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A unitary joint eigenfunction transform for the AΔOs $\exp(ia_{\pm}d/dz) + \exp(2\pi z/a_{\mp})$

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Abstract

For positive parameters a_+ and a_- the commuting difference operators $\exp(ia_{\pm}d/dz) + \exp(2\pi z/a_{\mp})$, acting on meromorphic functions $f(z), z = x + iy$, are formally self-adjoint on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$. Volkov showed that they admit joint eigenfunctions. We prove that the joint eigenfunctions for positive eigenvalues $\exp(2\pi p/a_{\mp}), p \in \mathbb{R}$, give rise to a unitary transform, thus associating commuting self-adjoint operators on $\mathcal{H}$ to the analytic difference operators.

Contents

1 Introduction

In a recent article [1] Volkov observes that the commuting analytic difference operators

$$A_1 = e^{-2i\pi w} + e^{\tau d/dw}, \quad A_2 = e^{-2i\pi w/\tau} + e^{d/dw}, \tag{1.1}$$

admit explicit joint eigenfunctions involving a special function that is a ratio of two double gamma functions, introduced and studied by Barnes a century ago [2]. Since the eigenvalues are given by

$$E_1 = e^{-2i\pi\lambda}, \quad E_2 = e^{-2i\pi\lambda/\tau}, \tag{1.2}$$

he suggests that within a suitable Hilbert space theory the above analytic difference operators (henceforth AΔOs) should be related by

$$(A_1)^{1/\tau} = A_2, \tag{1.3}$$

a relation that makes direct sense only when $1/\tau$ is a positive integer (and which can be verified to hold true in that case). However, Volkov's paper is mainly focused on features of the special function for arguments that are noncommuting operators.

The pertinent special function is often called the double sine function. Independently of the work by Barnes and later authors on the double sine, we introduced and studied essentially the same function in [3] and [4]. We dubbed it the hyperbolic gamma function for reasons explained in [4]. A close relative of the double sine/hyperbolic gamma was also introduced and studied by Faddeev [5] and Woronowicz [6], who refer to their functions as quantum dilogarithm and quantum exponential function, resp. We have collected some features of the hyperbolic gamma function that are important for our present purposes in Appendix A.

The parameter symmetry of the double sine and its avatars is intimately connected to Faddeev's notion of modular double of a quantum group [7], and to the occurrence of this 'modular symmetry' in various quantum integrable models. Besides the AΔO pair (1.1) at issue, the latter include the sine-Gordon and Liouville quantum field theories, and the relativistic Toda and Calogero-Moser N-particle systems, cf. e.g. [8, 9, 10, 11] and references given there.

The principal aim of this paper is to study the Hilbert space theory associated with AΔO pairs that amount to Volkov's AΔOs (1.1) for the special case of positive τ. This case is excluded from consideration in [1], but it is the only case in which we are able to give the relation (1.3) a precise meaning via the functional calculus for unbounded self-adjoint operators [12, 13].

To put the subject matter of this paper in a wider context, we point out that to date there exists no well-developed formalism dealing with the Hilbert space theory of AΔOs. Important heuristic guidance is provided by the transition from classical to quantum mechanics via the canonical quantization prescription

$$p \to -i\hbar d/dx. \tag{1.4}$$

Assume we start with a real-valued smooth Hamiltonian $H(x,p)$ on $\mathbb{R}^2$ that has a real-analytic dependence on x and a polynomial dependence on $\exp(\nu p)$ with ν real. The operator $\exp(-i\nu\hbar d/dx)$ associated to $\exp(\nu p)$ can be easily interpreted as a self-adjoint operator on the Hilbert space

$$\mathcal{H} \equiv L^2(\mathbb{R}, dx) \tag{1.5}$$

via Fourier transformation. The problem to find explicit eigenfunctions for $H(x, -i\hbar d/dx)$ now leads to consideration of the AΔO $A(z, -i\hbar d/dz)$ on a suitable space of analytic functions, the exponentials being defined by

$$(\exp(-i\nu\hbar d/dz)F)(z) = F(z - i\nu\hbar). \tag{1.6}$$

The main snag in using the AΔO-eigenfunctions for Hilbert space purposes is that they are highly non-unique. For the AΔOs just defined this amounts to the following: If $F(z)$ satisfies the analytic difference equation (henceforth AΔE)

$$A(z, -i\hbar d/dz)F(z) = \lambda F(z), \quad \lambda \in \mathbb{C}, \tag{1.7}$$

then clearly $\mu(z)F(z)$ also solves (1.7) for any $\mu(z)$ with period $i\nu\hbar$. The question therefore arises to single out the 'simplest' eigenfunctions, in the hope that the corresponding eigenfunction transform will yield the desired Hilbert space features (more specifically, 'orthogonality and completeness'). Even though this hope is borne out by various explicit examples (including the ones under consideration in this paper), no general theory exists at present.

In view of this state of affairs, we proceed at first in a somewhat more general setting in Section 2, studying a rather large class of AΔOs and their eigenfunctions, and specializing in several steps to the AΔO pairs at issue. Along the way we prove several propositions bearing on the existence or non-existence of joint eigenfunctions. In Prop. 2.4 we show in particular that for generic step size parameters joint eigenfunctions do not exist when the 'potentials' are multiplied by a positive coupling constant $g \neq 1$. We believe that the wider perspective thus gained may be helpful in further studies of the largely unexplored intersection of AΔE theory and Hilbert space theory.

On the other hand, for the detailed study of the Hilbert space aspects of the special AΔO pairs undertaken in Sections 3–5, we only need to know their joint eigenfunctions for positive eigenvalues. Using the first order AΔEs satisfied by the hyperbolic gamma function, it is a routine matter to check the joint eigenfunction property directly, so Section 2 might be skipped at first reading.

We proceed to summarize our main results for the AΔO pair

$$A_\delta \equiv \exp(ia_{-\delta}d/dz) + \exp(-2\pi z/a_\delta), \quad \delta = +, -. \tag{1.8}$$

Here and throughout this paper we choose

$$a_+, a_- \in (0, \infty). \tag{1.9}$$

As we already described above, the AΔOs $A_\pm$ are at first viewed as linear operators on the space $\mathcal{M}$ of meromorphic functions

$$f(z) = f(x + iy), \quad x, y \in \mathbb{R}. \tag{1.10}$$

Taking $z = x \in \mathbb{R}$ in the joint $A_\pm$-eigenfunctions for positive eigenvalues (which follow from Prop. 2.4) we then construct a unitary joint eigenfunction transform. It enables us to associate to the two commuting operators $A_{\pm}$ on $\mathcal{M}$ two commuting self-adjoint operators $H_{\pm}$ on the Hilbert space $\mathcal{H}$ (1.5). Since the latter are unitarily equivalent to multiplication by the functions $\exp(-2\pi p/a_{\pm})$ on the Hilbert space

$$\hat{\mathcal{H}} \equiv L^2(\mathbb{R}, dp), \tag{1.11}$$

it follows in particular that they satisfy the relation

$$(H_-)^{1/\tau} = H_+, \quad \tau \equiv a_+/a_-, \tag{1.12}$$

which gives a mathematically precise meaning to (1.3).

In fact, we consider together with the pair (1.8) a second pair, cf. (3.1)–(3.2). This is because the joint eigenfunction transforms $\mathcal{T}_R$ and $\mathcal{T}_L$ associated to the two pairs turn out to be inversely related (after the spectral representation space $\hat{\mathcal{H}}$ (1.11) is identified with $\mathcal{H}$ (1.5)). Just as in previous papers (cf. e.g. [14]) we use time-dependent scattering theory to show isometry of $\mathcal{T}_R$ and $\mathcal{T}_L$. A novel feature here is that the 'free' comparison dynamics for $t \to \infty$ differs from the one for $t \to -\infty$.

The groundwork for the study of the transforms is laid in Section 3. Their unitarity and wave operator features are dealt with in Section 4, with Appendix C supplying a key technical ingredient. Section 5 is concerned with various natural questions that arise after having shown that the transforms are unitary, entailing that the Hamiltonians are commuting self-adjoint operators. The main issue is to relate the Hamiltonians and their domains to that of the sum operator defined on the intersection of the domains of the pertinent multiplication operator and exponentiated momentum operator. In Appendix B we collect some properties of the latter in a self-contained setting, but with an eye on their occurrence in the main text.

2 A class of AΔOs and their eigenfunctions

A huge class of AΔOs with a well-defined action on the space $\mathcal{M}$ of meromorphic functions can be defined via the building block AΔO

$$(T_w F)(z) = F(z - w), \quad F \in \mathcal{M}, \quad w \in \mathbb{C}. \tag{2.1}$$

Specifically, any AΔO of the form

$$\sum_{j=1}^{M} C_j(z) T_{w_j}, \quad C_j \in \mathcal{M}, \quad w_j \in \mathbb{C}, \quad j = 1, \ldots, M, \tag{2.2}$$

leaves $\mathcal{M}$ invariant, so that its eigenvalue problem makes sense. For the AΔOs arising in the context of integrable systems and quantum groups the translation parameters w_j are far more special, though: They are of the form $kw, k \in \mathbb{Z}, w \in \mathbb{C}^*$. For T_w to admit a reinterpretation as a self-adjoint operator on the Hilbert space $\mathcal{H}$ (1.5), one needs to require $w \in i\mathbb{R}^*$, and so we specialize to this choice. In fact, in this section we only consider the case of nonnegative multiples, so that we may as well start from

$$A = \sum_{n=0}^{N} C_n(z) T_{ina}, \quad a > 0, \quad C_0, \ldots, C_N \in \mathcal{M}. \tag{2.3}$$

We proceed to study the eigenvalue problem

$$AF = \alpha F, \quad F \in \mathcal{M}, \quad \alpha \in \mathbb{C}, \tag{2.4}$$

specializing to the AΔO

$$T_{ia} + \exp(2\pi z/b), \quad a, b > 0, \tag{2.5}$$

in several steps. (Observe that the parity transforms of the AΔOs A_δ (1.8) are of the form (2.5).) First, it should be stressed that if F solves the Nth order AΔE (2.4), then for any μ in the space $\mathcal{P}_{ia}$, where

$$\mathcal{P}_w \equiv \{\mu \in \mathcal{M} \mid \mu(z+w) = \mu(z)\}, \quad w \in \mathbb{C}^*, \tag{2.6}$$

the function $\mu(z)F(z)$ solves (2.4) as well. It is therefore crucial to try and find the 'simplest' solutions in the infinite-dimensional solution space. In this generality, however, very little is known about this problem.

Next, we assume that the coefficients are of the form

$$C_n(z) = R_n(\exp(2\pi z/b)), \quad n = 0, \ldots, N, \quad b > 0, \tag{2.7}$$

where $R_n(w)$ is rational. Then the AΔO

$$B = \sum_{n=0}^{N} R_n(\exp(2\pi z/a))T_{inb} \tag{2.8}$$

clearly commutes with A, so that one may ask for joint solutions to (2.4) and to

$$BF = \beta F, \quad F \in \mathcal{M}, \quad \beta \in \mathbb{C}. \tag{2.9}$$

In this setting the multiplier ambiguity can be drastically reduced by requiring that the step size parameters a and b be rationally independent. Indeed, we have

$$a/b \notin \mathbb{Q} \Rightarrow \mathcal{P}_{ia} \cap \mathcal{P}_{ib} = \mathbb{C}, \tag{2.10}$$

cf. Prop. 2.1 below. Of course, this does not answer the question whether joint solutions exist.

Clearly, the most accessible case is the one of constant coefficients. Then all exponentials $\exp(\gamma z), \gamma \in \mathbb{C}$, are joint solutions, but since both eigenvalues α and β depend on γ, they cannot be freely chosen. In the simplest case

$$A = T_{ia}, \quad B = T_{ib}, \tag{2.11}$$

one obtains

$$\alpha = \exp(-ia\gamma), \quad \beta = \exp(-ib\gamma). \tag{2.12}$$

In particular, A admits any eigenvalue $\alpha \in \mathbb{C}^*$ (the case $\alpha = 0$ clearly yields $F = 0$). But even for this simplest choice of A and B it is not immediate whether or not eigenvalue pairs other than (2.12) are allowed.

To study this, let us start from the choice $\alpha = \exp(-ia\gamma)$, so that all A-eigenfunctions are of the form $\exp(\gamma z)\mu(z)$ with $\mu \in \mathcal{P}_{ia}$. Now we ask whether there exist functions in this infinite-dimensional solution space that solve (2.9) with $\beta \neq \exp(-ib\gamma)$. Thus we should consider the first order AΔE

$$\mu(z - ib)/\mu(z) = d, \quad d \in \mathbb{C}^*. \tag{2.13}$$

Proposition 2.1. Let $\mu \in \mathcal{P}_{ia}$ and $a/b \notin \mathbb{Q}$. Then (2.13) admits no solutions, unless

$$d = \exp(-2\pi i n b/a), \qquad n \in \mathbb{Z}. \tag{2.14}$$

Moreover, when d is given by (2.14), the only solutions are

$$\mu(z) = c\exp(2\pi n z/a), \qquad c \in \mathbb{C}^*. \tag{2.15}$$

Proof. Assume μ solves (2.13). Let $\Re z = x_0$ be a line on which $\mu(z)$ has no poles. Then we can write

$$\mu(x_0 + iy) = \sum_{n\in\mathbb{Z}} c_n \exp(2\pi i n y/a), \qquad y \in \mathbb{R}, \tag{2.16}$$

with the Fourier coefficients c_n having exponential decay as $|n| \to \infty$. From (2.13) we now deduce

$$c_n \exp(-2\pi i n b/a) = d c_n, \qquad \forall n \in \mathbb{Z}. \tag{2.17}$$

Since b/a is irrational, we have

$$n_1 \neq n_2 \Rightarrow \exp(-2\pi i n_1 b/a) \neq \exp(-2\pi i n_2 b/a), \qquad n_1, n_2 \in \mathbb{Z}. \tag{2.18}$$

Combining this with (2.17), the proposition follows. □

Note that for $d = 1$ this result amounts to (2.10). The proposition also makes clear that joint solutions for the special case (2.11) exist only for non-generic eigenvalue pairs

$$(\alpha, \beta) = (\exp(-ia\gamma), \exp(-ib\gamma)\exp(2\pi i k b/a)), \qquad \gamma \in \mathbb{C}, \qquad k \in \mathbb{Z}. \tag{2.19}$$

In general, therefore, one should at best expect that joint solutions to (2.4) and (2.9) exist for special eigenvalue pairs (α, β).

When the coefficients in the AΔO A (2.3) are not only non-constant, but N is also greater than 1, very little seems to be known about solutions to the single eigenvalue problem (2.4). In fact, we are only aware of results for quite special coefficients, which give rise to 'reflectionless' solutions [15, 16]. On the other hand, the first order case $N = 1$ is far more accessible, just as for linear ODEs. In particular, specializing again to coefficients of the form (2.7), the eigenvalue problem (2.4) becomes

$$[R_0(\exp(2\pi z/b)) + R_1(\exp(2\pi z/b))T_{ia}]F = \alpha F, \qquad \alpha \in \mathbb{C}. \tag{2.20}$$

As shown next, it can be solved explicitly in terms of the hyperbolic gamma function, in this section written as

$$G(z) = G(a, b; z). \tag{2.21}$$

(This function and related ones are the subject of Appendix A.)

Proposition 2.2. Assume $R_1(w) \neq 0$ and $R_0(w) \neq \alpha$. Then all solutions to (2.20) are of the form

$$F(z) = \mu(z)\exp[i(2n + M - N)\pi z^2/2ab + cz]\prod_{k=1}^{N} G(z - \delta_k)/\prod_{j=1}^{M} G(z - \gamma_j), \tag{2.22}$$

with $\mu \in \mathcal{P}_{ia}$. Moreover, the numbers $n \in \mathbb{Z}, M, N \in \mathbb{N}$ and $\delta_k, \gamma_j \in \mathbb{C}$ are uniquely determined, and $c \in \mathbb{C}$ is uniquely determined modulo $2\pi/a$.

Proof. We first rewrite (2.20) as

$$\frac{F(z-ia)}{F(z)} = \rho_\alpha(2\pi z/b), \tag{2.23}$$

where

$$\rho_\alpha(w) \equiv (\alpha - R_0(w))/R_1(w) \tag{2.24}$$

is rational. Thus we can factorize $\rho_\alpha(w)$ as

$$\rho_\alpha(w) = \gamma w^n \prod_{j=1}^{M}(w-z_j)/\prod_{k=1}^{N}(w-p_k), \tag{2.25}$$

where $n \in \mathbb{Z}, M, N \in \mathbb{N}, \gamma, z_1, \ldots, z_M, p_1, \ldots, p_N \in \mathbb{C}^*$ are uniquely determined (provided $z_j \neq p_k$, of course). We can now rewrite (2.23) as

$$\frac{F(z-ia)}{F(z)} = \eta \exp[(2n+M-N)\pi z/b] \frac{\prod_{j=1}^{M} 2\cosh[\pi(z-\alpha_j)/b]}{\prod_{k=1}^{N} 2\cosh[\pi(z-\beta_k)/b]}. \tag{2.26}$$

The general solution of (2.26) is therefore given by (2.22), with

$$\gamma_j = \alpha_j - ia/2, \quad j=1,\ldots,M, \quad \delta_k = \beta_k - ia/2, \quad k=1,\ldots,N, \tag{2.27}$$

cf. Appendix A. □

Using (2.22) it is straightforward to study the question whether among the solutions to (2.20) there are solutions to the second AΔE

$$[R_0(\exp(2\pi z/a)) + R_1(\exp(2\pi z/a))T_{ib}]F = \beta F, \quad \beta \in \mathbb{C}. \tag{2.28}$$

Indeed, from (2.22) and the G-AΔE (A.4) we have

$$\begin{aligned} \frac{F(z-ib)}{F(z)} &= \frac{\mu(z-ib)}{\mu(z)} \exp\left[\frac{\pi}{2a}(2n+M-N)(2z-ib) - icb\right] \\ &\quad \times \frac{\prod_{j=1}^{M} 2\cosh[\pi(z-\gamma_j - ib/2)/a]}{\prod_{k=1}^{N} 2\cosh[\pi(z-\delta_k - ib/2)/a]}. \end{aligned} \tag{2.29}$$

When we compare this to (2.28), we see that μ must satisfy an equation of the form

$$\frac{\mu(z-ib)}{\mu(z)} = R(\exp(2\pi z/a)), \quad \mu \in \mathcal{P}_{ia}, \tag{2.30}$$

with $R(w)$ a rational function whose dependence on α and β is suppressed.

We now focus on the solvability of equations of the form (2.30). In the special case that $R(w)$ is constant, we have already seen that this constant must take the values (2.14) for a solution to exist. For non-constant $R(w)$ the following observation is of considerable help.

Proposition 2.3. Assume the function

$$Q(z) \equiv \mu(z-ib)/\mu(z), \qquad \mu \in \mathcal{P}_{ia}, \qquad a/b \notin \mathbb{Q}, \tag{2.31}$$

has a zero at $z=z_0$. Then there exists $k \in \mathbb{Z}^*$ such that $Q(z)$ has a pole at $z=z_0+ikb$. Also, if $Q(z)$ has a pole at $z=z_0$, then there exists $l \in \mathbb{Z}^*$ such that $Q(z)$ has a zero at $z=z_0+ilb$.

Proof. The crux is that point sets of the form $\{z_0+ima+inb\}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ or $n \in -\mathbb{N}$ are dense on the line $\Re z = \Re z_0$. Therefore $\mu(z)$ cannot have zeros or poles in such point sets. To see how this yields the proposition, suppose $Q(z)$ vanishes at $z=z_0$. Then $\mu(z)$ has a pole at $z=z_0$ and/or a zero at $z=z_0-ib$.

First assume z_0 is a pole of $\mu(z)$. Then z_0+ib is a pole of $Q(z)$, unless $\mu(z)$ has a pole at z_0+ib whose multiplicity is at least equal to that of the pole at z_0. Repeating this argument, it easily follows that $Q(z)$ must have a pole at z_0+ikb for some $k \in \mathbb{N}^*$. (Indeed, if this is not the case, then $\mu(z)$ has poles at $z_0+ikb+ima, \forall (k,m) \in \mathbb{N}^* \times \mathbb{Z}$, yielding a contradiction.)

Next assume $\mu(z)$ vanishes at z_0-ib. Then z_0-ib is a pole of $Q(z)$, unless $\mu(z)$ has a zero at z_0-2ib whose multiplicity is at least that of the zero at z_0-ib. Hence it readily follows that $Q(z)$ has a pole at z_0-ikb for some $k \in \mathbb{N}^*$.

Clearly, the assumption that $Q(z)$ has a pole at z_0 leads in the same way to the existence of a zero at z_0+ilb for some $l \in \mathbb{Z}^*$. □

From this proposition it follows in particular that the eigenvalue equations (2.20) and (2.28) need not have joint solutions for *any* pair $(\alpha,\beta) \in \mathbb{C}^2$. The point is that the function on the rhs of (2.30) arising from (2.28) need not have pairs of zeros and poles that differ by imb for some $m \in \mathbb{Z}^*$ (modulo ia). This can already be seen from Prop. 2.2 and its proof, but it is more telling to inspect some simple cases. For example, the choice

$$R_0 = 0, \quad R_1(w) = w+1, \qquad a/b \notin \mathbb{Q}, \tag{2.32}$$

yields a joint eigenvalue problem not admitting any solutions, as can be easily checked by using Prop. 2.3.

The upshot is that the joint eigenvalue problem is 'overdetermined': Generically it will have no solutions, and even if joint solutions do exist, then for a given eigenvalue α one should not expect joint solutions to exist for arbitrary β, but only for a countable set of β's.

We now turn to the special case

$$R_0(w) = gw, \qquad R_1(w) = 1, \qquad g \in (0,\infty), \qquad a/b \notin \mathbb{Q}, \tag{2.33}$$

with which the remainder of this section is concerned. For $\alpha = 0$ the first eigenvalue AΔE can be written

$$\frac{F(z-ia)}{F(z)} = -e^{\lambda}\exp(2\pi z/b), \tag{2.34}$$

where we have set

$$g = e^{\lambda}, \qquad \lambda \in \mathbb{R}. \tag{2.35}$$

Obviously, it has the general solution

$$F(z) = \mu(z) \exp\left(\frac{i\pi z^2}{ab} + \left(-\frac{\pi}{a} - \frac{\pi}{b} + \frac{i\lambda}{a}\right) z\right), \quad \mu \in \mathcal{P}_{ia}, \tag{2.36}$$

which implies

$$\frac{F(z-ib)}{F(z)} = -\frac{\mu(z-ib)}{\mu(z)} \exp((2\pi z + \lambda b)/a). \tag{2.37}$$

Comparing this to the second eigenvalue AΔE

$$\frac{F(z-ib)}{F(z)} = \beta - e^{\lambda} \exp(2\pi z/a), \tag{2.38}$$

we deduce from Prop. 2.3 that for $\beta \neq 0$ there exist no joint solutions.

Choosing next $\beta = 0$, we should solve

$$\frac{\mu(z-ib)}{\mu(z)} = \exp(\lambda[1 - b/a]). \tag{2.39}$$

Thus we get a constraint on λ, cf. (2.13)–(2.14): It must satisfy

$$\exp(\lambda[1 - b/a]) = \exp(2\pi ikb/a), \quad k \in \mathbb{Z}. \tag{2.40}$$

Recalling (2.35), we infer $k = 0, \lambda = 0$. Thus we need $g = 1$ for joint solutions to exist. Combining this with (2.15) and (2.36), we see they are given by

$$F(z) = c \exp\left(\frac{i\pi}{ab}[z^2 + i(a+b)z]\right), \quad g = 1, \quad \alpha = \beta = 0, \quad c \in \mathbb{C}^*. \tag{2.41}$$

We proceed to consider the case $\alpha \neq 0$.

Proposition 2.4. The joint eigenvalue problem

$$(g \exp(2\pi z/b) + T_{ia})F = \exp(2\pi p/b)F, \quad g \in (0, \infty), \quad \Im p \in (-b/2, b/2], \tag{2.42}$$

$$(g \exp(2\pi z/a) + T_{ib})F = \beta F, \quad \beta \in \mathbb{C}, \tag{2.43}$$

(with a/b irrational) has no solutions for $g \neq 1$. For $g = 1$ it has no solutions unless β is given by

$$\beta = \exp(2\pi p_l/a), \quad p_l \equiv p - ilb, \quad l \in \mathbb{Z}, \tag{2.44}$$

and in that case all solutions are of the form

$$F(z) = c \exp\left(\frac{i\pi}{2ab}[z^2 + i(a+b)z + 2zp_l]\right) G(-z + p_l - i(a+b)/2), \quad c \in \mathbb{C}^*. \tag{2.45}$$

Proof. Instead of (2.34) we now get

$$\frac{F(z-ia)}{F(z)} = \exp(2\pi p/b) - e^{\lambda}\exp(2\pi z/b). \tag{2.46}$$

Rewriting this as

$$\frac{F(z-ia)}{F(z)} = 2i\cosh\left(\frac{\pi}{b}\left[z-p+\frac{ib}{2}+\frac{\lambda b}{2\pi}\right]\right)\exp\left(\frac{\lambda}{2}+\frac{\pi z}{b}+\frac{\pi p}{b}\right), \tag{2.47}$$

we obtain the general solution

$$F(z) = \mu(z)\frac{\exp\left(\frac{i\pi}{2ab}[z^2+(ia+ib+\lambda b/\pi+2p)z]\right)}{G(z-p+i(a+b)/2+\lambda b/2\pi)}, \quad \mu\in\mathcal{P}_{ia}, \tag{2.48}$$

to (2.42). This solution yields

$$\begin{aligned}\frac{F(z-ib)}{F(z)} &= 2i\frac{\mu(z-ib)}{\mu(z)}\cosh\left(\frac{\pi}{a}\left[z-p+\frac{ia}{2}+\frac{\lambda b}{2\pi}\right]\right)\\ &\quad\times\exp\left(\frac{\pi}{2a}\left[2z+\frac{\lambda b}{\pi}+2p\right]\right).\end{aligned} \tag{2.49}$$

In view of (2.43) we should also require

$$\frac{F(z-ib)}{F(z)} = \beta - e^{\lambda}\exp(2\pi z/a). \tag{2.50}$$

Demanding equality, we deduce from Prop. 2.3 that for $\beta=0$ no solution $\mu\in\mathcal{P}_{ia}$ exists. For $\beta\neq 0$ we may set

$$\beta = \exp(2\pi q/a), \tag{2.51}$$

and then μ should fulfil

$$\frac{\mu(z-ib)}{\mu(z)} = \exp\left(\frac{\lambda}{2}-\frac{\lambda b}{2a}+\frac{\pi}{a}(q-p)\right)\frac{\sinh(\pi[z-q+a\lambda/2\pi]/a)}{\sinh(\pi[z-p+b\lambda/2\pi]/a)}. \tag{2.52}$$

Invoking once more Prop. 2.3, we deduce that q must be of the form

$$q = p - \frac{(b-a)\lambda}{2\pi} - ilb, \qquad l\in\mathbb{Z}, \tag{2.53}$$

yielding

$$\frac{\mu(z-ib)}{\mu(z)} = \exp(\lambda[1-b/a])\exp(-i\pi lb/a)\frac{\sinh(\pi[z-p+ilb+b\lambda/2\pi]/a)}{\sinh(\pi[z-p+b\lambda/2\pi]/a)}. \tag{2.54}$$

Consider first the case $l>0$. Introducing

$$\mu_l(z) \equiv \exp(\pi lz/a)/\prod_{n=1}^{l}\sinh\left(\frac{\pi}{a}[z-p+inb+b\lambda/2\pi]\right), \tag{2.55}$$

we get $\mu_l\in\mathcal{P}_{ia}$; moreover, defining

$$\mu_r(z) \equiv \mu(z)/\mu_l(z), \tag{2.56}$$

it remains to solve

$$\frac{\mu_r(z - ib)}{\mu_r(z)} = \exp(\lambda[1 - b/a]), \qquad \mu_r \in \mathcal{P}_{ia}. \tag{2.57}$$

We have already seen that this entails that λ vanishes and $\mu_r(z)$ is constant, cf. (2.39)–(2.40). Hence we have $q = p_l$. To verify that the resulting solutions are of the form (2.45), it suffices to check

$$\frac{G(w + ilb + i(a + b)/2)}{G(w + i(a + b)/2)} = \rho \prod_{n=1}^{l} \sinh\left(\frac{\pi}{a}[w + inb]\right), \quad l \in \mathbb{N}^*. \tag{2.58}$$

This is a consequence of the G-AΔE (A.4) (yielding $\rho = (2i)^l$), so (2.45) follows for $l > 0$. The case $l \leq 0$ can be handled along the same lines. □

We would like to point out that the AΔO on the lhs of (2.42) also leaves the space $\mathcal{E}$ of *entire* functions invariant. Hence its eigenvalue problem (2.42) is well defined in $\mathcal{E}$. But in the course of the proof it becomes clear that (2.42) admits no solutions in $\mathcal{E}$. Indeed, this follows from the zero locations of the multiplier $\mu(z) \in \mathcal{P}_{ia}$ in the general solution (2.48) to (2.42): since they are ia-periodic, they cannot cancel all of the zeros of the G-function in (2.48), cf. (A.3).

It is also of interest to observe the relation of the solutions (2.45) to the zero-eigenvalue solutions (2.41): provided the former are multiplied by a suitable exponential depending only on p_l, they converge to the latter for $\Re p \to -\infty$ (by virtue of the G-asymptotics (A.11)).

To conclude this section we would like to stress that the above 'no-go' results hinge on the positivity of the parameters a and b. (Recall we are requiring this so that the AΔOs $T_{ima}, T_{inb}, m, n \in \mathbb{Z}^*$, are at least formally self-adjoint.) To explain what is involved here, let us replace ia and ib by arbitrary numbers $2\omega, 2\omega' \in \mathbb{C}^*$ with $\omega/\omega' \notin \mathbb{R}$. Then the intersection of $\mathcal{P}_{2\omega}$ and $\mathcal{P}_{2\omega'}$ consists of all elliptic functions with periods 2ω and $2\omega'$, which should be compared to (2.10). Moreover, using the Weierstrass σ-function $\sigma(\omega, \omega'; z)$ it is easy to construct for any $\lambda \in \mathbb{C}^*$ a meromorphic function $\mu_\lambda(z)$ such that

$$\mu_\lambda(z + 2\omega) = \mu_\lambda(z), \qquad \mu_\lambda(z + 2\omega') = \lambda\mu_\lambda(z). \tag{2.59}$$

From this it is easily seen that when a pair of commuting AΔOs

$$A_1 = C_1(z)T_{2\omega}, \quad A_2 = C_2(z)T_{2\omega'}, \quad C_1, C_2 \in \mathcal{M}, \quad \omega/\omega' \notin \mathbb{R}, \tag{2.60}$$

has a joint eigenfunction for *one* eigenvalue pair $(E_0, E_0') \in \mathbb{C}^{*2}$, then there also exist joint eigenfunctions for *all* $(E, E') \in \mathbb{C}^{*2}$, again in sharp contrast to the case $\omega/\omega' \in \mathbb{R}$.

3 Defining the eigenfunction transforms $\mathcal{T}_R$ and $\mathcal{T}_L$

We now embark on the program of associating commuting self-adjoint operators on the Hilbert space $\mathcal{H}$ (1.5) to the two pairs of AΔOs

$$A_{R\delta} = \exp(ia_{-\delta}d/dz) + \exp(-2\pi z/a_\delta), \qquad \delta = +, -, \tag{3.1}$$

$$A_{L\delta} = \exp(ia_{-\delta} d/dz) + \exp(2\pi z/a_\delta), \qquad \delta = +, -. \tag{3.2}$$

Replacing the parameters a and b employed in Section 2 by a_+ and a_-, resp., we deduce from Prop. 2.4 (flipping signs of z and p) that the joint eigenvalue problem

$$A_{R\delta} F = \exp(-2\pi p/a_\delta) F, \qquad p \in \mathbb{C}, \qquad \delta = +, -, \tag{3.3}$$

is solved by functions of the form

$$c_R(p) \exp\left(\frac{i\pi}{2a_+a_-}[z^2 - i(a_+ + a_-)z + 2zp]\right) G(z - p - i(a_+ + a_-)/2). \tag{3.4}$$

Likewise, the joint eigenvalue problem

$$A_{L\delta} F = \exp(2\pi p/a_\delta) F, \qquad p \in \mathbb{C}, \qquad \delta = +, -, \tag{3.5}$$

is solved by functions of the form

$$c_L(p) \exp\left(\frac{-i\pi}{2a_+a_-}[z^2 - i(a_+ + a_-)z + 2zp]\right) G(z - p - i(a_+ + a_-)/2). \tag{3.6}$$

(Using the G-AΔEs (A.1) and (A.4) it is quite easy to check the joint eigenfunction property directly.)

We now normalize the p-dependence of the constants such that we obtain

$$(A_{\sigma\delta}\mathcal{E}_\sigma)(z, p) = \exp(\mp 2\pi p/a_\delta)\mathcal{E}_\sigma(z, p), \quad p \in \mathbb{C}, \quad \delta = +, -, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right., \tag{3.7}$$

where

$$\mathcal{E}_R(z, p) \equiv (\alpha/2\pi)^{1/2} \exp(i\alpha zp) S_R(z - p), \quad \alpha = 2\pi/a_+a_-, \tag{3.8}$$

$$\mathcal{E}_L(z, p) \equiv (\alpha/2\pi)^{1/2} \exp(-i\alpha zp) S_L(z - p), \tag{3.9}$$

and S_R and S_L are defined by (A.12)–(A.13). In view of the asymptotics (A.14)–(A.15) we have

$$\mathcal{E}_\sigma(z, p) = \left(\frac{\alpha}{2\pi}\right)^{1/2} e^{\pm i\alpha zp}\left(1 + O(e^{-\rho|\Re z|})\right), \quad \Re z \to \pm\infty, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right., \tag{3.10}$$

$$\begin{aligned} \mathcal{E}_\sigma(z, p) \quad = \quad & \left(\frac{\alpha}{2\pi}\right)^{1/2} e^{\pm i[2\chi - i\alpha az + \alpha(z^2 + (p+ia)^2)/2]} \\ & \times \left(1 + O(e^{-\rho|\Re z|})\right), \quad \Re z \to \mp\infty, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right., \end{aligned} \tag{3.11}$$

with the bounds uniform for $(a_+, a_-, \Im z, p)$ in compacts of $(0, \infty)^2 \times \mathbb{R} \times \mathbb{C}$. In particular, choosing p and z real, the eigenfunction $\mathcal{E}_R$ reduces on the far right to the kernel of the Fourier transformation

$$\mathcal{F}_\alpha : \hat{\mathcal{H}} \to \mathcal{H}, \quad \phi(p) \mapsto \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dp e^{i\alpha xp}\phi(p) \tag{3.12}$$

from momentum space to position space,

$$\hat{\mathcal{H}} = L^2(\mathbb{R}, dp), \quad \mathcal{H} = L^2(\mathbb{R}, dx), \tag{3.13}$$

physically speaking. It is convenient (both from a notational and from a conceptual viewpoint) to distinguish $\hat{\mathcal{H}}$ and $\mathcal{H}$ until further notice. But it is useful to note already at this stage that when $\hat{\mathcal{H}}$ and $\mathcal{H}$ are identified in the obvious way, then the eigenfunction $\mathcal{E}_L(x,p)$ reduces on the far left to the kernel of $\mathcal{F}_\alpha^* = \mathcal{F}_\alpha^{-1}$.

We now define the space

$$\mathcal{C} \equiv C_0^\infty(\mathbb{R}), \tag{3.14}$$

which may be viewed as a dense subspace of $\mathcal{H}$ and $\hat{\mathcal{H}}$, and proceed to study the functions

$$I_{\sigma,\phi}(z) \equiv \int_{-\infty}^{\infty} dp \mathcal{E}_\sigma(z,p)\phi(p), \quad \sigma = R, L, \quad \phi \in \mathcal{C}. \tag{3.15}$$

Since ϕ has compact support, there exist $r_+, r_- \in \mathbb{R}$ such that

$$\text{supp}(\phi) \subset [r_-, r_+]. \tag{3.16}$$

The poles of $\mathcal{E}_\sigma(z,p)$ are located at

$$z = p - ika_+ - ila_-, \qquad k, l \in \mathbb{N}, \tag{3.17}$$

cf. Appendix A. Hence the integral (3.15) is well defined for z in the region

$$\mathcal{R}_\phi \equiv \{\Im z > 0\} \cup \{\Re z < r_-\} \cup \{\Re z > r_+\}. \tag{3.18}$$

Moreover, the function $I_{\sigma,\phi}(z)$ is analytic in $\mathcal{R}_\phi$ and satisfies

$$\begin{aligned} I_{\sigma,\phi}(z + ia_{-\delta}) + e^{\mp 2\pi z/a_\delta} I_{\sigma,\phi}(z) &= \int_{-\infty}^{\infty} dp \mathcal{E}_\sigma(z,p) e^{\mp 2\pi p/a_\delta} \phi(p), \\ & z \in \mathcal{R}_\phi, \quad \delta = +, -, \quad \sigma = \left\{ \begin{array}{l} R \\ L \end{array} \right., \end{aligned} \tag{3.19}$$

due to the eigenfunction property (3.7) of $\mathcal{E}_\sigma$.

Next, consider the behavior of $I_{\sigma,\phi}(x+iy), x, y \in \mathbb{R}$, for $|x| \to \infty$. In view of (3.10), $I_{\sigma,\phi}(x+iy)$ has Schwartz space decay as $x \to \infty / -\infty$ for $\sigma = R/L$, resp. Using (3.11), (3.16), (3.17) and the Schwarz inequality, we also deduce the bounds

$$I_{R,\phi}(x+iy) \leq C_R \|\phi\| \exp(-\alpha x(y-a)), \quad \forall x \leq r_- - 1, \tag{3.20}$$

$$I_{L,\phi}(x+iy) \leq C_L \|\phi\| \exp(\alpha x(y-a)), \quad \forall x \geq r_+ + 1, \tag{3.21}$$

where C_σ can be chosen uniformly for ϕ satisfying (3.16) and (a_+, a_-, y) in compacts of $(0,\infty)^2 \times \mathbb{R}$.

As a final preparation for Lemma 3.1 below, we focus on the behavior for $y \to 0$. The pole of $\mathcal{E}_\sigma(z,p)$ at $z = p$ is simple with residue

$$\rho_\sigma(p) = \frac{i}{2\pi} \exp[\pm i(\chi - \alpha a^2/4 + \alpha p^2)], \qquad \sigma = \left\{ \begin{array}{l} R \\ L \end{array} \right., \tag{3.22}$$

cf. (A.16). Also, letting $\Im z$ decrease, the next pole arises for $z = p - ia_s$, cf. (3.17). Thus the difference function

$$\mathcal{D}_\sigma(z,p) \equiv \mathcal{E}_\sigma(z,p) - \rho_\sigma(p)\phi(p)(z-p)^{-1}, \quad p \in \mathbb{R}, \quad \sigma = R, L, \tag{3.23}$$

is analytic for $\Im z > -a_s$. Now the comparison function

$$\mathcal{C}_{\sigma,\phi}(z) \equiv \int_{-\infty}^{\infty} dp \rho_\sigma(p)\phi(p)(z-p)^{-1}, \tag{3.24}$$

belongs to $\mathcal{H}$ for $y = \Im z \neq 0$. Indeed, it is routine to verify

$$(\mathcal{F}_\alpha^{-1}\mathcal{C}_{\sigma,\phi}(\cdot + iy))(p) = \mp 2\pi i \theta(\pm p)\exp(-\alpha y p)\psi_\sigma(p), \quad \pm y > 0, \tag{3.25}$$

$$\psi_\sigma(p) \equiv \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dq \exp(-i\alpha p q)\rho_\sigma(q)\phi(q), \tag{3.26}$$

(θ denotes the Heaviside function), and the function (3.25) is manifestly in $\hat{\mathcal{H}}$. Moreover, since $\psi_\sigma(p)$ is a Schwartz space function, the limits

$$\mathcal{C}^\pm_{\sigma,\phi}(x) \equiv \lim_{y \to 0_\pm} \mathcal{C}_{\sigma,\phi}(x+iy) \tag{3.27}$$

exist pointwise and in the strong $\mathcal{H}$-topology, and the resulting functions are smooth and vanish for $|x| \to \infty$.

Writing now

$$I_{\sigma,\phi}(z) = \int_{-\infty}^{\infty} dp \mathcal{D}_\sigma(z,p)\phi(p) + \mathcal{C}_{\sigma,\phi}(z), \tag{3.28}$$

the integral yields a function that is analytic in $z = x+iy$ for $y > -a_s$. From (3.25)–(3.27) it then follows that the $y \to 0_\pm$ limits exist pointwise, yielding

$$\lim_{y \to 0_\pm} I_{\sigma,\phi}(x+iy) = \int_{-\infty}^{\infty} dp \mathcal{D}_\sigma(x,p)\phi(p) + \mathcal{C}^\pm_{\sigma,\phi}(x) \equiv I^\pm_{\sigma,\phi}(x). \tag{3.29}$$

Furthermore, we have

$$I^+_{\sigma,\phi}(x) - I^-_{\sigma,\phi}(x) = \mathcal{C}^+_{\sigma,\phi}(x) - \mathcal{C}^-_{\sigma,\phi}(x) = -2\pi i \rho_\sigma(x)\phi(x). \tag{3.30}$$

In the next lemma we collect some of the features of $I_{\sigma,\phi}(z)$ we have just derived and obtain a few more.

Lemma 3.1. The function

$$x \mapsto I_{\sigma,\phi}(x+iy) = \int_{-\infty}^{\infty} dp \mathcal{E}_\sigma(x+iy,p)\phi(p), \quad \sigma \in \{R,L\}, \quad \phi \in \mathcal{C}, \quad y > 0, \tag{3.31}$$

belongs to $\mathcal{H}$ for all $y \in (0,a)$. The limit $y \to 0_+$ exists for all $x \in \mathbb{R}$ and in the $\mathcal{H}$-topology, yielding a function $I^+_{\sigma,\phi}(x)$ with the following properties. First, it is real-analytic for $x > r_+$ and $x < r_-$ (with (3.16) in force). Second, it has Schwartz space decay when $x \to \infty / -\infty$ for $\sigma = R/L$, resp., while

$$I^+_{\sigma,\phi}(x) = O(\exp[\pm \alpha a x]), \quad x \to \mp\infty, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right. . \tag{3.32}$$

Third, it is smooth. Fourth, it satisfies the bound

$$\|I^+_{\sigma,\phi}\| < C\|\phi\|, \tag{3.33}$$

with C uniform for ϕ obeying (3.16) and for (a_+,a_-) in compacts of $(0,\infty)^2$. Fifth, it is strongly continuous in a_+ and a_- for $(a_+,a_-)\in(0,\infty)^2$.

Proof. The first assertion is clear from the analyticity of $I_{\sigma,\phi}(z)$ in $\mathcal{R}_\phi$ (3.18) and the paragraph containing the bounds (3.20)–(3.21). Since $I^+_{\sigma,\phi}(x)$ coincides with $I_{\sigma,\phi}(z)$ for $z\in(-\infty,r_-)$ and $z\in(r_+,\infty)$, the latter paragraph also yields the first two properties of $I^+_{\sigma,\phi}(x)$ and smoothness for $x\notin[r_-,r_+]$. We have already shown that the limit $y\to 0_+$ exists for fixed x, cf. (3.29). Its existence in L^2-sense for x outside the interval $[r_--1,r_++1]$ (say) is clear from the features of $I_{\sigma,\phi}(z)$. For $x\in[r_--1,r_++1]$ it follows by using (3.28) and recalling the $y\to 0_+$ limit (3.27); smoothness of $\mathcal{C}^+_{\sigma,\phi}(x)$ also implies smoothness of $I^+_{\sigma,\phi}(x)$ for $x\in[r_--1,r_++1]$.

It remains to demonstrate the last two properties of $I^+_{\sigma,\phi}(x)$. We detail the case $\sigma=R$, the proof for the case $\sigma=L$ being similar. To prove (3.33), we split up the integral of $|I^+_{R,\phi}(x)|^2$ over $\mathbb{R}$ into integrals over $(-\infty,r_--1)$, $[r_--1,r_++1]$ and (r_++1,∞), and show that each of the three integrals is majorized by $C\|\phi\|^2$, with C of the asserted form.

To bound the first integral we use (3.20) with $y=0$, yielding an estimate of the announced type. To handle the second one we use (3.29), obtaining

$$\int_{r_--1}^{r_++1} dx|I^+_{R,\phi}(x)|^2 \le \int_{r_--1}^{r_++1} dx\left|\int_{r_-}^{r_+} dp\mathcal{D}_R(x,p)\phi(p)\right|^2 + \int_{r_--1}^{r_++1} dx|\mathcal{C}^+_{R,\phi}(x)|^2. \tag{3.34}$$

The first integral on the rhs is majorized by $\|\phi\|^2$ times the maximum of $|\mathcal{D}_R(x,p)|^2$ for (x,p) in the square $[r_--1,r_++1]\times[r_-,r_+]$, which is of the required form. The second one is bounded by $\|\phi\|^2$, as readily follows from (3.22)–(3.27).

To handle the third integral we invoke the bound (3.10). It implies

$$\int_{r_++1}^{\infty} dx|I^+_{R,\phi}(x)|^2 \le \int_{r_++1}^{\infty} dx|(\mathcal{F}_\alpha\phi)(x)|^2 + \int_{r_++1}^{\infty} dx\left|\int_{r_-}^{r_+} dpO(e^{-\rho x})\phi(p)\right|^2. \tag{3.35}$$

The first integral on the rhs is bounded by $\|\phi\|^2$, since $\mathcal{F}_\alpha$ is unitary. In the second integral the uniformity properties of the remainder function $O(\exp(-\rho x))$ in (3.10) imply a bound $C\|\phi\|^2$ of the required form too. Hence we have now proved (3.33).

Finally, the strong continuity of $I^+_{\sigma,\phi}(x)$ in a_+ and a_- can be deduced from the continuity of $\mathcal{E}_\sigma(x,p)$ in (a_+,a_-) and dominated convergence. More in detail, the pertinent dominating function can be chosen uniformly for (a_+,a_-) in compacts of $(0,\infty)^2$, since the error terms in (3.10)–(3.11) have this uniformity feature. Furthermore, when (3.10) is invoked to handle the relevant intervals, the strong continuity of $\mathcal{F}_\alpha$ (3.12) in α for $\alpha\in(0,\infty)$ should be used. □

We are now prepared to define transforms $\mathcal{T}_R$ and $\mathcal{T}_L$ by

$$\mathcal{T}_\sigma : \mathcal{C}\subset\hat{\mathcal{H}}\to\mathcal{H},\quad \phi(p)\mapsto I^+_{\sigma,\phi}(x) = \int_{-\infty}^{\infty} dp\mathcal{E}_\sigma(x+i0,p)\phi(p),\quad \sigma=R,L, \tag{3.36}$$

and Hamiltonians $H_{\sigma\pm}$ on

$$\mathcal{P}_\sigma \equiv \mathcal{T}_\sigma \mathcal{C}, \tag{3.37}$$

by setting

$$H_{\sigma\delta}\mathcal{T}_\sigma\phi \equiv \mathcal{T}_\sigma M_{\sigma\delta}\phi, \quad \phi\in\mathcal{C}, \quad \delta=+,-, \tag{3.38}$$

where $M_{\sigma\delta}$ denotes the multiplication operators given by

$$(M_{\sigma\delta}\phi)(p) \equiv \exp(\mp 2\pi p/a_\delta)\phi(p), \quad \delta=+,-, \quad \sigma=\left\{\begin{array}{c} R \\ L \end{array}\right. . \tag{3.39}$$

To verify that the definition (3.38) makes sense, note first that $M_{\sigma\delta}$ leaves $\mathcal{C}$ invariant. Next, assume $\mathcal{T}_\sigma\phi=0$. Then we have in particular $I^+_{\sigma,\phi}(x)=0$ in $\mathcal{R}_\phi$ (3.18). This implies that the lhs of (3.19) vanishes, entailing $\mathcal{T}_\sigma M_{\sigma\delta}\phi=0$. Hence (3.38) indeed gives rise to well-defined linear operators from $\mathcal{P}_\sigma$ to $\mathcal{H}$.

From (3.19) we deduce

$$(H_{\sigma\delta}I^+_{\sigma,\phi})(x) = I_{\sigma,\phi}(x+ia_{-\delta}) + \exp(\mp\alpha a_{-\delta}x)I^+_{\sigma,\phi}(x), \quad \sigma=\left\{\begin{array}{c} R \\ L \end{array}\right. . \tag{3.40}$$

Although this shows that the Hamiltonian action corresponds to the AΔO action (recall (3.1)–(3.2)), it should be stressed that none of the functions $I^+_{\sigma,\phi}$ extends to a meromorphic function (except when $\phi=0$ of course). Indeed, this is clear from (3.30).

To proceed, we point out that the two functions on the rhs of (3.40) belong to $\mathcal{H}$, provided that $a_+\neq a_-$ and the Hamiltonian with the smallest step size a_s is chosen. Indeed, in that case $I_{\sigma,\phi}(x+ia_s)$ belongs to $\mathcal{H}$ by virtue of Lemma 3.1. Since the lhs of (3.40) belongs to $\mathcal{H}$, also the second function on the rhs is in $\mathcal{H}$. But it is more telling to deduce this directly from the bound (3.32).

From the latter bound it is plausible that the two functions are not in $\mathcal{H}$ for the Hamiltonian with the largest step size a_l. (We cannot prove this in general, but from Prop. 5.2 it follows that there do exist $\phi\in\mathcal{C}$ for which the two summands are not in $\mathcal{H}$.) In view of this different behavior, it is expedient to work from now on with Hamiltonians $H_{\sigma s}$ and $H_{\sigma l}$ corresponding to the small and large step size a_s and a_l, resp.

We are now in the position to make contact with the results in Appendix B. Indeed, from the domain characterization in Lemma B.2 and the properties of $I_{\sigma,\phi}(z)$ established above it is evident that we have

$$I^+_{\sigma,\phi}(x) \in \mathcal{D}(E(a_s)), \quad a_s<a_l, \tag{3.41}$$

cf. (B.3). Moreover, as we already pointed out, the bound (3.32) entails

$$I^+_{\sigma,\phi}(x) \in \mathcal{D}(M(\mp\alpha a_s)), \quad a_s<a_l, \quad \sigma=\left\{\begin{array}{c} R \\ L \end{array}\right. . \tag{3.42}$$

Therefore we have

$$\mathcal{P}_\sigma \subset \mathcal{D}(S(a_s,\mp\alpha a_s)), \quad a_s<a_l, \quad \sigma=\left\{\begin{array}{c} R \\ L \end{array}\right. , \tag{3.43}$$

and

$$H_{\sigma s} = S(a_s,\mp\alpha a_s)\restriction \mathcal{P}_\sigma, \quad a_s<a_l, \quad \sigma=\left\{\begin{array}{c} R \\ L \end{array}\right. . \tag{3.44}$$

It is now easy to prove the following lemma.

Lemma 3.2. The operator $H_{\sigma s}$ is symmetric.

Proof. On account of (3.44), $H_{\sigma s}$ equals the restriction of a symmetric operator to $\mathcal{P}_\sigma$, provided that $a_s < a_l$. Thus we have

$$(\mathcal{T}_\sigma \phi_1, \mathcal{T}_\sigma M_{\sigma s}\phi_2) = (\mathcal{T}_\sigma M_{\sigma s}\phi_1, \mathcal{T}_\sigma \phi_2), \quad \forall \phi_1, \phi_2 \in \mathcal{C}. \tag{3.45}$$

Now $M_{\sigma s}\phi$ is strongly continuous in a_+ and a_-. Due to the fifth property in Lemma 3.1, the same is true for $\mathcal{T}_\sigma M_{\sigma s}\phi$ and $\mathcal{T}_\sigma \phi$. Thus we can let a_s converge to a_l, obtaining (3.45) for $a_s = a_l$. □

Later we will show that $H_{\sigma l}$ is symmetric too. For the moment, however, we have not even shown that the Hamiltonians are densely defined. To prove directly that $\mathcal{P}_\sigma$ is dense seems hard at this point. Denseness will be a corollary of our results concerning scattering theory in the next section. We conclude this section by introducing an interacting dynamics (unitary one-parameter group on $\mathcal{H}$) that will serve as the starting point for time-dependent scattering theory.

To this end we need a property of $\mathcal{P}_\sigma$ that easily follows from the bound (3.33); specifically, all vectors in $\mathcal{P}_\sigma$ are analytic vectors for $H_{\sigma s}$. Indeed, from its definition it is clear that $H_{\sigma s}$ leaves $\mathcal{P}_\sigma$ invariant, and we have

$$\|H_{\sigma s}^n \mathcal{T}_\sigma \phi\| = \|\mathcal{T}_\sigma M_{\sigma s}^n \phi\| \le C c^n \|\phi\|, \quad \phi \in \mathcal{C}, \quad n \in \mathbb{N}, \tag{3.46}$$

with C and c depending only on r_- and r_+, cf. (3.16).

Since $\mathcal{P}_\sigma$ consists of analytic vectors for $H_{\sigma s}$ and is left invariant, and since $H_{\sigma s}$ is symmetric on $\mathcal{P}_\sigma$ (as shown in the previous lemma), it follows from Nelson's analytic vector theorem that $H_{\sigma s}$ is essentially self-adjoint on $\mathcal{P}_\sigma$. Denoting the self-adjoint closure again by $H_{\sigma s}$, we obtain a unitary one-parameter group $\exp(-itH_{\sigma s}), t \in \mathbb{R}$, on the closure of $\mathcal{P}_\sigma$. Since we have not yet shown that the latter equals $\mathcal{H}$, we extend $H_{\sigma s}$ provisionally to a self-adjoint operator on $\mathcal{H}$ by choosing it equal to an arbitrary bounded self-adjoint operator on the orthogonal complement of $\mathcal{P}_\sigma$.

4 Scattering theory and unitarity of $\mathcal{T}_R$ and $\mathcal{T}_L$

From the definition of the unitary one-parameter group $\exp(-itH_{\sigma s})$ just given it is clear that it satisfies the intertwining relation

$$\exp(-itH_{\sigma s})\mathcal{T}_\sigma \phi = \mathcal{T}_\sigma \exp(-itM_{\sigma s})\phi, \quad \phi \in \mathcal{C}, \tag{4.1}$$

cf. (3.38)–(3.39) with $a_\delta = a_l$. At this point we have not yet shown that $\mathcal{T}_\sigma$ is bounded and that $\mathcal{P}_\sigma$ is dense in $\mathcal{H}$. In this section we prove in particular that these two properties hold true. We are going to make use of time-dependent scattering theory [18]. In order to avoid the repeated use of an identification operator, we henceforth identify $\hat{\mathcal{H}}$ with $\mathcal{H}$ in the natural way.

It is a remarkable feature of the 'interacting' evolution $\exp(-itH_{\sigma s})$ that it resembles two distinct 'free' evolutions for $t \to \pm\infty$, in the sense that the corresponding wave operators exist. The following lemma contains the key relations implying existence. It makes use of the self-adjoint operator $M(\eta)$ introduced in Appendix B and of the unitary multiplication operator m_α defined by

$$(m_\alpha f)(x) \equiv \exp(-i\pi/4 - 2i\chi + i\alpha x^2)f(x). \tag{4.2}$$

(Recall χ is given by (A.10).)

Lemma 4.1. For all $\phi \in \mathcal{C}$ and $c > 0$ we have

$$\lim_{t\to-\infty} \|(\mathcal{T}_R - \mathcal{F}_\alpha)\exp(-itM(-c))\phi\| = 0, \tag{4.3}$$

$$\lim_{t\to\infty} \|(\mathcal{T}_R - m_\alpha)\exp(-itM(-c))\phi\| = 0, \tag{4.4}$$

$$\lim_{t\to-\infty} \|(\mathcal{T}_L - m_\alpha^*)\exp(-itM(c))\phi\| = 0, \tag{4.5}$$

$$\lim_{t\to\infty} \|(\mathcal{T}_L - \mathcal{F}_\alpha^*)\exp(-itM(c))\phi\| = 0. \tag{4.6}$$

Proof. We only give the proofs of (4.3) and (4.4), as the limits (4.5) and (4.6) can be handled by making suitable sign changes. We assume from now on that ϕ satisfies (3.16).

To prove (4.3) we introduce

$$\mathcal{J}(z,p) \equiv [\mathcal{E}_R(z,p) - (\alpha/2\pi)^{1/2}\exp(i\alpha zp)]\exp(it\omega(p))\phi(p), \tag{4.7}$$

$$\omega(p) \equiv -\exp(-cp), \tag{4.8}$$

and write

$$\|(\mathcal{T}_R - \mathcal{F}_\alpha)e^{-itM(-c)}\phi\|^2 = \int_{-\infty}^{\infty} dx \left| \int_{r_-}^{r_+} dp \mathcal{J}(x+i0,p)\right|^2. \tag{4.9}$$

We now split up the x-integral into integrals over $(-\infty, r_- - 1), [r_- - 1, r_+ + 1]$, and $(r_+ + 1, \infty)$, obtaining three summands I_-, I_0, and I_+, resp. To estimate I_- we first use

$$\begin{aligned} \left|\int_{r_-}^{r_+} dp \mathcal{J}(x,p)\right|^2 \quad &\le \quad \left|\int_{r_-}^{r_+} dp \mathcal{E}_R(x,p)\exp(it\omega(p))\phi(p)\right|^2 \\ &+ \frac{\alpha}{2\pi}\left|\int_{r_-}^{r_+} dp\exp(i\alpha xp + it\omega(p))\phi(p)\right|^2. \end{aligned} \tag{4.10}$$

In the contribution

$$\int_{-\infty}^{r_- -1} dx \int_{r_-}^{r_+} dp_1 \int_{r_-}^{r_+} dp_2 \overline{\phi}(p_1)\phi(p_2)\overline{\mathcal{E}_R}(x,p_1)\mathcal{E}_R(x,p_2)\exp(it[\omega(p_1) - \omega(p_2)]) \tag{4.11}$$

of the first term to I_- we now change variables

$$p_j = -c^{-1}\ln v_j \Rightarrow \omega(p_j) = -v_j, \quad j = 1,2. \tag{4.12}$$

In view of the uniform exponential decay of $\mathcal{E}_R(x,p)$ as $x \to -\infty$ (cf. (3.11)), the integrand belongs to $L^1(\mathbb{R}^3, dxdv_1dv_2)$. Hence the Riemann-Lebesgue lemma applies, yielding limit 0 for $t \to -\infty$. In the second term on the rhs of (4.10) we use the stationary phase formula

$$\exp(i\alpha xp + it\omega(p)) = \frac{\partial_p \exp(i\alpha xp + it\omega(p))}{i(\alpha x - ct\omega(p))}, \tag{4.13}$$

to deduce that its contribution to I_- also yields limit 0 for $t \to -\infty$. Hence I_- vanishes for $t \to -\infty$.

Next we invoke (3.10), the variable change (4.12) and the Riemann-Lebesgue lemma to infer that I_+ vanishes for $t \to -\infty$. Turning to I_0, we employ again the estimate (4.10) (with $x \to x + i0$). Then we invoke once more the stationary phase formula (4.13) to conclude that the contribution to I_0 from the second term on the rhs of (4.10) vanishes for $t \to -\infty$. Thus we are left with

$$\int_{r_- -1}^{r_+ +1} dx \left| \int_{r_-}^{r_+} dp \mathcal{E}_R(x + i0, p) \exp(it\omega(p))\phi(p) \right|^2 . \tag{4.14}$$

To control this term we use the estimate (recall (3.23))

$$\left| \int_{r_-}^{r_+} dp \mathcal{E}_R(x + i0, p) e^{it\omega(p)} \phi(p) \right|^2 \leq \left| \int_{r_-}^{r_+} dp \mathcal{D}_R(x, p) e^{it\omega(p)} \phi(p) \right|^2 + \left| \int_{r_-}^{r_+} dp \frac{\rho_R(p)}{x + i0 - p} e^{it\omega(p)} \phi(p) \right|^2 . \tag{4.15}$$

In the contribution to I_0 from the first term we change variables using (4.12), and invoke the Riemann-Lebesgue lemma to get limit 0 for $t \to -\infty$. To treat the contribution from the second term we choose r such that

$$[r_- - 1, r_+ + 1] \subset [-r, r], \tag{4.16}$$

and set $\psi(x) = \rho_R(x)\phi(x)$. Then the assumptions of Lemma C.1 are satisfied, so we may use (C.34) with s replaced by $-t$ to deduce that the second term also yields a vanishing contribution for $t \to -\infty$. As a result we have now proved (4.3).

The proof of (4.4) proceeds along similar lines. Thus we first write

$$\|(\mathcal{T}_R - m_\alpha) e^{-itM(-c)} \phi\|^2 = K_- + K_+ + K_0, \tag{4.17}$$

with K_-, K_+ and K_0 defined by

$$\int_{-\infty}^{r_- -1} dx \left| \int_{r_-}^{r_+} dp \mathcal{E}_R(x, p) \exp(it\omega(p))\phi(p) \right|^2 , \tag{4.18}$$

$$\int_{r_+ +1}^{\infty} dx \left| \int_{r_-}^{r_+} dp \mathcal{E}_R(x, p) \exp(it\omega(p))\phi(p) \right|^2 , \tag{4.19}$$

$$\int_{r_- -1}^{r_+ +1} dx \left| \int_{r_-}^{r_+} dp \mathcal{E}_R(x + i0, p) \exp(it\omega(p))\phi(p) - m_\alpha(x) \exp(it\omega(x))\phi(x) \right|^2 . \tag{4.20}$$

respectively. Since K_- is of the form (4.11), it can be treated as before, yielding limit 0 for $t \to \infty$. For K_+ we use

$$\left| \int_{r_-}^{r_+} dp \mathcal{E}_R(x, p) e^{it\omega(p)} \phi(p) \right|^2 \leq \left| \int_{r_-}^{r_+} dp \left(\mathcal{E}_R(x, p) - (\alpha/2\pi)^{1/2} e^{i\alpha xp} \right) e^{it\omega(p)} \phi(p) \right|^2 + \frac{\alpha}{2\pi} \left| \int_{r_-}^{r_+} dp e^{i\alpha xp} e^{it\omega(p)} \phi(p) \right|^2 . \tag{4.21}$$

Invoking (3.10) we conclude as before from the Riemann-Lebesgue lemma that the first term yields vanishing contribution for $t \to \infty$. To verify that the same is true for the second term, the stationary phase formula (4.13) can again be used.

It remains to consider K_0. Just as in (4.15) we use (3.23), obtaining an integral involving $\mathcal{D}_R(x,p)$, whose contribution vanishes for $t \to \infty$. The second contribution reads

$$\int_{r_- -1}^{r_+ +1} dx \left| \int_{r_-}^{r_+} dp \frac{\rho_R(p)}{x + i0 - p} e^{it\omega(p)} \phi(p) - m_\alpha(x) e^{it\omega(x)} \phi(x) \right|^2 . \tag{4.22}$$

Now we see from (4.2), (A.10) and (3.22) that we have

$$m_\alpha(x) = -2i\pi\rho_R(x). \tag{4.23}$$

From (C.35) we therefore infer that the $t \to \infty$ limit of (4.22) vanishes, concluding the proof of (4.4). □

Our next aim is to obtain wave operators for the interacting evolutions $\exp(-itH_{\sigma s}), \sigma = R, L$, by using Lemma 4.1. To this end we choose $c = \alpha a_s$ in (4.3)–(4.6). Now we recall (B.5)–(B.7) and note the relations

$$\mathcal{F}_\alpha = S_\alpha \mathcal{F} = \mathcal{F} S_\alpha^*, \tag{4.24}$$

cf. (3.12) and (B.10). Using also (B.13) we deduce

$$\exp(-itE(a_s)) = \mathcal{F}_\alpha^* \exp(-itM(\alpha a_s)) \mathcal{F}_\alpha, \tag{4.25}$$

and the well-known relations

$$\mathcal{F}_\alpha^2 = \mathcal{F}_\alpha^{*2} = P \tag{4.26}$$

(cf. (B.9)) yield

$$\exp(-itE(a_s)) = \mathcal{F}_\alpha \exp(-itM(-\alpha a_s)) \mathcal{F}_\alpha^*. \tag{4.27}$$

The three unitary one-parameter groups occurring in (4.25) and (4.27) now serve as the free comparison dynamics for $\exp(-itH_{Rs})$ and $\exp(-itH_{Ls})$, in the precise sense specified next.

Corollary 4.2. We have

$$s \cdot \lim_{t \to -\infty} \exp(itH_{Rs}) \exp(-itE(a_s)) = \mathcal{T}_R \mathcal{F}_\alpha^*, \tag{4.28}$$

$$s \cdot \lim_{t \to \infty} \exp(itH_{Rs}) \exp(-itM(-\alpha a_s)) = \mathcal{T}_R m_\alpha^*, \tag{4.29}$$

$$s \cdot \lim_{t \to -\infty} \exp(itH_{Ls}) \exp(-itM(\alpha a_s)) = \mathcal{T}_L m_\alpha, \tag{4.30}$$

$$s \cdot \lim_{t \to \infty} \exp(itH_{Ls}) \exp(-itE(a_s)) = \mathcal{T}_L \mathcal{F}_\alpha, \tag{4.31}$$

and the operators $\mathcal{T}_R$ and $\mathcal{T}_L$ are isometric.

Proof. From (4.1) and (4.27) we have

$$\|(\mathcal{T}_R - e^{itH_{Rs}}e^{-itE(a_s)}\mathcal{F}_\alpha)\phi\| = \|(\mathcal{T}_R - \mathcal{F}_\alpha)e^{-itM(-\alpha a_s)}\phi\|, \quad \forall \phi \in \mathcal{C}. \tag{4.32}$$

By (4.3) this vanishes for $t \to -\infty$. This implies

$$\|\mathcal{T}_R\phi\| = \lim_{t\to-\infty} \|e^{itH_{Rs}}e^{-itE(a_s)}\mathcal{F}_\alpha\phi\| = \|\phi\|, \quad \forall \phi \in \mathcal{C}. \tag{4.33}$$

Hence $\mathcal{T}_R$ is isometric, and (4.28) also follows. Likewise, from (4.1) and (4.25) we get

$$\|(\mathcal{T}_L - e^{itH_{Ls}}e^{-itE(a_s)}\mathcal{F}_\alpha^*)\phi\| = \|(\mathcal{T}_L - \mathcal{F}_\alpha^*)e^{-itM(\alpha a_s)}\phi\|, \tag{4.34}$$

so by (4.6) we get limit 0 for $t \to \infty$. Thus isometry of $\mathcal{T}_L$ and (4.31) follow.

To prove (4.29) we use (4.1) once more, obtaining

$$\|(\mathcal{T}_R m_\alpha^* - e^{itH_{Rs}}e^{-itM(-\alpha_s)})m_\alpha\phi\| = \|(\mathcal{T}_R - m_\alpha)e^{-itM(-\alpha a_s)}\phi\|. \tag{4.35}$$

This has limit 0 for $t \to \infty$ in view of (4.4), yielding (4.29). Similarly, (4.30) follows from

$$\|(\mathcal{T}_L m_\alpha - e^{itH_{Ls}}e^{-itM(\alpha a_s)})m_\alpha^*\phi\| = \|(\mathcal{T}_L - m_\alpha^*)e^{-itM(\alpha a_s)}\phi\|, \tag{4.36}$$

by using (4.5). □

Now that the isometry of the joint eigenfunction transforms $\mathcal{T}_R$ and $\mathcal{T}_L$ has been established, we reconsider their kernels. Combining the definitions (3.8)–(3.9) and (A.12)–(A.13) with the relation

$$\overline{G(a_+, a_-; z)} = G(a_+, a_-; -\overline{z}), \qquad a_+, a_- > 0, \quad z \in \mathbb{C} \tag{4.37}$$

(which follows from (A.2)), we deduce not only the relation

$$\mathcal{E}_R(x, x') = \overline{\mathcal{E}_L(x', x)}, \qquad x, x' \in \mathbb{R}, \quad x \neq x', \tag{4.38}$$

but also that for all $\phi, \psi \in \mathcal{C}$ we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dx\overline{\phi}(x)\left(\int_{-\infty}^{\infty} dx'\mathcal{E}_R(x + i0, x')\psi(x')\right) \\ &= \int_{-\infty}^{\infty} dx' \left(\int_{-\infty}^{\infty} dx\mathcal{E}_L(x' + i0, x)\phi(x)\right)^{-} \psi(x'). \end{aligned} \tag{4.39}$$

This can be rewritten as

$$(\phi, \mathcal{T}_R\psi) = (\mathcal{T}_L\phi, \psi), \qquad \forall \phi, \psi \in \mathcal{C}. \tag{4.40}$$

We are now prepared for the following theorem, which we view as the principal result of this paper.

Theorem 4.3. The transforms $\mathcal{T}_R$ and $\mathcal{T}_L$ (defined by (3.36)) are unitary operators related by

$$\mathcal{T}_L = \mathcal{T}_R^*. \tag{4.41}$$

They are strongly continuous in a_+ and a_+ for $(a_+, a_-) \in (0, \infty)^2$. The Hamiltonians $H_{\sigma\pm}$ are densely defined and essentially self-adjoint on $\mathcal{P}_\sigma, \sigma = R, L$. Their closures are commuting self-adjoint operators, related by

$$(H_{\sigma+})^{a_+/a_-} = H_{\sigma-}, \qquad \sigma = R, L. \tag{4.42}$$

With $H_{\sigma s}$ replaced by $H_{\sigma\delta}$ and a_s by $a_{-\delta}$, the wave operator relations (4.28)–(4.31) hold true for $\delta = +$ and for $\delta = -$.

Proof. Since $\mathcal{C}$ is dense and the transforms are isometries by Corollary 4.2, their unitarity and inverse relation (4.41) easily follow from (4.40). Recalling that $\mathcal{T}_\sigma$ is strongly continuous in (a_+, a_-) for all $\phi \in \mathcal{C}$ (by the last assertion of Lemma 3.1), it follows that $\mathcal{T}_\sigma(a_+, a_-)$ is a strongly continuous family of unitaries.

Turning to the Hamiltonians $H_{\sigma\delta}$, we first note unitarity implies that $\mathcal{P}_\sigma$ is dense in $\mathcal{H}$, and that we have

$$H_{\sigma\delta} f = \mathcal{T}_\sigma M_{\sigma\delta} \mathcal{T}_\sigma^* f, \quad \forall f \in \mathcal{P}_\sigma, \quad \delta = +, -, \tag{4.43}$$

cf. (3.38)–(3.39). Also, since $\mathcal{C}$ is a core for $M_{\sigma\pm}$, it follows that $\mathcal{P}_\sigma$ is a core for the Hamiltonians $H_{\sigma\pm}$ and that their closures (denoted again $H_{\sigma+}$ and $H_{\sigma-}$) commute in the usual sense of unbounded self-adjoint operators. Moreover, $H_{\sigma+}$ and $H_{\sigma-}$ are related by (4.42), since $M_{\sigma+}$ and $M_{\sigma-}$ are manifestly related in the same fashion, cf. (3.39). Finally, using Lemma 4.1 with c equal to αa_l, we readily obtain the limit relations of Corollary 4.2 for the Hamiltonians with the largest step size a_l too. □

5 Further developments

Even though Theorem 4.3 yields a rather complete picture of the Hilbert space status of the transforms and Hamiltonians, it leaves some natural questions open. Furthermore, a few conclusions of interest can be drawn from it that are not immediate. This final section is devoted to these questions and conclusions. We should mention at the outset that in the process we arrive at some further questions we are not able to answer.

First, now that we have Theorem 4.3 available, we should reconsider domain issues. Throughout this section $H_{\sigma\delta}$ denotes the self-adjoint closure of $H_{\sigma\delta}$ as defined initially on the dense subspace $\mathcal{P}_\sigma$, and $\mathcal{D}(H_{\sigma\delta})$ denotes its definition domain. Recall also that the sum operator $S(\nu, \eta)$ is defined and studied in Appendix B.

Proposition 5.1. We have

$$\mathcal{D}(S(a_\delta, \mp\alpha a_\delta)) \subset \mathcal{D}(H_{\sigma\delta}), \quad \delta = +, -, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right. , \tag{5.1}$$

$$H_{\sigma\delta} \restriction \mathcal{D}(S(a_\delta, \mp\alpha a_\delta)) = S(a_\delta, \mp\alpha a_\delta), \quad \delta = +, -, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right. . \tag{5.2}$$

Next, assume $a_s < a_l$. Then $\mathcal{Q}$ (B.14) is a core for $H_{\sigma s}$, and we have

$$\mathcal{D}(H_{\sigma s}) = \mathcal{D}(S(a_s, \mp\alpha a_s)), \quad H_{\sigma s} = S(a_s, \mp\alpha a_s), \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right. . \tag{5.3}$$

Proof. We only prove the assertions for $\sigma = L$, the proof for $\sigma = R$ being essentially the same. Let $f \in \mathcal{Q}$ and $g \in \mathcal{P}_L$. Then we have by (3.40)

$$(f, H_{L\delta} g) = \int_{-\infty}^{\infty} dx \overline{f}(x)(g(x + ia_{-\delta}) + \exp(\alpha a_{-\delta} x) g(x)). \tag{5.4}$$

In view of the properties of $g(z)$ for $\Im z \in [0, a_{-\delta}]$, we can now shift contours in the first term, which yields

$$\int_{-\infty}^{\infty} dx\,(f^*(x - ia_{-\delta}) + \exp(\alpha a_{-\delta} x) f^*(x))\, g(x), \quad f^*(z) \equiv \overline{f(\overline{z})}. \tag{5.5}$$

Hence we obtain

$$(f, H_{L\delta} g) = (S(a_{-\delta}, \alpha a_{-\delta}) f, g), \quad f \in \mathcal{Q}, \quad \forall g \in \mathcal{P}_L. \tag{5.6}$$

Since $\mathcal{P}_L$ is a core for $H_{L\delta}$, we deduce that $\mathcal{Q}$ belongs to $\mathcal{D}(H_{L\delta})$ and that the action of $H_{L\delta}$ on $\mathcal{Q}$ coincides with the action of $S(a_{-\delta}, \alpha a_{-\delta})$. In view of Lemma B.3 this implies (5.1) and (5.2).

Now assume $a_s < a_l$. We have already seen that H_{Ls} coincides with $S(a_s, \alpha a_s)$ on $\mathcal{P}_L$, cf. (3.44). From Theorem 4.3 we know that $\mathcal{P}_L$ is a core for H_{Ls}, so that the relations (5.3) follow from Lemma B.4. The core property of $\mathcal{Q}$ for H_{Ls} is then clear from Lemma B.3. □

It should be recalled at this point that the functions in $\mathcal{P}_\sigma$ do not extend to meromorphic functions, cf. (3.30). But of course $\mathcal{Q}$ consists of meromorphic (even entire) functions. In view of (5.2) the action of the four Hamiltonians $H_{\sigma\delta}$ on $\mathcal{Q}$ coincides with the (restriction of the) action of the AΔOs $A_{\sigma\delta}$. This implies in particular the AΔO identities

$$a_+/a_- \in \mathbb{N}^* \Rightarrow (A_{\sigma+})^{a_+/a_-} = A_{\sigma-}, \quad \sigma = R, L. \tag{5.7}$$

(Indeed, the difference of the AΔOs occurring in this formula vanishes on $\mathcal{Q}$ due to (4.42), and it is easy to see that an AΔO annihilating $\mathcal{Q}$ vanishes identically.) Admittedly, this way to prove the identities (5.7) is not exactly the simplest one.

An obvious question that arises next is whether $\mathcal{Q}$ is also a core for the Hamiltonians H_{Rl} and H_{Ll}. In view of (5.1)–(5.2) this is equivalent to $\mathcal{Q}$ being a core for $S(a_l, \pm\alpha a_l)$, and due to Lemma B.3 this is also equivalent to essential self-adjointness of $S(a_l, \pm\alpha a_l)$ on its domain (B.53). We leave this question unanswered for the special case $a_s = a_l$, just as we leave the question open whether $S(a_l, \pm\alpha a_l)$ is closed for the special cases

$$\exp(i\pi a_l/a_s) = -1, \tag{5.8}$$

cf. Lemma B.4. But we do answer the core question for $a_s < a_l$.

Proposition 5.2. Assume $a_s < a_l$. Then $\mathcal{Q}$ is not a core for $H_{\sigma l}$. Moreover, we have

$$\mathcal{P}_\sigma \not\subseteq \mathcal{D}(E(a_l)), \quad \mathcal{P}_\sigma \not\subseteq \mathcal{D}(M(\mp\alpha a_l)), \quad \sigma = \begin{cases} R \\ L \end{cases}. \tag{5.9}$$

Proof. We show this for $\sigma = L$, the proof for $\sigma = R$ being similar. We first prove that the second assertion follows from the first one. Indeed, assuming $\mathcal{Q}$ is not a core for H_{Ll}, suppose $\mathcal{P}_L$ is a subspace of $\mathcal{D}(E(a_l))$. By (3.40) $\mathcal{P}_L$ is then also a subspace of $\mathcal{D}(M(\alpha a_l))$, hence of $\mathcal{D}(S(a_l, \alpha a_l))$. In view of (5.2) it now follows that $\mathcal{Q}$ is a core for H_{Ll}, a contradiction. Likewise, if $\mathcal{P}_L$ were a subspace of $\mathcal{D}(M(\alpha a_l))$, then by (3.40) it would be a subspace of $\mathcal{D}(E(a_l))$ as well, yielding again a contradiction.

It remains to prove that $\mathcal{Q}$ is not a core for H_{Ll}. Defining

$$v \equiv \exp(2\pi i\rho/a_l), \quad \rho \in (0, a_s/2), \tag{5.10}$$

we obtain $\Im v > 0$. Hence we need only show that the subspace $(H_{Ll} - \overline{v})\mathcal{Q}$ is not dense in $\mathcal{H}$. To this end we proceed to define a nonzero vector ψ_v that is orthogonal to $(H_{Ll} - \overline{v})\mathcal{Q}$.

Consider the function

$$\psi_v(z) \equiv \mu_\rho(z)\mathcal{E}_L(z, i\rho), \tag{5.11}$$

where

$$\mu_\rho(z) \equiv \exp(2\pi z/a_l) - v \in \mathcal{P}_{ia_l}. \tag{5.12}$$

Recalling (3.7) we obtain

$$(A_{Ll}\psi_v)(z) = v\psi_v(z). \tag{5.13}$$

Since we have $a_s < a_l$ and $\mu_\rho(z)$ takes out the simple pole of $G(z - i\rho - ia)$ at $z = i\rho$, the function $\psi_v(z)$ is analytic for $\Im z \in [0, a_l]$. On account of (3.10)–(3.11) it also satisfies

$$\psi_v(z) = O(\exp(\rho\alpha\Re z)), \qquad \Re z \to -\infty, \tag{5.14}$$

$$\psi_v(z) = O(\exp(\alpha[a_s - a + \Im z]\Re z)), \quad \Re z \to \infty, \tag{5.15}$$

where the bounds are uniform for $(a_+, a_-, \Im z)$ in compacts of $(0, \infty)^2 \times \mathbb{R}$. This implies in particular that $\psi_v(x)$ has exponential decay for $x \to \pm\infty$, so that $\psi_v \in \mathcal{H}$.

Next, choosing $f \in \mathcal{Q}$, consider

$$((H_{Ll} - \overline{v})f, \psi_v) = \int_{-\infty}^{\infty} dx(f^*(x - ia_l) + \exp(\alpha a_l x)f^*(x) - vf^*(x))\psi_v(x). \tag{5.16}$$

Shifting contours in the first term, we see this equals

$$\int_{-\infty}^{\infty} dx\overline{f}(x)(\psi_v(x + ia_l) + \exp(\alpha a_l x)\psi_v(x) - v\psi_v(x)). \tag{5.17}$$

By (5.13) the function in brackets vanishes. Therefore ψ_v is orthogonal to $(H_{Ll} - \overline{v})\mathcal{Q}$, as announced. □

From this proposition we see that the operators $S(a_l, \mp\alpha a_l), a_l > a_s$, admit self-adjoint extensions that differ from the self-adjoint operators $H_{\sigma l}, \sigma = R, L$. It would be of interest to obtain more information on this, but we do not pursue this here. We do add one observation, though. Assume the deficiency index is finite. (This is very likely the case.) Then the self-adjoint extension $H_{\sigma l}$ is the only one one that commutes with $H_{\sigma s}$. Indeed, it is not difficult to deduce this from the spectral characteristics of the Cayley transform of $H_{\sigma s}$.

Now that we have the relations (5.3) available, it is not hard to see that

$$\mathcal{F}_\alpha \exp(itH_{Rs})\mathcal{F}_\alpha^* = \exp(itH_{Ls}), \qquad a_s < a_l. \tag{5.18}$$

Indeed, this is tantamount to

$$\mathcal{F}_\alpha S(a_s, -\alpha a_s)\mathcal{F}_\alpha^* = S(a_s, \alpha a_s), \tag{5.19}$$

a relation that is clear from

$$\mathcal{F}_\alpha E(a_s)\mathcal{F}_\alpha^* = M(\alpha a_s), \qquad \mathcal{F}_\alpha M(-\alpha a_s)\mathcal{F}_\alpha^* = E(a_s), \tag{5.20}$$

cf. also (4.24)–(4.25). We are now prepared to obtain a second relation between $\mathcal{T}_L$ and $\mathcal{T}_R$ (in adition to the adjoint relation (4.41)).

Proposition 5.3. We have

$$\mathcal{T}_L = \mathcal{F}_\alpha \mathcal{T}_R P m_\alpha^*, \tag{5.21}$$

where P is given by (B.9) and m_α is given by (4.2).

Proof. We first assume $a_s < a_l$. Multiplying (4.28) and (4.29) from the left by $\mathcal{F}_\alpha$ and from the right by $\mathcal{F}_\alpha^*$, we can use (5.18), (4.27) and (4.25) to infer

$$s \cdot \lim_{t \to -\infty} \exp(itH_{Ls}) \exp(-itM(\alpha a_s)) = \mathcal{F}_\alpha \mathcal{T}_R P, \tag{5.22}$$

$$s \cdot \lim_{t \to -\infty} \exp(itH_{Ls}) \exp(-itE(a_s)) = \mathcal{F}_\alpha \mathcal{T}_R m_\alpha^* \mathcal{F}_\alpha^*. \tag{5.23}$$

Comparing this to (4.30) and (4.31), we obtain

$$\mathcal{T}_L m_\alpha = \mathcal{F}_\alpha \mathcal{T}_R P, \quad \mathcal{T}_L \mathcal{F}_\alpha = \mathcal{F}_\alpha \mathcal{T}_R m_\alpha^* \mathcal{F}_\alpha^*. \tag{5.24}$$

Each of these relations amounts to (5.21).

By strong continuity in (a_+, a_-) we can let a_s converge to a_l to obtain (5.21) for $a_+ = a_-$. □

The relation (5.21) implies a Fourier transform formula that is of independent interest, cf. Appendix A. To derive it, we set

$$\phi_\alpha(x) \equiv \overline{m_\alpha}(x)\phi(-x), \quad \phi \in \mathcal{C}. \tag{5.25}$$

Then we have $\phi_\alpha \in \mathcal{C}$, and from (3.36) and (5.21) we obtain

$$I_{L,\phi}^+ = \mathcal{F}_\alpha I_{R,\phi_\alpha}^+. \tag{5.26}$$

Recalling the analyticity properties of $I_{L,\phi}(z)$ and Lemma 3.1, this implies in particular

$$I_{L,\phi}(z) = (\mathcal{F}_\alpha I_{R,\phi_\alpha}^+)(z), \qquad \Im z \in [0, a). \tag{5.27}$$

Consider now the integral

$$\left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx' e^{i\alpha z(x'+is)} I_{R,\phi_\alpha}(x' + is), \quad \Im z \in (0, a), \quad s \in (0, a - \Im z). \tag{5.28}$$

Due to the decay properties of $I_{R,\phi_\alpha}(z)$ (cf. Lemma 3.1) it is well defined and independent of s. Taking $s \to 0_+$ we obtain the function $(\mathcal{F}_\alpha I_{R,\phi_\alpha}^+)(z)$. Using (5.27) we then deduce

$$I_{L,\phi}(x+iy) = \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx' e^{i\alpha(x+iy)(x'+is)} \int_{-\infty}^{\infty} dp \mathcal{E}_R(x'+is, -p)\overline{m_\alpha}(p)\phi(p), \tag{5.29}$$

where $y \in (0, a)$ and $s \in (0, a - y)$. The integrand is in $L^1(\mathbb{R}^2, dx'dp)$, so by Fubini's theorem we may write

$$I_{L,\phi}(z) = \int_{-\infty}^{\infty} dp K(z, p)\phi(p), \quad \Im z \in (0, a), \tag{5.30}$$

where

$$K(z,p) \equiv \left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx' e^{i\alpha z(x'+is)} \mathcal{E}_R(x'+is,-p)\overline{m_\alpha}(p),$$
$$\Im z \in (0,a), \quad s \in (0, a-\Im z). \tag{5.31}$$

Now we also have

$$I_{L,\phi}(z) = \int_{-\infty}^{\infty} dp \mathcal{E}_L(z,p)\phi(p), \quad \Im z \in (0,a), \tag{5.32}$$

so a comparison to (5.30) gives

$$\mathcal{E}_L(z,p) = K(z,p), \quad \Im z \in (0,a). \tag{5.33}$$

Finally, substituting (3.8)–(3.9) and changing variables, we obtain

$$S_L(x+iy) = e^{i\pi/4+2i\chi}\left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} du e^{i\alpha(x+iy)(u+iv)} S_R(u+iv),$$
$$y \in (0,a), \quad v \in (0,a-y). \tag{5.34}$$

More information on this formula can be found in Appendix A, cf. (A.21).

Appendix A. The hyperbolic gamma function and its relatives

In [4] we studied the hyperbolic gamma function in the framework of a general theory of minimal solutions to first order AΔEs: It is the minimal solution of the AΔE

$$\frac{G(z+ia_+/2)}{G(z-ia_+/2)} = 2\cosh(\pi z/a_-), \quad a_+, a_- \in (0,\infty), \tag{A.1}$$

rendered unique by requiring $G(0) = 1$. It is explicitly given by

$$G(a_+,a_-;z) = \exp\left(i\int_0^\infty \frac{dy}{y}\left(\frac{\sin 2yz}{2\sinh(a_+y)\sinh(a_-y)} - \frac{z}{a_+a_-y}\right)\right),$$
$$|\Im z| < (a_+ + a_-)/2, \tag{A.2}$$

and extends to a meromorphic function with zeros at

$$z_{kl} \equiv i(a_+ + a_-)/2 + ika_+ + ila_-, \quad k,l \in \mathbb{N}, \tag{A.3}$$

and poles at $z = -z_{kl}$.

The manifest symmetry of the integrand in (A.2) under interchange of a_+ and a_- implies that $G(a_+,a_-;z)$ also satisfies the AΔE

$$\frac{G(z+ia_-/2)}{G(z-ia_-/2)} = 2\cosh(\pi z/a_+). \tag{A.4}$$

For our present purposes we only need a few more features of $G(z)$. (We suppress the parameter dependence when no confusion can arise.) First, it satisfies the reflection equation

$$G(-z) = 1/G(z), \tag{A.5}$$

as is plain from (A.2). Secondly, the pole at $z = -ia$, where

$$a \equiv (a_+ + a_-)/2, \tag{A.6}$$

is simple and has residue

$$\lim_{z\to -ia}(z+ia)G(z) = \frac{i}{2\pi}(a_+a_-)^{1/2}. \tag{A.7}$$

(This is Eq. (3.28) in [4].)

Thirdly, we need the $\Re z \to \pm\infty$ asymptotics in the strong form detailed in Appendix A of [17], but only for $a_+, a_- > 0$. To specify it, we introduce the quantities

$$\alpha \equiv 2\pi/a_+a_-, \tag{A.8}$$

$$a_s \equiv \min(a_+, a_-), \qquad a_l \equiv \max(a_+, a_-), \tag{A.9}$$

$$\chi \equiv \frac{\pi}{24}\left(\frac{a_+}{a_-} + \frac{a_-}{a_+}\right). \tag{A.10}$$

Then we have

$$G(a_+, a_-; z)\exp(\pm i(\chi + \alpha z^2/4)) = 1 + O\left(e^{-\rho|\Re z|}\right), \quad \Re z \to \pm\infty, \quad \rho < \alpha a_s, \tag{A.11}$$

where the implied constant can be chosen uniformly for $(a_+, a_-, \Im z)$ varying over compact subsets of $(0,\infty)^2 \times \mathbb{R}$.

In this paper it is convenient to work with the two functions

$$S_R(a_+, a_-; z) \equiv G(a_+, a_-; z - ia)\exp[i\chi + i\alpha(z-ia)^2/4], \tag{A.12}$$

$$S_L(a_+, a_-; z) \equiv G(a_+, a_-; z - ia)\exp[-i\chi - i\alpha(z-ia)^2/4]. \tag{A.13}$$

On account of (A.11) their asymptotics is given by

$$S_\sigma(z) = 1 + O\left(e^{-\rho|\Re z|}\right), \quad \Re z \to \pm\infty, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right., \tag{A.14}$$

$$\begin{aligned} S_\sigma(z) &= \exp(\pm i[2\chi + \alpha(z-ia)^2/2]) \\ &\quad \times \left(1 + O\left(e^{-\rho|\Re z|}\right)\right), \quad \Re z \to \mp\infty, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right. . \end{aligned} \tag{A.15}$$

Also, they both have simple poles at the origin with residues

$$\lim_{z\to 0} zS_\sigma(z) = i\exp(\pm i[\chi - \alpha a^2/4])(a_+a_-)^{1/2}/2\pi, \quad \sigma = \left\{ \begin{array}{c} R \\ L \end{array} \right., \tag{A.16}$$

cf. (A.7).

Next we detail the relation of the special functions mentioned in the introduction to the hyperbolic gamma function. The double sine function (used for example in [20, 21, 22, 23, 10]) is given by

$$S_2(z \mid a_+, a_-) = G(a_+, a_-; -iz + ia). \tag{A.17}$$

(For $a_+ = a_- = 1$ this function was already introduced and studied in 1886 by Hölder [24].) Woronowicz [6] works with a function $V_\theta(z)$ that can be written

$$V_\theta(z) = G(a_+, a_-; z)\exp(-i\alpha z^2/4 - i\chi), \quad a_+ = 2\pi, \quad a_- = 2\pi/\theta. \tag{A.18}$$

Faddeev and collaborators [8] use a function $e_b(z)$ that satisfies

$$e_b(z) = G(a_+, a_-; -z)\exp(i\alpha z^2/4 + i\chi), \quad a_+ = 1/b, \quad a_- = b, \tag{A.19}$$

and in [1] Volkov uses a function

$$\gamma(z) = G(a_+, a_-; iz - ia)\exp(-i\alpha(z-a)^2/4 + i\chi), \quad a_+ = 1, \quad a_- = \tau. \tag{A.20}$$

Fourier transform formulas involving the above functions are obtained in particular in [6, 8, 19], cf. also [10, 1]. As a corollary of the results in the main text we arrive at the Fourier transform formula

$$\begin{aligned}\left(\frac{\alpha}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\alpha z w} S_R(w)\,d\Re w &= e^{-i\pi/4 - 2i\chi} S_L(z),\\ &\Im z, \Im w > 0, \quad \Im z + \Im w < a,\end{aligned} \tag{A.21}$$

cf. (5.34). (The analyticity and decay properties of the integrand $I(\Re w)$ ensure that the integral is absolutely convergent and independent of $\Im w$. Specifically, letting

$$z = x + iy, \quad w = u + iv, \quad x, y, u, v \in \mathbb{R}, \tag{A.22}$$

as in (5.34), we have

$$I(u) = O(\exp(-\alpha y u)), \quad u \to \infty, \tag{A.23}$$

$$I(u) = O(\exp(-\alpha[y + v - a]u)), \quad u \to -\infty, \tag{A.24}$$

cf. (A.14)–(A.15) with $\sigma = R$.)

This formula is substantially equivalent to the previous ones. Indeed, it can be extended to complex a_+ and a_- by using the results on the hyperbolic gamma function collected in Appendix A of [17], and then it amounts to the formulas in [8, 10]. Staying with the case $a_+, a_- \in (0, \infty)$ to which we restrict attention in this paper, (A.21) has two limits in the sense of tempered distributions that are of particular interest.

First, taking y and v to 0 we obtain

$$\left(\frac{\alpha}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\alpha x u} S_R(u + i0)\,du = e^{-i\pi/4 - 2i\chi} S_L(x + i0). \tag{A.25}$$

A second specialization arises by taking y to 0 and v to a. Using (A.12)–(A.13) this yields

$$\begin{aligned}&\left(\frac{\alpha}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{i\alpha x u} G(u) e^{i\alpha u^2/4}\,du\\ &= e^{-i\pi/4 - 4i\chi}\exp[-i\alpha(x + ia)^2/4]G(x - ia + i0).\end{aligned} \tag{A.26}$$

The specialization of (A.21) obtained by taking y to a and v to 0 amounts to the inverse Fourier transform of (A.26). Restricting his parameter θ in (A.18) to $(2, \infty)$, Woronowicz obtains a formula equivalent to (A.26) in Appendix B of [6].

We conclude this appendix by supplying a short proof of (A.21) that is independent of previous ones. First, we need only prove (A.21) for $a_s < a_l$, since the case $a_s = a_l$ then follows by continuity. Now we set

$$a_l - a_s = \delta > 0, \tag{A.27}$$

and choose

$$v \in (0, \delta/4). \tag{A.28}$$

Thus we may take y in the interval

$$I_\delta \equiv (0, a_s + \delta/4) \tag{A.29}$$

in the integral on the lhs of (A.21). Denoting the latter integral by $F(z)$, we proceed to study it by making use of the A∆Es

$$S_R(w + ia_s) = (1 - \exp(-\alpha a_s w))S_R(w), \tag{A.30}$$

$$S_L(z + ia_s) = (1 - \exp(\alpha a_s z))S_L(z), \tag{A.31}$$

which follow from (A.12)–(A.13) and the G-A∆Es (A.1) and (A.4). Letting $y \in (0, \delta/4)$, we have

$$\begin{aligned} F(z + ia_s) &= \int_{-\infty}^{\infty} e^{i\alpha zw} e^{-\alpha a_s w} S_R(w) du \\ &= \int_{-\infty}^{\infty} e^{i\alpha zw} [S_R(w) - S_R(w + ia_s)] du \\ &= F(z) - e^{\alpha a_s z} F(z), \end{aligned} \tag{A.32}$$

where we used (A.30) in the first step, and shifted contours in the second one. As a consequence $F(z)$ extends to an analytic function for $y > 0$, which satisfies the same A∆E (A.31) as $S_L(z)$. Therefore the quotient

$$Q(z) \equiv F(z)/S_L(z) \tag{A.33}$$

is ia_s-periodic.

Focusing on $Q(z)$ from now on, we first note that $Q(z)$ is entire. (Indeed, I_δ (A.29) contains the period interval

$$\Pi \equiv [\delta/8, a_s + \delta/8], \tag{A.34}$$

and $F(z)$ has no poles in I_δ, while $S_L(z)$ has no zeros in I_δ.) We continue to study the $x \to \pm\infty$ asymptotics of $Q(z)$ for $y \in \Pi$, choosing from now on v equal to

$$v_0 \equiv \delta/8. \tag{A.35}$$

From (A.14)–(A.15) we have

$$1/S_L(z) = 1 + O(e^{\rho x}), \qquad x \to -\infty, \tag{A.36}$$

$$1/S_L(z) = \exp(i[2\chi + \alpha(z - ia)^2/2])(1 + O(e^{-\rho x})) = O(e^{-\alpha(y-a)x}), \qquad x \to \infty, \tag{A.37}$$

uniformly for $y \in \Pi$. Also, using (A.23)–(A.24) we get

$$\begin{aligned} |F(z)| &\leq e^{-\alpha v_0 x}\left(C_+\int_0^\infty e^{-\alpha y u}du + C_-\int_{-\infty}^0 e^{-\alpha(y+v_0-a)u}du\right) \\ &= O(e^{-\alpha v_0 x}), \quad x\to\pm\infty, \end{aligned} \tag{A.38}$$

uniformly for $y \in \Pi$. Combining this with (A.35)–(A.37) we deduce

$$Q(z) = O(e^{-\alpha\delta x/8}), \quad x \to -\infty, \tag{A.39}$$

$$Q(z) = o(e^{\alpha a x}), \quad x \to \infty, \tag{A.40}$$

uniformly for $y \in \Pi$.

We are now prepared to infer that $Q(z)$ must be constant. Indeed, Q is ia_s-periodic and entire, so it can be written as a Fourier series

$$Q(z) = \sum_{n\in\mathbb{Z}} c_n \exp(n\alpha a_l z). \tag{A.41}$$

Since $\delta/8 < a_l$, we obtain from (A.39)

$$\lim_{x\to-\infty} \exp(\alpha a_l z)Q(z) = 0, \tag{A.42}$$

uniformly in y. Thus $c_n = 0, \forall n < 0$. Likewise, since $a < a_l$, we get from (A.40)

$$\lim_{x\to\infty} \exp(-\alpha a_l z)Q(z) = 0, \tag{A.43}$$

uniformly in y. Hence $c_n = 0, \forall n > 0$, so that $Q(z)$ is constant, as announced.

We mention in passing that uniformity in y is critical for this conclusion. Indeed, there exist entire functions $E(z)$ satisfying

$$\lim_{r\to0} E(re^{i\phi}) = \lim_{r\to\infty} E(re^{i\phi}) = 0, \quad \forall\phi\in(0,2\pi], \tag{A.44}$$

a fact that is not widely known.

It remains to show

$$Q(z) = (2\pi/\alpha)^{1/2}e^{-i\pi/4-2i\chi}. \tag{A.45}$$

To this end we write

$$\begin{aligned} F(z) &= \int_{-\infty}^{\infty} e^{i\alpha z w}\left(S_R(w) - \frac{1}{1+\exp(-\alpha a_s w)}\right)du + B(z), \\ &\quad \Im z \in (0, a_s), \quad \Im w \in (0, \delta/4), \end{aligned} \tag{A.46}$$

so that $B(z)$ equals the elementary integral

$$\int_{-\infty}^{\infty} e^{i\alpha z u}\frac{\exp(\alpha a_s u/2)}{2\cosh(\alpha a_s u/2)}du = \frac{i\pi}{\alpha a_s \sinh(\pi z/a_s)}, \quad \Im z\in(0,a_s). \tag{A.47}$$

Also, for $|\Im z| < a_s$ the integrand in (A.46) has exponential decay as $|u| \to \infty$, which entails that the integral yields a function that is analytic in $|\Im z| < a_s$. As a consequence,

$F(z)$ extends to a function that is analytic for $|\Im z| < a_s$, but for a simple pole at $z = 0$ with residue i/α. Moreover, $S_L(z)$ has residue

$$r_L = \frac{i}{\alpha}(\alpha/2\pi)^{1/2} e^{i\pi/4+2i\chi} \tag{A.48}$$

at $z = 0$, cf. (A.16) and (A.10). Thus we have

$$Q = \lim_{z\to 0} \frac{F(z)}{S_L(z)} = \frac{i}{\alpha} \cdot \frac{1}{r_L} = (2\pi/\alpha)^{1/2} e^{-i\pi/4-2i\chi}, \tag{A.49}$$

and so the proof of (A.21) is complete.

Appendix B. The operators $M(\eta)$, $E(\nu)$, and their sum

This appendix is mostly concerned with various dense subspaces of the Hilbert space

$$\mathcal{H} \equiv L^2(\mathbb{R}, dx), \tag{B.1}$$

in relation to the multiplication operators

$$M(\eta) = \exp(\eta x), \qquad \eta \in \mathbb{R}^*, \tag{B.2}$$

and exponentiated momentum operators

$$E(\nu) = \exp(i\nu d/dx), \qquad \nu \in \mathbb{R}^*. \tag{B.3}$$

Clearly $M(\eta)$ is self-adjoint on its natural definition domain

$$\mathcal{D}(M(\eta)) \equiv \{f \in \mathcal{H} \mid \exp(\eta x) f(x) \in \mathcal{H}\}, \tag{B.4}$$

whereas we use Fourier transformation to define $E(\nu)$ as a self-adjoint operator. Specifically, in this appendix we work with the unitary operator

$$\mathcal{F} : \mathcal{H} \to \mathcal{H}, \qquad f(x) \mapsto (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx' \exp(ixx') f(x'), \tag{B.5}$$

and define

$$E(\nu) \equiv \mathcal{F}^* M(\nu) \mathcal{F}. \tag{B.6}$$

Thus the definition domain of $E(\nu)$ is given by

$$\mathcal{D}(E(\nu)) \equiv \mathcal{F}^*(\mathcal{D}(M(\nu)). \tag{B.7}$$

Defining time reversal, parity and scaling by

$$(Tf)(x) \equiv \overline{f}(x), \tag{B.8}$$

$$(Pf)(x) \equiv f(-x), \tag{B.9}$$

$$(S_\lambda f)(x) \equiv \lambda^{1/2} f(\lambda x), \qquad \lambda \in (0, \infty), \tag{B.10}$$

we clearly have

$$TE(\nu)T = E(-\nu), \quad TM(\eta)T = M(\eta), \tag{B.11}$$

$$PE(\nu)P = E(-\nu), \quad PM(\eta)P = M(-\eta), \tag{B.12}$$

$$S_\lambda^* E(\nu) S_\lambda = E(\lambda\nu), \quad S_\lambda^* M(\eta) S_\lambda = M(\lambda^{-1}\eta). \tag{B.13}$$

Consider now the function space

$$\mathcal{Q} \equiv \mathrm{Span}\{\exp(-\kappa x^2 + \xi x) \mid \Re\kappa > 0, \xi \in \mathbb{C}\}. \tag{B.14}$$

It is easily verified that $\mathcal{Q}$ is a dense subspace of $\mathcal{H}$ satisfying

$$\mathcal{F}\mathcal{Q} = \mathcal{Q}. \tag{B.15}$$

Moreover, any $f(x) \in \mathcal{Q}$ extends to an entire function $f(z) = f(x + iy), x, y \in \mathbb{R}$, and one has

$$\mathcal{Q} \subset \mathcal{D}(M(\eta)), \quad \mathcal{Q} \subset \mathcal{D}(E(\nu)), \tag{B.16}$$

$$(E(\nu)f)(x) = f(x + i\nu), \quad f \in \mathcal{Q}. \tag{B.17}$$

It is also not hard to see that $\mathcal{Q}$ is a core (domain of essential self-adjointness [12]) for $M(\eta)$ and $E(\nu)$. (In view of (B.6) and (B.15), this need only be shown for $M(\eta)$. To this end assume f is orthogonal to $(M(\eta) + i)\mathcal{Q}$. Then the Fourier transform of the function $\overline{f}(x)(e^{\eta x} + i)e^{-\kappa x^2}$ vanishes. Thus $f = 0$, implying $(M(\eta) + i)\mathcal{Q}$ is dense. Likewise, $(M(\eta) - i)\mathcal{Q}$ is dense, so that $\mathcal{Q}$ is a core.)

We mention in passing that the operators $E(\nu)$ and $M(2\pi/\nu)$ and their common core $\mathcal{Q}$ furnish a quite simple example of the Nelson phenomenon [12]. Indeed, both operators leave $\mathcal{Q}$ invariant, and in view of (B.17) they commute on $\mathcal{Q}$. Even so, the operators do not commute in the usual sense of unbounded self-adjoint operators. (If they did, the translation group $\exp(td/dx)$, $t \in \mathbb{R}$, and the bounded operator $(1 + \exp(2\pi x/\nu))^{-1}$ would commute, which is plainly false.)

For a given f it is easy to recognize whether it belongs to $\mathcal{D}(M(\eta))$, cf. (B.4). The definition (B.7) of $\mathcal{D}(E(\nu))$ is far less explicit, however. We are going to characterize the domain and action of $E(\nu)$ in a more illuminating way. In particular, we shall obtain necessary and sufficient conditions for a function $f(x)$ to belong to $\mathcal{D}(E(\nu))$ that are weak enough to be of practical use, in the sense that they can be readily verified in concrete cases. For this purpose it is expedient to introduce the auxiliary multiplication operators

$$M_a(\eta) \equiv \cosh(\eta x), \quad \mathcal{D}(M_a(\eta)) \equiv \{f \in \mathcal{H} \mid \cosh(\eta x) f(x) \in \mathcal{H}\}, \quad \eta > 0, \tag{B.18}$$

and their Fourier transforms

$$E_a(\nu) \equiv \mathcal{F}^* M_a(\nu) \mathcal{F} = \cosh(i\nu d/dx), \quad \nu > 0. \tag{B.19}$$

By contrast to $M(\eta)$ and $E(\nu)$, the latter operators have bounded inverses. Clearly $\mathcal{Q}$ is also a core for $M_a(\eta)$ and $E_a(\nu)$. First, we render the domain and action of $E_a(\nu)$ more explicit.

Lemma B.1. Assume $f(x) \in \mathcal{H}$ extends to an analytic function $f(z) = f(x+iy)$ in the strip $|y| < \nu$ with the following properties:
(i) for all $\epsilon > 0$ one has

$$\exp(-\epsilon z^2)f(z) = O(1), \qquad |y| < \nu, \qquad |x| \to \infty, \tag{B.20}$$

with the bound uniform for $|y| \leq \nu - \delta, \forall \delta \in (0,\nu)$;
(ii) there exist functions $f_{\pm\nu}(x) \in \mathcal{H}$ such that for all $\epsilon > 0$ one has

$$s \cdot \lim_{y\uparrow\nu} \exp(-\epsilon(x \pm iy)^2) f(x \pm iy) = \exp(-\epsilon(x \pm i\nu)^2) f_{\pm\nu}(x). \tag{B.21}$$

Then $f(x)$ belongs to $\mathcal{D}(E_a(\nu))$, and the action of $E_a(\nu)$ is given by

$$E_a(\nu)f = (f_\nu + f_{-\nu})/2. \tag{B.22}$$

Conversely, let $f \in \mathcal{D}(E_a(\nu))$. Then $f(x)$ extends to an analytic function $f(z)$ in the strip $|y| < \nu$ with the above properties. Furthermore, one has

$$|f(z)| \leq C_\delta, \qquad |y| \leq \nu - \delta, \qquad \forall \delta \in (0,\nu), \tag{B.23}$$

$$\lim_{x\to\pm\infty} f(x+iy) = 0, \qquad \forall y \in (-\nu,\nu), \tag{B.24}$$

$$f(x+iy) \in \mathcal{H}, \quad \forall y \in (-\nu,\nu), \tag{B.25}$$

$$f_{\pm\nu}(x) = s \cdot \lim_{y\uparrow\nu} f(x \pm iy), \tag{B.26}$$

and the map

$$[-\nu,\nu] \to \mathcal{H}, \qquad y \mapsto f(\cdot + iy), \tag{B.27}$$

is strongly continuous.

Proof. To prove the first assertion let $\phi \in \mathcal{Q}$. Using (B.17) we deduce

$$2(E_a(\nu)\phi, f) = \int_{-\infty}^{\infty} dx[\phi^*(x-i\nu) + \phi^*(x+i\nu)]f(x), \tag{B.28}$$

where

$$\phi^*(z) \equiv \overline{\phi(\overline{z})}. \tag{B.29}$$

Since $\phi \in \mathcal{Q}$, there exists $\epsilon > 0$ such that the function $\exp(\epsilon x^2)\phi^*(x+i\alpha)$ belongs to $\mathcal{H}$ for all $\alpha \in \mathbb{R}$ (recall (B.14)). Thus we can use the bound (B.20) and Cauchy's theorem to shift contours, obtaining

$$\begin{aligned} 2(E_a(\nu)\phi, f) \quad = \quad & \int_{-\infty}^{\infty} dx e^{\epsilon(x+iy)^2} \phi^*(x+iy-i\nu) \left(e^{-\epsilon(x+iy)^2} f(x+iy)\right) \\ & + \int_{-\infty}^{\infty} dx e^{\epsilon(x-iy)^2} \phi^*(x-iy+i\nu) \left(e^{-\epsilon(x-iy)^2} f(x-iy)\right), \end{aligned} \tag{B.30}$$

where $y \in (0,\nu)$. Now we may and will view the integrals as inner products of y-dependent vectors in $\mathcal{H}$. (Indeed, from (B.20) with $\epsilon \to \epsilon/2$ it is clear that the vectors $\exp[-\epsilon(\cdot \pm$

$iy)^2]f(\cdot \pm iy)$ depend continuously on $y \in (0, \nu)$ in the strong $\mathcal{H}$-topology.) Taking $y \uparrow \nu$ and invoking (B.21), we deduce

$$2(E_a(\nu)\phi, f) = (\phi, f_\nu + f_{-\nu}), \quad \forall \phi \in \mathcal{Q}. \tag{B.31}$$

Because $\mathcal{Q}$ is a core for the self-adjoint operator $E_a(\nu)$, this implies not only our claim $f \in \mathcal{D}(E_a(\nu))$, but also the action formula (B.22).

In order to prove the necessary conditions, let $f \in \mathcal{D}(E_a(\nu))$. Then we have (cf. (B.19))

$$\hat{f} \equiv \mathcal{F}f \in \mathcal{D}(M_a(\nu)), \tag{B.32}$$

so that

$$\hat{f}(x) = \frac{1}{\cosh \nu x} g(x), \quad g \equiv M_a(\nu)\mathcal{F}f \in \mathcal{H}. \tag{B.33}$$

Acting with $\mathcal{F}^*$, we may now write

$$f(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dx' \frac{\exp(-ixx')}{\cosh(\nu - \delta)x'} \left(\frac{\cosh(\nu - \delta)x'}{\cosh \nu x'} g(x') \right). \tag{B.34}$$

Fixing $\delta \in (0, \nu)$, the function in parentheses is in $L^1(\mathbb{R})$. Hence $f(x)$ extends to an analytic function $f(z)$ in the strip $|y| < \nu - \delta$. As δ is arbitrary, the features (B.23)–(B.24) readily follow, and we also deduce

$$f(\cdot + iy) = \mathcal{F}^* M(y) M_a(\nu)^{-1} g. \tag{B.35}$$

Since the family of bounded multiplication operators $M(y)/M_a(\nu)$, $y \in [-\nu, \nu]$, is strongly continuous in y, the remaining assertions follow from (B.35). (Note that (B.20) is evident from (B.23), and (B.21) from (B.26).) □

We are now prepared to study $E(\nu)$. For convenience we choose $\nu > 0$. The corresponding results for negative ν can be derived by using P or T.

Lemma B.2. Assume

$$f(z) = f(x + iy), \quad x \in \mathbb{R}, \quad y \in (0, \nu), \tag{B.36}$$

is an analytic function with the following properties:

(i) for all $\epsilon > 0$ one has

$$\exp(-\epsilon z^2) f(z) = O(1), \quad y \in (0, \nu), \quad |x| \to \infty, \tag{B.37}$$

with the bound uniform for $y \in [\delta, \nu - \delta]$, $\forall \delta \in (0, \nu/2)$;

(ii) there exist functions $f(x), f_\nu(x) \in \mathcal{H}$ such that for all $\epsilon > 0$ one has

$$s \cdot \lim_{y \downarrow 0} \exp(-\epsilon(x + iy)^2) f(x + iy) = \exp(-\epsilon x^2) f(x), \tag{B.38}$$

$$s \cdot \lim_{y \uparrow \nu} \exp(-\epsilon(x + iy)^2) f(x + iy) = \exp(-\epsilon(x + i\nu)^2) f_\nu(x). \tag{B.39}$$

Then $f(x)$ belongs to $\mathcal{D}(E(\nu))$, and the action of $E(\nu)$ is given by

$$E(\nu) f = f_\nu. \tag{B.40}$$

Conversely, let $f\in\mathcal{D}(E(\nu))$. Then there exists a function $f(z)$ that is analytic for $y\in(0,\nu)$ and that has the above properties. Moreover, it fulfils

$$|f(z)|\le C_{\delta},\quad y\in(\delta,\nu-\delta),\quad \forall\delta\in(0,\nu/2), \tag{B.41}$$

$$\lim_{x\to\pm\infty} f(x+iy)=0,\quad \forall y\in(0,\nu), \tag{B.42}$$

$$f(\cdot+iy)\in\mathcal{H},\quad \forall y\in(0,\nu), \tag{B.43}$$

$$f=s\cdot\lim_{y\downarrow 0} f(\cdot+iy), \tag{B.44}$$

$$f_{\nu}=s\cdot\lim_{y\uparrow\nu} f(\cdot+iy), \tag{B.45}$$

and $f(\cdot+iy)$ is strongly continuous for $y\in[0,\nu]$.

Proof. To prove the sufficient conditions we modify the argument in the proof of Lemma B.1. Thus we fix $\phi\in\mathcal{Q}$ and choose $\epsilon>0$ such that the function $\exp(\epsilon x^2)\phi^*(x+i\alpha)$ belongs to $\mathcal{H}$ for all real α. By (B.37) the integral

$$\int_{-\infty}^{\infty} dx e^{\epsilon(x+iy)^2}\phi^*(x+iy-i\nu)\left(e^{-\epsilon(x+iy)^2}f(x+iy)\right),\quad y\in(0,\nu), \tag{B.46}$$

is well defined and does not depend on y. Viewing it as the inner product of two y-dependent vectors in $\mathcal{H}$, we now use (B.17) and the boundary values (B.38)–(B.39) to obtain

$$(E(\nu)\phi,f)=(\phi,f_{\nu}),\quad \forall\phi\in\mathcal{Q}. \tag{B.47}$$

Since $\mathcal{Q}$ is a core, this yields $f(x)\in\mathcal{D}(E(\nu))$, and also the action formula (B.40).

Now let $f(x)\in\mathcal{D}(E(\nu))$, so that (cf. (B.7))

$$\hat{f}\equiv\mathcal{F}f\in\mathcal{D}(M(\nu)). \tag{B.48}$$

As we obviously have a definition domain equality

$$\mathcal{D}(M(\nu))=\mathcal{D}(M_a(\nu/2)M(\nu/2)), \tag{B.49}$$

we deduce

$$g\equiv\mathcal{F}^*M(\nu/2)\hat{f}\in\mathcal{D}(E_a(\nu/2)). \tag{B.50}$$

Hence g satisfies (B.23)–(B.27) with $f\to g$ and $\nu\to\nu/2$. Setting

$$f(x+iy)\equiv g(x-i\nu/2+iy),\quad y\in[0,\nu], \tag{B.51}$$

the converse assertions easily follow. □

We proceed to study the sum operator

$$S(\nu,\eta)\equiv E(\nu)+M(\eta),\quad \nu,\eta\in\mathbb{R}^*, \tag{B.52}$$

on its natural initial domain

$$\mathcal{D}(S(\nu,\eta)) \equiv \mathcal{D}(E(\nu)) \cap \mathcal{D}(M(\eta)). \tag{B.53}$$

It is obviouly symmetric. It is not obvious, but true that the closure of the restricted operator

$$S_r(\nu,\eta) \equiv S(\nu,\eta) \upharpoonright \mathcal{Q} \tag{B.54}$$

is an extension of $S(\nu,\eta)$. This is the content of the following lemma, which we invoke in Section 5.

Lemma B.3. The domain of the closure $\overline{S_r}(\nu,\eta)$ of the operator (B.54) contains the domain $\mathcal{D}(S(\nu,\eta))$ (B.53).

Proof. Let $f \in \mathcal{D}(S(\nu,\eta))$. Then $f \in \mathcal{D}(E(\nu))$, and since $\mathcal{Q}$ is a core for $E(\nu)$, there exists a sequence

$$f_n \in \mathcal{Q}, \quad f_n \to f, \quad E(\nu)f_n \to E(\nu)f, \quad n \to \infty. \tag{B.55}$$

Now consider the sequence

$$f_{n,\epsilon}(x) \equiv \exp(-\epsilon x^2) f_n(x) \in \mathcal{Q}, \quad \epsilon > 0. \tag{B.56}$$

We have

$$(E(\nu)f_{n,\epsilon})(x) = \exp(-\epsilon(x+i\nu)^2)(E(\nu)f_n)(x), \tag{B.57}$$

$$(M(\eta)f_{n,\epsilon})(x) = \exp(-\epsilon x^2 + \eta x) f_n(x), \tag{B.58}$$

so that

$$f_{n,\epsilon}(x) \to f_\epsilon(x) \equiv \exp(-\epsilon x^2) f(x), \; E(\nu)f_{n,\epsilon} \to E(\nu)f_\epsilon, \; M(\eta)f_{n,\epsilon} \to M(\eta)f_\epsilon. \tag{B.59}$$

(We used (B.40) to rewrite the second limit.) Hence we obtain

$$S_r(\nu,\eta)f_{n,\epsilon} \to S(\nu,\eta)f_\epsilon. \tag{B.60}$$

Therefore f_ϵ belongs to the domain of $\overline{S_r}(\nu,\eta)$ and we have

$$\overline{S_r}(\nu,\eta)f_\epsilon = S(\nu,\eta)f_\epsilon = \exp(-\epsilon(\cdot + i\nu)^2)E(\nu)f + \exp(-\epsilon(\cdot)^2)M(\eta)f. \tag{B.61}$$

Letting $\epsilon \downarrow 0$, the rhs has the strong limit $S(\nu,\eta)f$, so that the assertion follows. □

The next lemma is concerned with the question whether $S(\nu,\eta)$ is closed. It seems quite likely that this is not the case whenever the phase

$$q \equiv \exp(-i\nu\eta/2) \tag{B.62}$$

equals -1, whereas we shall prove that $S(\nu,\eta)$ is closed for $q \neq -1$. The special character of this phase is due to the operator $1 + qE(\nu)$ not having a bounded inverse for $q = -1$. To appreciate the role of this operator, we need a few preliminaries.

First, we define the unitary multiplication operator

$$(\mathcal{U}(s)f)(x) \equiv \exp(isx^2)f(x), \quad s \in \mathbb{R}. \tag{B.63}$$

Next we note that we have

$$\mathcal{U}(\eta/2\nu)M(\eta)\mathcal{D}(S(\nu,\eta)) \subset \mathcal{D}(E(\nu)). \tag{B.64}$$

Indeed, this inclusion readily follows from Lemma B.2: for $g \in \mathcal{D}(S(\nu,\eta))$ the analytic function

$$f(z) \equiv \exp(i\eta z^2/2\nu)\exp(\eta z)g(z) \tag{B.65}$$

has the properties (i) and (ii), with

$$f_\nu(x) \equiv \overline{q}\exp(i\eta x^2/2\nu)g_\nu(x). \tag{B.66}$$

As a consequence, we obtain an identity

$$S(\nu,\eta)g = \mathcal{U}(-\eta/2\nu)(1+qE(\nu))\mathcal{U}(\eta/2\nu)M(\eta)g, \quad \forall g \in \mathcal{D}(S(\nu,\eta)), \tag{B.67}$$

in which the above operator $1+qE(\nu)$ features. This prepares us for the last lemma of this appendix.

Lemma B.4. Assume $q \neq -1$. Then $S(\nu,\eta)$ is closed.

Proof. Consider a sequence

$$g_n \in \mathcal{D}(S(\nu,\eta)), \quad g_n \to g, \quad S(\nu,\eta)g_n \to h. \tag{B.68}$$

Rewriting $S(\nu,\eta)g_n$ by using (B.67) with $g \to g_n$, we multiply by the bounded operator

$$\mathcal{U}(-\eta/2\nu)(1+qE(\nu))^{-1}\mathcal{U}(\eta/2\nu), \tag{B.69}$$

concluding that $M(\eta)g_n$ has a strong limit. Hence $E(\nu)g_n$ has a strong limit as well. Since $M(\eta)$ and $E(\nu)$ are closed, we deduce $g \in \mathcal{D}(M(\eta))$ and $g \in \mathcal{D}(E(\nu))$. Therefore $S(\nu,\eta)$ is closed. □

The results in the main text imply that there is a remarkable difference between the cases $|\nu\eta| < 2\pi$ and $|\nu\eta| > 2\pi$: In the first case $\mathcal{Q}$ is a core for $S(\nu,\eta)$, whereas $\mathcal{Q}$ is not a core in the second one (cf. Propositions 5.1 and 5.2). In view of Lemmas B.3 and B.4 it follows that $S(\nu,\eta)$ is self-adjoint for $|\nu\eta| < 2\pi$, and not essentially self-adjoint for $|\nu\eta| > 2\pi$. The state of affairs for $|\nu\eta| = 2\pi$ is an interesting open question.

Appendix C. Limits of principal value integrals

This appendix concerns an auxiliary result that is used in the proof of Lemma 4.1, but which is also of some interest in itself. It deals with the asymptotic behavior of a class of principal value integrals (Hilbert transforms) as a parameter goes to infinity. As such it is related to Lemmas 5.1 and 5.2 in our article [25]. But the latter lemmas do not imply Lemma C.1. Conversely, Lemma C.1 does not imply the previous lemmas, but its proof is inspired by the proofs given in *loc. cit.*

Lemma C.1. Let $\psi(x)$ be a $C_0^\infty(\mathbb{R})$-function with

$$\mathrm{supp}(\psi) \subset (-r, r), \quad r > 0. \tag{C.1}$$

Let $\omega(x)$ be a real-valued $C^\infty(\mathbb{R})$-function such that

$$\omega'(x) \geq C_1 > 0, \quad \forall x \in [-r-1, r+1], \tag{C.2}$$

$$|\omega''(x)| \geq C_2 > 0, \quad \forall x \in [-r-1, r+1]. \tag{C.3}$$

Finally, let

$$H(x) \equiv P \int_{-\infty}^{\infty} dv \frac{\psi(v) e^{-is\omega(v)}}{x - v}, \quad s \in \mathbb{R}. \tag{C.4}$$

Then we have

$$\max_{|x| \leq r} |H(x) \mp i\pi\psi(x) e^{-is\omega(x)}| = O(|s|^{-1/3}), \quad s \to \pm\infty. \tag{C.5}$$

Proof. Letting at first $e \in (0,1)$, we write

$$H(x) = H^-(x) - H^+(x) + H^0(x), \tag{C.6}$$

where we have

$$H^{\pm}(x) \equiv \int_{|s|^{e-1}}^{\infty} \frac{du}{u} I(x \pm u) \tag{C.7}$$

$$H^0(x) \equiv \int_0^{|s|^{e-1}} \frac{du}{u} [I(x-u) - I(x+u)], \tag{C.8}$$

with

$$I(v) \equiv \psi(v) \exp(-is\omega(v)). \tag{C.9}$$

Now we consider the three summands in (C.6).

First, integration by parts in

$$H^-(x) = \int_{|s|^{e-1}}^{\infty} \frac{du}{u} \psi(x-u) \frac{\partial_u e^{-is\omega(x-u)}}{is\omega'(x-u)} \tag{C.10}$$

yields a boundary term

$$\frac{i}{s} \cdot \frac{\psi(x - |s|^{e-1})}{|s|^{e-1}} \cdot \frac{\exp(-is\omega(x - |s|^{e-1}))}{\omega'(x - |s|^{e-1})} = O(|s|^{-e}), \quad |s| \to \infty, \tag{C.11}$$

and the sum of two integrals

$$I_1 \equiv \frac{1}{is} \int_{|s|^{e-1}}^{\infty} \frac{du}{u^2} \frac{I(x-u)}{\omega'(x-u)}, \tag{C.12}$$

$$I_2 \equiv -\frac{1}{is} \int_{|s|^{e-1}}^{\infty} \frac{du}{u} \exp(-is\omega(x-u)) \partial_u \frac{\psi(x-u)}{\omega'(x-u)}. \tag{C.13}$$

From (C.1)–(C.3) we deduce

$$|I_1| \leq \frac{C}{|s|}\int_{|s|^{e-1}}^{2r}\frac{du}{u^2} = O(|s|^{-e}), \quad |s|\to\infty, \tag{C.14}$$

$$|I_2| \leq \frac{C}{|s|}\int_{|s|^{e-1}}^{2r}\frac{du}{u} = O(\ln(|s|)/|s|), \quad |s|\to\infty. \tag{C.15}$$

Clearly, the implied constants in (C.11), (C.14) and (C.15) can be chosen uniformly for $|x|\leq r$, so that

$$\max_{|x|\leq r}|H^-(x)| \leq C|s|^{-e}, \quad |s|\to\infty. \tag{C.16}$$

It is plain that $H^+(x)$ can be handled in the same fashion, yielding

$$\max_{|x|\leq r}|H^+(x)| \leq C|s|^{-e}, \quad |s|\to\infty. \tag{C.17}$$

Turning to $H^0(x)$, we set $u=k/|s|$ in (C.8), so that we have

$$H^0(x) = \varepsilon(s)\int_0^{|s|^e}\frac{dk}{k}\left(I(x-\frac{k}{s}) - I(x+\frac{k}{s})\right), \tag{C.18}$$

where $\varepsilon(s)$ denotes the sign of s. We now write

$$\omega(x\pm\frac{k}{s}) = \omega(x) \pm \frac{k}{s}\omega'(x) + \frac{k^2}{s^2}R_{\pm}(x,\frac{k}{s}), \tag{C.19}$$

with $R_{\pm}(x,v)\in C^{\infty}(\mathbb{R}^2)$. Then we set

$$H^0(x) = \varepsilon(s)e^{-is\omega(x)}(J_-(x) - J_+(x) + J_0(x)), \tag{C.20}$$

where we have introduced

$$J_{\pm}(x) \equiv \int_0^{|s|^e}\frac{dk}{k}\psi(x\pm\frac{k}{s})\exp(\mp ik\omega'(x))\left(\exp[-i\frac{k^2}{s}R_{\pm}(x,\frac{k}{s})] - 1\right), \tag{C.21}$$

$$J_0(x) \equiv \int_0^{|s|^e}\frac{dk}{k}\left(\psi(x-\frac{k}{s})e^{ik\omega'(x)} - \psi(x+\frac{k}{s})e^{-ik\omega'(x)}\right). \tag{C.22}$$

To estimate $J_{\pm}$ we note that

$$\left|\frac{k^2}{s}R_{\pm}(x,\frac{k}{s})\right| \leq C|s|^{2e-1}, \quad \forall k\in(0,|s|^e), \tag{C.23}$$

uniformly for $|x|\leq r$. Choosing from now on $e\in(0,1/2)$, we deduce

$$\left|\exp[-i\frac{k^2}{s}R_{\pm}(x,\frac{k}{s})] - 1\right| \leq C\frac{k^2}{|s|}, \quad |s|\geq\Lambda, \quad \forall k\in(0,|s|^e), \tag{C.24}$$

for sufficiently large Λ, with C uniform for $|x|\leq r$. Hence,

$$\max_{|x|\leq r}|J_{\pm}(x)| \leq C\int_0^{|s|^e}\frac{dk}{k}\frac{k^2}{|s|} = C|s|^{2e-1}, \quad |s|\to\infty. \tag{C.25}$$

It remains to handle $J_0(x)$ To this end we use

$$\psi(x \pm \frac{k}{s}) = \psi(x) + \frac{k}{s}\rho_\pm(x, \frac{k}{s}), \tag{C.26}$$

with $\rho_\pm(x,v) \in C^\infty(\mathbb{R}^2)$. Then we may write

$$J_0(x) = K_0(x) + R_0(x), \tag{C.27}$$

where

$$K_0(x) \equiv 2i\psi(x)\int_0^{|s|^e\omega'(x)} \frac{dv}{v}\sin v, \tag{C.28}$$

so that the remainder term reads

$$R_0(x) = \frac{1}{s}\int_0^{|s|^e} dk\left(\rho_-(x, \frac{k}{s})e^{ik\omega'(x)} - \rho_+(x, \frac{k}{s})e^{-ik\omega'(x)}\right). \tag{C.29}$$

Obviously, the latter satisfies

$$\max_{|x|\le r} |R_0(x)| \le C|s|^{e-1}. \tag{C.30}$$

With the final choice $e = 1/3$ we now combine the bounds (C.16), (C.17), (C.25) and (C.30) to deduce

$$|H(x) - i\pi\varepsilon(s)e^{-is\omega(x)}\psi(x)| \le |K_0(x) - i\pi\psi(x)| + O(|s|^{-1/3}), \quad |s| \to \infty, \tag{C.31}$$

with the implied constant uniform for $|x| \le r$. Since we also have (cf. (C.28))

$$\begin{aligned} K_0(x) - i\pi\psi(x) &= -2i\psi(x)\int_{|s|^{1/3}\omega'(x)}^\infty dv\frac{\sin v}{v} \\ &= -2i\psi(x)\left(\frac{\cos(|s|^{1/3}\omega'(x))}{|s|^{1/3}\omega'(x)} - \int_{|s|^{1/3}\omega'(x)}^\infty fracdvv^2\cos v\right) \\ &= O(|s|^{-1/3}, \qquad |s| \to \infty, \end{aligned} \tag{C.32}$$

uniformly for $|x| \le r$, the lemma now follows. □

In Section 4 we use this lemma in the following more convenient guise.

Corollary C.2. With the assumptions of Lemma C.1, let

$$L_+(x) \equiv \lim_{\epsilon\downarrow 0}\int_{-\infty}^\infty dv\frac{\psi(v)e^{-is\omega(v)}}{x + i\epsilon - v}. \tag{C.33}$$

Then we have

$$\max_{|x|\le r} |L_+(x)| = O(s^{-1/3}), \qquad s \to \infty, \tag{C.34}$$

$$\max_{|x|\le r} |L_+(x) + 2i\pi\psi(x)e^{-is\omega(x)}| = O(|s|^{-1/3}), \qquad s \to -\infty. \tag{C.35}$$

Proof. Using the well-known relation

$$\frac{1}{u+i0} = P\frac{1}{u} - i\pi\delta(u) \tag{C.36}$$

between tempered distributions, we obtain

$$L_+(x) = H(x) - i\pi\psi(x)\exp(-is\omega(x)). \tag{C.37}$$

Hence (C.34)–(C.35) follow from (C.5). □

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References

[1] Volkov A Yu, Noncommutative hypergeometry, `arXiv:math.QA/0312084`.

[2] Barnes E W, The theory of the double gamma function, *Phil. Trans. Roy. Soc. A* **196** (1901), 265–388.

[3] Ruijsenaars S N M, Systems of Calogero-Moser type, in Proceedings of the 1994 Banff summer school Particles and fields, CRM Ser. in Math. Phys., Editors: Semenoff G and Vinet L, Springer, New York, 1999, 251–352.

[4] Ruijsenaars S N M, First order analytic difference equations and integrable quantum systems, *J. Math. Phys.* **38** (1997), 1069–1146.

[5] Faddeev L D, Discrete Heisenberg-Weyl group and modular group, *Lett. Math. Phys.* **34** (1995), 249–254.

[6] Woronowicz S L, Quantum exponential function, *Rev. Math. Phys.* **12** (2000), 873–920.

[7] Faddeev L D, Modular double of a quantum group, in Conférence Moshé Flato 1999 (Dijon), Vol I, *Math. Phys. Stud.* **21**, Kluwer, Dordrecht, 2000, 149–156.

[8] Faddeev L D, Kashaev R M and Volkov A Yu, Strongly coupled quantum discrete Liouville theory. I: Algebraic approach and duality, *Commun. Math. Phys.* **219** (2001), 199–219.

[9] Ruijsenaars S N M, Sine-Gordon solitons vs. relativistic Calogero-Moser particles, in Proceedings of the Kiev NATO Advanced Study Institute "Integrable structures of exactly solvable two-dimensional models of quantum field theory", Editors: Pakuliak S and von Gehlen G, *NATO Science Series* **35**, Kluwer, Dordrecht, 2001, 273–292.

[10] Kharchev S, Lebedev D and Semenov-Tian-Shansky M, Unitary representations of $U_q(sl(2,\mathbb{R})$, the modular double and the multiparticle q-deformed Toda chains, *Commun. Math. Phys.* **225** (2002), 573–609.

[11] Van de Bult F J, Ruijsenaars' hypergeometric function and the modular double of $U_q(sl_2(\mathbb{C}))$, `arXiv:math.QA/0501405`.

[12] Reed M and Simon B, Methods of Modern Mathematical Physics. I. Functional analysis, Academic Press, New York, 1972.

[13] Reed M and Simon B, Methods of Modern Mathematical Physics. II. Fourier analysis, Self-adjointness, Academic Press, New York, 1975.

[14] Ruijsenaars S N M, A generalized hypergeometric function III. Associated Hilbert space transform, *Commun. Math. Phys.* **243** (2003), 413–448.

[15] Ruijsenaars S N M, A new class of reflectionless second-order AΔOs and its relation to nonlocal solitons, *Regular and Chaotic Dynamics* **7** (2002), 351–391.

[16] Ruijsenaars S N M, Isometric reflectionless eigenfunction transforms for higher-order AΔOs, Calogero Festschrift issue, *J. Nonlinear Math. Phys.* **12** Suppl. 1 (2005), 565–598.

[17] Ruijsenaars SNM, A generalized hypergeometric function satisfying four analytic difference equations of Askey-Wilson type, *Commun. Math. Phys.* **206** (1999), 639–690.

[18] Reed M and Simon B, Methods of Modern Mathematical Physics. III. Scattering theory, Academic Press, New York, 1979.

[19] Bytsko A G and Teschner J, R-operator, co-product and Haar-measure for the modular double of $U_q(sl(2,\mathbb{R}))$, *Commun. Math. Phys.* **240** (2003), 171–196.

[20] Shintani T, On a Kronecker limit formula for real quadratic fields, *J. Fac. Sci. Univ. Tokyo* **24** (1977), 167–199.

[21] Kurokawa N, Multiple sine functions and Selberg zeta functions, *Proc. Japan Acad.* **A67** (1991), 61–64.

[22] Jimbo M and Miwa T, Quantum K Z equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime, *J. Phys. A* **29** (1996), 2923–2958.

[23] Nishizawa M and Ueno K, Integral solutions of q-difference equations of the hypergeometric type with $|q| = 1$, in Proceedings of the Workshop Infinite analysis (1996) IIAS, Japan, `arXiv:q-alg/9612014`.

[24] Hölder O, Ueber eine transcendente Function, *Göttingen Nachr.* **16** (1886), 514–522.

[25] Ruijsenaars S N M, Scattering theory for the Federbush, massless Thirring and continuum Ising models, *J. Func. Anal.* **48** (1982), 135–171.