{-\pi+1}, ..., x_0$, $\pi = max\{p_k, q_m\}$, *there exists a unique positive solution* x_n *of (1.2) with initial values* $x_{-\pi}, x_{-\pi+1}, ..., x_0$.

Since A, B, a_i, b_j, $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., m\}$ are positive fuzzy numbers there exist positive constants K, L, M, N, P_i, Q_i, S_j, T_j, $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., m\}$, such that

$$\begin{aligned}
&[A]_a = [A_{l,a}, A_{r,a}] \subset \overline{\bigcup_{a\in(0,1]} [A_{l,a}, A_{r,a}]} \subset [K, L],\\
&[B]_a = [B_{l,a}, B_{r,a}] \subset \overline{\bigcup_{a\in(0,1]} [B_{l,a}, B_{r,a}]} \subset [M, N],\\
&[a_i]_a = [a_{i,l,a}, a_{i,r,a}] \subset \overline{\bigcup_{a\in(0,1]} [a_{i,l,a}, a_{i,r,a}]} \subset [P_i, Q_i],\ i \in \{1, 2, ..., k\},\\
&[b_j]_a = [b_{j,l,a}, b_{j,r,a}] \subset \overline{\bigcup_{a\in(0,1]} [b_{j,l,a}, b_{j,r,a}]} \subset [S_j, T_j],\ j \in \{1, 2, ..., m\},\ a \in (0, 1].
\end{aligned} \tag{3.1}$$

If x_n is the unique positive solution of (1.2) with initial values $x_{-\pi}, x_{-\pi+1}, ..., x_0$, such that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \quad a \in (0, 1], \quad n = -\pi, -\pi + 1, ... \tag{3.2}$$

then arguing as in Proposition 1 of [15], we get that $(L_{n,a}, R_{n,a})$, $n = 0, 1, ...$ satisfies the following family of systems of parametric ordinary difference equations

$$L_{n+1,a} = \frac{A_{l,a} + \sum_{i=1}^{k} a_{i,l,a} L_{n-p_i,a}}{B_{r,a} + \sum_{j=1}^{m} b_{j,r,a} R_{n-q_j,a}}, R_{n+1,a} = \frac{A_{r,a} + \sum_{i=1}^{k} a_{i,r,a} R_{n-p_i,a}}{B_{l,a} + \sum_{j=1}^{m} b_{j,l,a} L_{n-q_j,a}}, \quad n = 0, 1, ... \tag{3.3}$$

The following lemma and proposition, are a slight generalization of Lemma 2 and Proposition 2 of [15].

Lemma 1. *Consider the system of difference equations*

$$y_{n+1} = \frac{V + \sum_{i=1}^{k} v_i y_{n-p_i}}{U + \sum_{j=1}^{m} u_j z_{n-q_j}}, \quad z_{n+1} = \frac{C + \sum_{i=1}^{k} c_i z_{n-p_i}}{D + \sum_{j=1}^{m} d_j y_{n-q_j}}, \quad n = 0, 1, ... \tag{3.4}$$

where $k, m \in \{1, 2, ...\}$, V, U, C, D, v_i, u_j, c_i, d_j, $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., m\}$ *are positive real constants,* p_i, $i \in \{1, 2..., k\}$, q_j, $j \in \{1, 2..., m\}$ *are positive integers such that* $p_1 < p_2 < ... < p_k$, $q_1 < q_2 < ... < q_m$ *and the initial values* y_i, z_i, $i = -\pi, -\pi + 1, ..., 0$ *are positive real numbers. If*

$$\frac{\sum_{i=1}^{k} v_i}{U} < 1 \quad \text{and} \quad \frac{\sum_{i=1}^{k} c_i}{D} < 1$$

then the following statements are true.

(a) *Every positive solution of (3.4) is bounded and persists.*

(b) *System (3.4) has a unique positive equilibrium.*

(c) *Every positive solution of (3.4) tends to the unique positive equilibrium of (3.4).*

Proposition 2. *Consider equation (1.2) where* $k, m \in \{1, 2, ...\}$, A, B, a_i, b_j, $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., m\}$ *are positive fuzzy numbers,* p_i, $i \in \{1, 2, ..., k\}$, q_j, $j \in \{1, 2, ..., m\}$ *are positive integers such that* $p_1 < p_2 < ... < p_k$, $q_1 < q_2 < ... < q_m$. *If (3.1) and*

$$\frac{\sum_{i=1}^{k} Q_i}{M} < 1 \tag{3.5}$$

hold, then the following statements are true.

(a) *Every positive solution of (1.2) is bounded and persists.*

(b) *Equation (1.2) has a unique positive equilibrium.*

(c) *Every positive solution of (1.2) nearly converges to the unique positive equilibrium* x *with respect to* D *as* $n \to \infty$ *and converges to* x *with respect to* D_1 *as* $n \to \infty$.

If for the unique positive equilibrium x of (1.2), relation (2.3) holds, then arguing as in the proof of Proposition 2 of [15], we have the following relations:

$$L_a = \frac{A_{l,a} + \sum_{i=1}^{k} a_{i,l,a} L_a}{B_{r,a} + \sum_{j=1}^{m} b_{j,r,a} R_a}, \quad R_a = \frac{A_{r,a} + \sum_{i=1}^{k} a_{i,r,a} R_a}{B_{l,a} + \sum_{j=1}^{m} b_{j,l,a} L_a}. \tag{3.6}$$

and

$$\lambda \leq L_a \leq R_a \leq \mu, \quad a \in (0, 1], \tag{3.7}$$

where

$$\lambda = \frac{K}{N - \sum_{i=1}^{k} P_i + \mu \sum_{j=1}^{m} T_j}, \quad \mu = \frac{L}{M(1-\theta)}, \quad \theta = \frac{\sum_{i=1}^{k} Q_i}{M}. \tag{3.8}$$

In the next proposition, we study the asymptotic stability of the unique positive equilibrium of (1.2).

Proposition 3. *Consider the fuzzy difference equation (1.2) where* $k, m \in \{1, 2, ...\}$, A, B, a_i, b_j, $i \in \{1, 2, ..., k\}$, $j \in \{1, 2, ..., m\}$ *are positive fuzzy numbers,* p_i, $i \in \{1, 2, ..., k\}$, q_j, $j \in \{1, 2, ..., m\}$ *are positive integers such that* $p_1 < p_2 < ... < p_k$, $q_1 < q_2 < ... < q_m$. *If relation*

$$M > \sum_{i=1}^{k} Q_i + (L \sum_{j=1}^{m} T_j)^{\frac{1}{2}} \tag{3.9}$$

where L, M, Q_i, T_j, $i = 1, 2, ...k$, $j = 1, 2, ...m$ *are defined in (3.1), holds, then the unique positive equilibrium* x *of (1.2) is nearly asymptotically stable.*

Proof From (3.9) it is obvious that (3.5) holds and so, from statement (b) of Proposition 2, equation (1.2) has a unique positive equilibrium x.

Let ϵ be a positive real number. Since (3.9) holds, we consider the positive real number δ as follows

$$\delta < \min \left\{ \epsilon, \lambda, \frac{M - \sum_{i=1}^{k} Q_i}{\sum_{j=1}^{m} T_j} - \mu \right\}, \tag{3.10}$$

where λ, μ was defined in (3.8).

Let x_n be a positive solution of (1.2) such that

$$D(x_{-i}, x) \leq \delta < \epsilon, \quad i = 0, 1, ..., \pi. \tag{3.11}$$

From (3.11) we have

$$|L_{-i,a} - L_a| \leq \delta, \quad |R_{-i,a} - R_a| \leq \delta, \quad i = 0, 1, ..., \pi, \quad a \in (0, 1]. \tag{3.12}$$

In addition, from (3.1), (3.3), (3.6), (3.10) and (3.12) we get

$$
\begin{aligned}
L_{1,a} - L_a &= \frac{A_{l,a} + \sum_{i=1}^{k} a_{i,l,a} L_{-p_i,a}}{B_{r,a} + \sum_{j=1}^{m} b_{j,r,a} R_{-q_j,a}} - L_a \le \frac{A_{l,a} + \sum_{i=1}^{k} a_{i,l,a}(L_a + \delta)}{B_{r,a} + \sum_{j=1}^{m} b_{j,r,a}(R_a - \delta)} - L_a = \\
&\delta \frac{\sum_{i=1}^{k} a_{i,l,a} + L_a \sum_{j=1}^{m} b_{j,r,a}}{B_{r,a} + \sum_{j=1}^{m} b_{j,r,a}(R_a - \delta)} \le \delta \frac{\sum_{i=1}^{k} Q_i + R_a \sum_{j=1}^{m} b_{j,r,a}}{M + R_a \sum_{j=1}^{m} b_{j,r,a} - \sum_{j=1}^{m} T_j \delta}.
\end{aligned} \tag{3.13}
$$

From (3.10) and (3.13), it is obvious that

$$|L_{1,a} - L_a| < \delta < \epsilon. \tag{3.14}$$

Moreover, from (3.1), (3.3), (3.6), (3.7), (3.10), (3.12) and arguing as above we have that

$$R_{1,a} - R_a = \delta \frac{\sum_{i=1}^{k} a_{i,r,a} + R_a \sum_{j=1}^{m} b_{j,l,a}}{B_{l,a} + \sum_{j=1}^{m} b_{j,l,a}(L_a - \delta)} < \delta \frac{\sum_{i=1}^{k} Q_i + \mu \sum_{j=1}^{m} T_j}{M - \delta \sum_{j=1}^{m} T_j}. \tag{3.15}$$

From (3.10) and (3.15) we get that

$$|R_{1,a} - R_a| < \delta < \epsilon. \tag{3.16}$$

From (3.14), (3.16) and working inductively we can easily prove that

$$|L_{n,a} - L_a| < \epsilon, \quad |R_{n,a} - R_a| < \epsilon, \quad a \in (0,1], \quad n = 0, 1, ...,$$

and so

$$D(x_n, x) < \epsilon, \quad n \ge 0.$$

Therefore, the positive equilibrium x is stable. Moreover, from the statement (c) of Proposition 2 we have that every positive solution of (1.2) nearly tends to x with respect to D as $n \to \infty$. So, x is nearly asymptotically stable. The proof of Proposition 3 is completed.

Now, we study the oscillatory behavior of the positive solutions of the system

$$y_{n+1} = \frac{V + \sum_{i=1}^{k} v_i y_{n-2i+1}}{U + \sum_{i=1}^{k} u_i z_{n-2i+2}}, \qquad z_{n+1} = \frac{C + \sum_{i=1}^{k} c_i z_{n-2i+1}}{D + \sum_{i=1}^{k} d_i y_{n-2i+2}}, \quad n = 0, 1, ... \tag{3.17}$$

where $k \in \{1, 2, ...\}$, $V, U, C, D, v_i, u_i, c_i, d_i$, $i \in \{1, 2, ..., k\}$ are positive real numbers, in order to study the oscillatory behavior of the fuzzy difference equation

$$x_{n+1} = \frac{A + \sum_{i=1}^{k} a_i x_{n-2i+1}}{B + \sum_{i=1}^{k} b_i x_{n-2i+2}}, \tag{3.18}$$

where $A, B, a_i, b_i, i \in \{1, 2, ..., k\}$ are positive fuzzy numbers.
Obviously, equation (3.18) is a special case of (1.2).

We give the following definition.

If y_n, z_n are sequences of positive numbers, we say that (y_n, z_n) oscillates about (y, z), $y, z \in I\!R^+$, if for every $n_0 \in I\!N$ there exist $s, m \in I\!N$, $s, m \geq n_0$ such that

$$\begin{aligned} &(y_s - y)(y_m - y) \leq 0 \quad \textit{and} \quad (z_s - z)(z_m - z) \leq 0 \\ &(y_s - y)(z_s - z) \geq 0 \quad \textit{and} \quad (y_m - y)(z_m - z) \geq 0. \end{aligned} \tag{3.19}$$

Lemma 2. *Consider the system of difference equations (3.17) where V, U, C, D, v_i, u_i, c_i, d_i, $i \in \{1, 2, ..., k\}$ are positive real numbers. A positive solution (y_n, z_n) of system (3.17) oscillates about the unique positive equilibrium (μ_1, μ_2) of system (3.17), if there exists an $m \in \{0, 1, ...\}$ such that one of the following conditions is satisfied:*

$$\begin{aligned} (i) \qquad & y_{m-2i+1} \leq \mu_1, \quad z_{m-2i+1} \leq \mu_2, \\ & y_{m-2i+2} > \mu_1, \quad z_{m-2i+2} > \mu_2, \quad i = 1, 2, ..., k. \end{aligned} \tag{3.20}$$

$$\begin{aligned} (ii) \qquad & y_{m-2i+1} > \mu_1, \quad z_{m-2i+1} > \mu_2, \\ & y_{m-2i+2} \leq \mu_1, \quad z_{m-2i+2} \leq \mu_2, \quad i = 1, 2, ..., k. \end{aligned}$$

Proof (i) For any $\rho = 0, 1, ...$, we prove that

$$y_{m+2\rho+1} < \mu_1, \quad z_{m+2\rho+1} < \mu_2, \quad y_{m+2\rho+2} > \mu_1, \quad z_{m+2\rho+2} > \mu_2. \tag{3.21}$$

From (3.17) and (3.20) we have:

$$y_{m+1} = \frac{V + \sum_{i=1}^{k} v_i y_{m-2i+1}}{U + \sum_{i=1}^{k} u_i z_{m-2i+2}} < \frac{V + \sum_{i=1}^{k} v_i \mu_1}{U + \sum_{i=1}^{k} u_i \mu_2} = \mu_1. \tag{3.22}$$

Similarly, we can prove that

$$z_{m+1} < \mu_2, \quad y_{m+2} > \mu_1, \quad z_{m+2} > \mu_2. \tag{3.23}$$

Using relations (3.20), (3.22), (3.23) and working inductively we can prove that (3.21) hold for any $\rho = 0, 1, ...,$ which means that the solution (y_n, z_n) of system (3.17) oscillates about (μ_1, μ_2), if there exists an $m \in \{0, 1, ...\}$ such that (3.20) hold.

(ii) The proof is similar to the proof of (i).

In what follows we need the following lemma, which has been proved in [6].

Lemma 3. *Let X, Y be fuzzy numbers and*

$$[X]_a = [X_{l,a}, X_{r,a}], \quad [Y]_a = [Y_{l,a}, Y_{r,a}], \quad a \in (0, 1]$$

be the a-cuts of X, Y respectively. Let Z be a fuzzy number such that

$$[Z]_a = [Z_{l,a}, Z_{r,a}], a \in (0, 1].$$

Then

$$MIN\{X, Y\} = Z \quad (resp. \quad MAX\{X, Y\} = Z)$$

if and only if

$$min\{X_{l,a}, Y_{l,a}\} = Z_{l,a}, \quad min\{X_{r,a}, Y_{r,a}\} = Z_{r,a}, \quad a \in (0, 1]$$

$$(resp. \quad max\{X_{l,a}, Y_{l,a}\} = Z_{l,a}, \quad max\{X_{r,a}, Y_{r,a}\} = Z_{r,a}, \quad a \in (0, 1]).$$

Now, we study the oscillatory behavior of the fuzzy difference equation (3.18).

Proposition 4. *Consider equation (3.18) where $k \in \{1, 2, ...\}$, A, B, a_i, b_i, $i \in \{1, 2, ..., k\}$ are positive fuzzy numbers. Then a positive solution x_n of (3.18) oscillates about the positive equilibrium x of (3.18) which satisfies (2.3) , if there exists $m \in \{0, 1, ...\}$ such that one of the following conditions is satisfied:*

$$\begin{aligned} (i) \qquad & L_{m-2i+1,a} \le L_a, R_{m-2i+1,a} \le R_a, \\ & L_{m-2i+2,a} > L_a, R_{m-2i+2,a} > R_a, i \in \{1, 2, ..., k\}. \end{aligned} \tag{3.24}$$

$$\begin{aligned} (ii) \qquad & L_{m-2i+1,a} > L_a, R_{m-2i+1,a} > R_a, \\ & L_{m-2i+2,a} \le L_a, R_{m-2i+2,a} \le R_a, i \in \{1, 2, ..., k\}. \end{aligned} \tag{3.25}$$

Proof Let x_n be a positive solution of (3.18). From (3.19), (3.24) (resp. (3.25)), statement (i) (resp. statement (ii)) of Lemma 2, Lemma 3 and arguing as in Proposition 2.4 of [11] we can easily prove that the solution x_n of (3.18) oscillates about x.

In the following proposition we study the periodicity of the positive solutions of system (3.4).

Lemma 4. *Consider system (3.4) where V, U, C, D are positive constants such that*

$$U = \sum_{i=1}^{k} v_i, \quad D = \sum_{i=1}^{k} c_i, \quad V \sum_{j=1}^{m} d_j = C \sum_{j=1}^{m} u_j. \tag{3.26}$$

Then the following statements are true.

I. Every positive solution of (3.4) is bounded and persists.

II. Let r be a common divisor of the integers p_i+1, q_j+1, $i=1,2,...,k$, $j=1,2,...,m$ such that

$$p_i + 1 = r r_i, \ i = 1,2,...,k, \quad q_j + 1 = r s_j, \ j = 1,2,...,m, \tag{3.27}$$

then system (3.4) has periodic solutions of prime period r. Moreover, if all r_i, $i=1,2,...,k$ (resp. s_j, $j=1,2,...,m$) are even (resp. odd) positive integers then system (3.4) has periodic solutions of prime period $2r$.

III. Let r be the greatest common divisor of the integers p_i+1, q_j+1, $i=1,2,...,k$, $j=1,2,...,m$ such that (3.27) hold, then if all r_i, $i=1,2,...,k$ (resp. s_j, $j=1,2,...,m$) are even (resp. odd) positive integers every positive solution of (3.4) tends to periodic solution of period $2r$, otherwise every positive solution of (3.4) tends to periodic solution of period r.

Proof I. Let (y_n, z_n) be a positive solution of (3.4).
There exist positive numbers L_1, L_2 such that

$$L_1 < y_{-i} < \frac{V}{\sum_{j=1}^{m} u_j L_2}, \quad L_2 < z_{-i} < \frac{C}{\sum_{j=1}^{m} d_j L_1}, \quad i = 0,1,...,\pi. \tag{3.28}$$

Then in view of (3.4), (3.26) and (3.28) we take

$$\begin{aligned} y_1 &= \frac{V + \sum_{i=1}^{k} v_i y_{-p_i}}{U + \sum_{j=1}^{m} u_j z_{-q_j}} > \frac{V + U L_1}{V \sum_{j=1}^{m} d_j + U L_1 \sum_{j=1}^{m} d_j} \sum_{j=1}^{m} d_j L_1 = L_1, \\ y_1 &< \frac{V \sum_{j=1}^{m} u_j L_2 + VU}{\sum_{j=1}^{m} u_j L_2 + U} \frac{1}{\sum_{j=1}^{m} u_j L_2} = \frac{V}{\sum_{j=1}^{m} u_j L_2}. \end{aligned} \tag{3.29}$$

Similarly, we take

$$z_1 > L_2, \quad z_1 < \frac{C}{\sum_{j=1}^{m} d_j L_1}. \tag{3.30}$$

Then using (3.4), (3.26), (3.28), (3.29), (3.30) and arguing as above we take

$$y_n \in [L_1, \frac{V}{\sum_{j=1}^{m} u_j L_2}], \quad z_n \in [L_2, \frac{C}{\sum_{j=1}^{m} d_j L_1}], \quad n = 1, 2, ...$$

and so (y_n, z_n) is bounded and persists, if (3.26) holds.

II. From (3.27), if $\phi = \max\{r_k, s_m\}$ we get

$$\pi + 1 = r\phi$$

Let (y_n, z_n) be a positive solution of (3.4) with initial values satisfying

$$y_{-r\phi+r\lambda+\theta} = y_{-r+\theta}, \quad z_{-r\phi+r\lambda+\theta} = z_{-r+\theta}, \quad \lambda = 0, 1, ..., \phi - 1,$$

$$\theta = 1, 2, ..., r, \quad z_w = \frac{V}{y_w \sum_{j=1}^{m} u_j}, \quad w = -r+1, -r+2, ..., 0. \tag{3.31}$$

Using (3.4), (3.27), (3.31) and arguing as in Proposition 2 of [13] we can easily prove that

$$y_{vr+\theta} = y_{-r+\theta}, \quad z_{vr+\theta} = z_{-r+\theta}, \quad v = 0, 1, ..., \quad \theta = 1, 2, ..., r$$

and so (y_n, z_n) is periodic of period r.

Now, we prove that system (3.4) has periodic solutions of prime period $2r$, if all r_i, $i = 1, 2, ..., k$ (resp. s_j, $j = 1, 2, ..., m$) are even (resp. odd) positive integers.

Firstly, suppose that $p_k < q_m$. Let (y_n, z_n) be a positive solution of (3.4) with initial values satisfying

$$y_{-rs_m+2r\lambda+\zeta} = y_{-r+\zeta}, \quad z_{-rs_m+2r\lambda+\zeta} = z_{-r+\zeta},$$

$$y_{-rs_m+2r\nu+r+\zeta} = y_{-2r+\zeta}, \quad z_{-rs_m+2r\nu+r+\zeta} = z_{-2r+\zeta}, \tag{3.32}$$

$$\lambda = 0, 1, ..., \frac{s_m - 1}{2}, \quad \nu = 0, 1, ..., \frac{s_m - 3}{2}, \quad \zeta = 1, 2, ..., r,$$

and in addition for $\zeta = 1, 2, ..., r$

$$z_{-r+\zeta} = \frac{V}{y_{-2r+\zeta} \sum_{j=1}^{m} u_j}, \; z_{-2r+\zeta} = \frac{V}{y_{-r+\zeta} \sum_{j=1}^{m} u_j}. \tag{3.33}$$

Using (3.4), (3.26), (3.27), (3.32), (3.33) and arguing as in Lemma 2 of [16] we can prove that system (3.4) has periodic solutions of prime period $2r$.

Now, suppose that $q_m < p_k$. Let (y_n, z_n) be a positive solution of (3.4) such that the initial values satisfy relations (3.33) and for $\omega = 0, 1, ..., \frac{r_k}{2} - 1$, $\theta = 1, 2, ..., 2r$

$$y_{-rr_k+2r\omega+\theta} = y_{-2r+\theta}, \quad z_{-rr_k+2r\omega+\theta} = z_{-2r+\theta}. \tag{3.34}$$

Arguing as in Lemma 2 of [16], we can similarly prove that, system (3.4) has periodic solutions of prime period $2r$, if $q_m < p_k$. This completes the proof of the statement II.

III. Now, we prove that every positive solution of system (3.4) tends to a periodic solution of period κr, where

$$\kappa = \begin{cases} 2, & \text{if } r_i, i = 1,2...,k \text{ are even, } s_j, j = 1,2...,m \text{ are odd} \\ 1, & \text{otherwise} \end{cases} \tag{3.35}$$

Let (y_n, z_n) be an arbitrary positive solution of (3.4). We prove that there exist the

$$\lim_{n\to\infty} y_{\kappa nr+i} = \epsilon_i, \quad \lim_{n\to\infty} z_{\kappa nr+i} = \xi_i, \quad i = 0,1,...,\kappa r - 1. \tag{3.36}$$

We fix a $\tau \in \{0,1,...,\kappa r - 1\}$. Since from statement I the solution (y_n, z_n) is bounded and persists, we have

$$\begin{aligned} &\liminf_{n\to\infty} y_{\kappa nr+\tau} = l_\tau > 0, \quad \liminf_{n\to\infty} z_{\kappa nr+\tau} = m_\tau > 0, \\ &\limsup_{n\to\infty} y_{\kappa nr+\tau} = L_\tau < \infty, \quad \limsup_{n\to\infty} z_{\kappa nr+\tau} = M_\tau < \infty. \end{aligned} \tag{3.37}$$

Therefore, from relations (3.4), (3.26) and (3.37) we take

$$m_\tau L_\tau = l_\tau M_\tau = \frac{C}{\sum_{j=1}^{m} d_j}. \tag{3.38}$$

Using (3.4), (3.26), (3.38), Lemma 1 of [16] and arguing as in Lemma 2 of [16], we can prove (3.36). This completes the proof of the lemma.

In the next proposition we study the periodicity of the positive solutions of (1.2).

Proposition 5. *Consider equation (1.2) where $k, m \in \{1,2,...\}$, A, B, a_i, b_j, $i \in \{1,2...,k\}$, $j \in \{1,2...,m\}$ are positive real numbers, such that*

$$B = \sum_{i=1}^{k} a_i, \tag{3.39}$$

and p_i, $i \in \{1,2...,k\}$, q_j, $j \in \{1,2...,m\}$ are positive integers. Then the following statements are true.

(i) If r is a common divisor of the integers p_i+1, q_j+1, $i = 1,2,...,k$, $j = 1,2,...,m$ then equation (1.2) has periodic solutions of prime period r. Moreover, if r_i, $i = 1,2...,k$ (resp. s_j, $j = 1,2...,m$), r_i, s_j are defined in (3.27) are even (resp. odd) integers then equation (1.2) has periodic solutions of prime period $2r$.

(ii) If r is the greatest common divisor of the integers $p_i + 1$, $q_j + 1$, $i = 1,2,...,k$, $j = 1,2,...,m$ such that (3.27) hold, then every positive solution of (1.2) nearly converges to a period κr solution of (1.2) with respect to D as $n \to \infty$ and converges to a period κr solution of (1.2) with respect to D_1 as $n \to \infty$, κ is defined in (3.35).

Proof (i) Since A, B, a_i, b_j, $i \in \{1, 2..., k\}$, $j \in \{1, 2..., m\}$ are positive real numbers we have that

$$\begin{aligned} &A = A_{l,a} = A_{r,a}, \quad B = B_{l,a} = B_{r,a}, \\ &a_i = a_{i,l,a} = a_{i,r,a}, \quad i = 1, 2, ..., k, \quad b_j = b_{j,l,a} = b_{j,r,a}, \quad j = 1, 2, ..., m. \end{aligned} \tag{3.40}$$

We consider functions $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi + 1, ..., 0$ such that for $\lambda = 0, 1, ..., \phi - 1$, $\theta = 1, 2, ..., r$ and $a \in (0, 1]$

$$L_{-r\phi+r\lambda+\theta,a} = L_{-r+\theta,a}, \quad R_{-r\phi+r\lambda+\theta,a} = R_{-r+\theta,a}, \tag{3.41}$$

the functions $L_{w,a}$, $w = -r + 1, -r + 2, ..., 0$ are increasing, left continuous and for all $w = -r + 1, -r + 2, ..., 0$ we have

$$\gamma \leq L_{w,a} \leq \gamma + \epsilon, \quad R_{w,a} = \frac{A}{L_{w,a}\sum_{j=1}^{m} b_j}, \quad \gamma = \delta - \epsilon, \quad \delta = \sqrt{\frac{A}{\sum_{j=1}^{m} b_j}}, \tag{3.42}$$

where ϵ is a positive number such that $\epsilon < \delta$. Using (3.42) and since the functions $L_{w,a}$, $w = -r+1, -r+2, ..., 0$ are increasing and A, b_j, $j = 1, 2, ..., m$, are positive real numbers, we get that $R_{w,a}$, $w = -r+1, -r+2, ..., 0$ are decreasing functions. Moreover, from (3.42) we get

$$L_{w,a} \leq R_{w,a}, \quad \gamma \leq L_{w,a}, R_{w,a} \leq \frac{A}{\gamma\sum_{j=1}^{m} b_j}$$

and so from Theorem 2.1 of [17] $L_{w,a}, R_{w,a}$, $w = -r + 1, -r + 2, ..., 0$ determine fuzzy numbers x_w, $w = -r + 1, -r + 2, ..., 0$ such that $[x_w]_a = [L_{w,a}, R_{w,a}]$, $w = -r + 1, -r + 2, ..., 0$. Let x_n be a positive solution of (1.2) with initial values $x_{-\pi}, x_{-\pi+1}, ..., x_0$, such that (3.2) hold and the functions $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi + 1, ..., 0$, $a \in (0, 1]$ are defined in (3.41), (3.42) and $L_{i,a}$, $i = -\pi, -\pi + 1, ..., 0$, $a \in (0, 1]$ are increasing and left continuous. Then from statement II of Lemma 4 we have that for any $a \in (0, 1]$, the system (3.3), where (3.39) and (3.40) holds, has periodic solutions of prime period r, which means that there exists solution $(L_{n,a}, R_{n,a})$, $a \in (0, 1]$ of the system (3.3) such that

$$L_{n+r,a} = L_{n,a} \ \ \textit{and} \ \ R_{n+r,a} = R_{n,a}, \quad a \in (0, 1]. \tag{3.43}$$

Therefore, from (2.5) and (3.43) we have that equation (1.2) has periodic solutions of prime period r.

Now, suppose that r_i, $i = 1, 2..., k$ (resp. s_i, $j = 1, 2..., m$) are even (resp. odd) integers. We consider functions $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi + 1, ..., 0$ such that analogous relations (3.32), (3.33), (3.34) hold, $L_{w,a}$, $w = -2r+1, -2r+2, ..., 0$ are increasing, left continuous functions and the first relation of (3.42) holds. Arguing as above, the solution x_n of (1.2) with initial values x_i, $i = -\pi, -\pi + 1, ..., 0$ satisfying (3.2), where $L_{i,a}, R_{i,a}$, $i = -\pi, -\pi + 1, ..., 0$ are defined above, is a periodic solution of prime period $2r$.

(ii) Suppose that (3.40) holds. Let x_n be a positive solution of (1.2) such that (2.2) holds. Since $(L_{n,a}, R_{n,a})$ is a positive solution of system which is defined by (3.3) and (3.39), (3.40), from Lemma 4 we have that

$$\lim_{n\to\infty} L_{\kappa nr+l,a} = \epsilon_{l,a}, \quad \lim_{n\to\infty} R_{\kappa nr+l,a} = \xi_{l,a}, \quad a \in (0,1], \quad l = 0, 1, ..., \kappa r - 1, \tag{3.44}$$

where κ is defined in (3.35). Using (3.44) and arguing as in Proposition 2 of [12], we can prove that every positive solution of (1.2) nearly converges to a period κr solution of (1.2) with respect to D as $n \to \infty$ and converges to a period κr solution of (1.2) with respect to D_1 as $n \to \infty$. Thus, the proof of the proposition is completed.

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