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A factorization for $\mathbb{Z} \times \mathbb{Z}$-matrices yielding solutions of Toda-type hierarchies

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Abstract

In this paper one considers the problem of finding solutions to a number of Toda-type hierarchies. All of them are associated with a commutative subalgebra of the $k \times k$-matrices. The first one is formulated in terms of upper triangular $\mathbb{Z} \times \mathbb{Z}$-matrices, the second one in terms of lower triangular ones and the third is a combination of the two foregoing types. It is shown that in an appropriate group setting solutions of the linearization of these Lax equations can be constructed by using a Birkhoff-type decomposition in the relevant group.

1 Introduction

A well-known example of a system of differential difference equations is the equations of motion of an infinite number of particles on a straight line, the so-called infinite Toda-chain. Recall from [11] that these equations in dimensionless form have the form

$$\frac{dq_n}{dt} = p_n \text{ and } \frac{dp_n}{dt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \quad n \in \mathbb{Z}. \tag{1.1}$$

Here q_n is the displacement of the n-th particle. One can reformulate these equations as an equality between infinite matrices by defining

$$a_n := \frac{1}{2} e^{-(q_n - q_{n-1})} \text{ and } b_n := \frac{1}{2} p_n.$$

The equations (1.1) get then the form

$$\frac{da_n}{dt} = a_n(b_n - b_{n-1}) \text{ and } \frac{db_n}{dt} = 2(a_{n-1}^2 - a_n^2), \quad n \in \mathbb{Z}. \tag{1.2}$$

If one introduces the $\mathbb{Z} \times \mathbb{Z}$-matrices L and B by

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & b_{n-1} & a_n & 0 & \ddots \\ \ddots & a_n & b_n & a_{n+1} & \ddots \\ & 0 & a_{n+1} & b_{n+1} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix} \text{ and } B = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & 0 & -a_n & 0 & \ddots \\ \ddots & a_n & 0 & -a_{n+1} & \ddots \\ & 0 & a_{n+1} & 0 & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix},$$

then a direct computation shows that the equations (1.2) amount to the matrix equation

$$\frac{dL}{dt} = BL - LB = [B, L]. \tag{1.3}$$

This is an example of a so-called *Lax equation*, because it suggests that the matrix L is obtained by conjugating a matrix that does not depend of t with a t-dependent one, a phenomenon first observed in the KdV-setting by P.Lax, see [8]. Several variations and extensions on the above situation have been considered, see e.g. [2] ,[12] and [6], the last one describing the algebraic structure behind various formulations. These systems of equations play a role in various parts of mathematics, like random matrices and orthogonal polynomials, see [1] and [5], but also in a diversity of subjects from theoretical physics, such as matrix models, quantum gravity and string theory. To get an impression of these connections we refer to [10], [7], [3] and [9].

In this paper one considers the problem of finding solutions to a number of Toda-type hierarchies. All of them are associated with a commutative subalgebra of the $k \times k$-matrices. The first one is formulated in terms of upper triangular $\mathbb{Z} \times \mathbb{Z}$-matrices, the second one in terms of lower triangular ones and the third is a combination of the two foregoing types. It is shown that in an appropriate group setting solutions of the linearization of these Lax equations can be constructed by using the Birkhoff decomposition in the relevant group. A description of the various sections is as follows: the first introduces the relevant notations and general properties of $\mathbb{Z} \times \mathbb{Z}$-matrices. The next section discusses the various hierarchies with their corresponding linearizations and oscillating matrices of a certain type. It also gives a useful sufficiency criterion for oscillating matrices to yield solutions of the hierarchies. In the subsequent section, a Banach setting is presented in which the formal products from the linearization become genuine products. Also the decomposition of the group of commuting flows is discussed there. In the final section one concludes with the construction of the solutions

2 The space $M_{\mathbb{Z}}(R)$

The hierarchies of Toda-type that form the subject of this paper consist of nonlinear equations for a number of $\mathbb{Z} \times \mathbb{Z}$-matrices whose coefficients are depending of the flow parameters. Therefore the basic prerequisites of this space will be recalled.

Let R be a commutative ring. The ring of $k \times k-$matrices with coefficients from R is denoted by $M_k(R)$. Likewise one writes $M_{\mathbb{Z}}(R)$ for the R-module of $\mathbb{Z} \times \mathbb{Z}$-matrices with coefficients from R. The ordering of the columns and rows in $M_{\mathbb{Z}}(R)$ that will be used is the one that is compatible with the finite dimensional case, i.e. any matrix $A = (\alpha_{ij})$ in $M_{\mathbb{Z}}(R)$ is denoted by

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \alpha_{n-1\,n-1} & \alpha_{n-1\,n} & \alpha_{n-1\,n+1} & \ddots \\ \ddots & \alpha_{n\,n-1} & \alpha_{n\,n} & \alpha_{n\,n+1} & \ddots \\ \ddots & \alpha_{n+1\,n-1} & \alpha_{n+1\,n} & \alpha_{n+1\,n+1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

There are a number of special elements in $M_{\mathbb{Z}}(R)$ that will be used frequently. First of all, there is the basic matrix $E_{(i,j)}$, i and $j \in \mathbb{Z}$, given by

$$(E_{(i,j)})_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}.$$

Thus one can describe every $A = (A_{ij}) \in M_{\mathbb{Z}}(R)$ as a formal linear combination of the basic matrices

$$A = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} A_{ij}E_{(i,j)}.$$

A dominant role in this paper is played by the shift matrix Λ given by

$$\Lambda = \sum_{i\in\mathbb{Z}} E_{(i-1,i)}.$$

With every collection $\{d(ks)|s \in \mathbb{Z}\}$ of matrices in $M_k(R)$ one associates the diagonal of k-blocks $\mathrm{diag}(d(ks))$ in $M_{\mathbb{Z}}(R)$ given by

$$\mathrm{diag}(d(ks)) := \sum_{s\in\mathbb{Z}}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} d(ks)_{\alpha\beta}E_{(s+\alpha-1,s+\beta-1)}.$$

Its matrix looks as follows

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & d(kn-k) & 0 & 0 & \ddots \\ \ddots & 0 & d(kn) & 0 & \ddots \\ \ddots & 0 & 0 & d(kn+k) & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

For each $k \geq 1$, denote the ring of k-block diagonal matrices in $M_{\mathbb{Z}}(R)$ by

$$\mathcal{D}_k(R) = \{d = \mathrm{diag}(d(ks))|d(ks) \in M_k(R) \text{ for all } s \in \mathbb{Z}\}.$$

One has a ringhomomorphism i_k from $M_k(R)$ into $\mathcal{D}_k(R)$ by taking for an $A \in M_k(R)$ all diagonal blocks of $i_k(A)$ equal to A. In particular every $\mathcal{D}_k(R)$ becomes a $M_k(R)$-algebra in this way. The elements Λ^{km}, $m \in \mathbb{Z}$, act on $\mathcal{D}_k(R)$ according to

$$\Lambda^{km} diag(d(ks))\Lambda^{-km} = diag(d(ks + km)). \tag{2.1}$$

Therefore the image of i_k consists of all matrices in $\mathcal{D}_k(R)$ that commute with Λ^k. This relation implies further that for each $m \neq 0$ any $d\Lambda^{km}$ with $d = diag(d(ks)) \in \mathcal{D}_k(R)$ invertible can be written as

$$diag(d(ks))\Lambda^{km} = w_0\Lambda^{km}w_0^{-1}, \text{ with } w_0 \in \mathcal{D}_k(R).$$

A specific choice of the element $w_0 = diag(w_0(ks))$ is as follows: first of all one takes for all s_0 with $0 \leq s_0 < |m|$, the matrix $w_0(s_0k)$ equal to the identity. Next one defines for all these s_0 and all $t \geq 1$

$$\begin{aligned} w_0((s_0+tm)k) &:= d((s_0+(t-1)m)k)^{-1}\cdots d(s_0)^{-1}, \\ w_0((s_0-tm)k) &:= d((s_0-tm)k)\cdots d((s_0-m)k) \end{aligned} \tag{2.2}$$

Each matrix in $M_{\mathbb{Z}}(R)$ can be divided into so-called k-block diagonals. For, one defines namely

Definition 1. For each $j \in \mathbb{Z}$, the *j-th k-block diagonal* of any matrix $A = (A_{ij}) \in M_{\mathbb{Z}}(R)$, is the $\mathbb{Z}\times\mathbb{Z}$-matrix

$$\sum_{i\in\mathbb{Z}}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} A_{(ki-kj+\alpha-1,ki+\beta-1)}E_{(ki-kj+\alpha-1,ki+\beta-1}.$$

From equation (2.1) it is clear that the j-th k-block diagonal of a $\mathbb{Z}\times\mathbb{Z}$-matrix A can uniquely be written in the form $diag(d(ks))\Lambda^{kj}$ or $\Lambda^{kj}diag(c(ks))$ with $diag(d(ks))$ and $diag(c(ks)) \in \mathcal{D}_k(R)$. Thus each $A = (A_{(i,j)}) \in M_{\mathbb{Z}}(R)$ can uniquely be written as

$$A = \sum_{j\in\mathbb{Z}} d_j\Lambda^{kj} \text{ or } A = \sum_{j\in\mathbb{Z}} \Lambda^{kj}c_j, \tag{2.3}$$

with d_j and c_j in $\mathcal{D}_k(R)$. In particular any matrix that commutes with Λ^k has the form (2.3) with d_j and c_j in the image of i_k.

To the first decomposition in (2.3) one links two notations: if $A = \sum_{j\in\mathbb{Z}} d_j\Lambda^j$ as in (2.3) then one writes

$$A_+(k) = \sum_{j\geq 0} d_j\Lambda^{kj} \text{ and } A_-(k) = \sum_{j<0} d_j\Lambda^{kj}.$$

Inside $M_{\mathbb{Z}}(R)$ two subspaces are considered that are rings w.r.t. the usual product

Definition 2. An element A in $M_{\mathbb{Z}}(R)$ is called *upper k-block triangular of level m*, if it can be written as

$$A = \sum_{j\geq m} d_j\Lambda^{kj}, \text{ with } d_j \in \mathcal{D}_k(R).$$

One calls m the *order* of A in Λ^k, if d_m is nonzero. The collection of all these elements is denoted by UT_m, $UT_m(R)$ or $UT_m^{(k)}(R)$, depending, if one has to stress where the coefficients come from, what the size of the blocks along the diagonal is or both. Likewise one uses the notations

$$UT(R) := \bigcup_{k\in\mathbb{Z}} UT_k =: UT$$

for the set of all uppertriangular matrices.

One verifies directly that UT with the product forms an R-algebra. All commutative R-subalgebras of UT that contain the element Λ^k have the following form: choose any commutative R-subalgebra C of $M_k(R)$, then the required algebra consists of the

$$U(C) := \{\sum_{i \geq N} i_k(c_i)\Lambda^{ki} \mid \text{with } c_i \in C \text{ for all } i\}.$$

Likewise one introduces the opposite class of matrices

Definition 3. An element A in $M_{\mathbb{Z}}(R)$is called *lower k-block triangular of level m*, if it can be written as

$$A = \sum_{j \leq m} d_j \Lambda^{kj}, \quad \text{with} \quad d_j \in \mathcal{D}(R).$$

Like for UT we call m the *order* of A in Λ^k, if d_m is nonzero. The collection of all these elements is denoted by LT_m, $LT_m(R)$ or $LT_m^{(k)}(R)$, depending of the dependence that has to be stressed. Similarly, the notations

$$LT(R) := \bigcup_{k \in \mathbb{Z}} LT_k =: LT$$

are used for the set of all lowertriangular matrices.

Again one verifies easily that LT with the usual product forms an algebra over R. The commutative R-subalgebras of UT containing the element Λ^k can be described as follows: let C as above be any commutative R-subalgebra of $M_k(R)$, then

$$L(C) := \{\sum_{i \leq N} i_k(c_i)\Lambda^{ki} \mid \text{with } c_i \in C \text{ for all } i\}$$

is the required algebra. If C is maximal commutative inside $M_k(R)$, then the same holds for $L(C)$.

Note that, if $U \in UT$ and $V \in LT$ have the form respectively

$$U = \sum_{i \geq 0} u_i \Lambda^{ik} \text{ and } V = \sum_{i \leq 0} v_i \Lambda^{ik},$$

with u_0 and v_0 invertible in $\mathcal{D}_k(R)$, then the elements U and V are invertible and the diagonal k-block components of their inverses can be computed recursively.

3 The hierarchies of Toda-type

From now on one assumes that R is an algebra over $\mathbb{C}$ such that $\mathbb{C}$ is isomorphic to the subalgebra $\{\alpha.1 \mid \alpha \in \mathbb{C}\}$ of R. Then $M_k(\mathbb{C})$ is naturally a subring of $M_k(R)$.

The first hierarchy is similar to the multicomponent KP-hierarchy and its variations. Only now the hierarchy is formulated in terms of $\mathbb{Z} \times \mathbb{Z}$-matrices, not of pseudodifferential operators. One starts with a commutative subalgebra h_L of $M_k(\mathbb{C})$. Let $\{E_\alpha \mid 1 \leq \alpha \leq m_L\}$ be a basis of h_L. Inside LT one is interested in matrices of the form

$$\mathcal{L} := \sum_{i \leq 1} l_i \Lambda^{ki} \text{ and } U_\alpha = \sum_{i \leq 0} u_{i,\alpha} \Lambda^{ki}, \ 1 \leq \alpha \leq m_L, \tag{3.1}$$

with $l_1 = i_k(\text{Id})$ and $u_{0,\alpha} = i_k(E_\alpha)$. They should be seen as perturbations inside LT of the special cases $\mathcal{L} = \Lambda^k$ and $U_\alpha = i_k(E_\alpha)$. As such the perturbation should first of all respect the commutation relations that existed between the basic generators

$$[\mathcal{L}, U_\alpha] = 0 \text{ and } [U_\alpha, U_\beta] = 0 \text{ for all } \alpha \text{ and } \beta,$$

and secondly the algebraic relations inside h_L

$$U_\alpha U_\beta = \sum_\gamma C^\gamma_{\alpha\beta} U_\gamma, \text{ where } E_\alpha E_\beta = \sum_\gamma C^\gamma_{\alpha\beta} E_\gamma.$$

More importantly, the matrices $\mathcal{L}$ and the U_α should satisfy a number of nonlinear differential equations. For all the indices $i \geq 0$ and all $\alpha, 1 \leq \alpha \leq m_L$, one writes $P_{i\alpha} := \mathcal{L}^i U_\alpha$. Then the algebra R should be equiped with a collection of commuting $\mathbb{C}$-linear derivations $\partial_{P_{i\alpha}} : R \mapsto R$. In principle one thinks of $\partial_{P_{i\alpha}}$ as representing differentiation in the direction $\Lambda^{ki} i_k(E_\alpha)$. The so-called *Lax equations of the lower triangular* h_L*-hierarchy* are then by definition

$$\partial_{P_{i\alpha}}(\mathcal{L}) = [(P_{i\alpha})_+, \mathcal{L}] \text{ and } \partial_{P_{i\alpha}}(U_\beta) = [(P_{i\alpha})_+, U_\beta]. \tag{3.2}$$

These equations are equivalent to zero curvature relations for all the finite-band matrices $\{B_{i\alpha} := (P_{i\alpha})_+\}$. The Lax equations also result from the so-called linearization. Thereto one considers the function

$$\psi_0 := \exp(\sum_{i=0}^\infty t_{i\alpha} i_k(E_\alpha)\Lambda^{ki}).$$

It belongs to $UT(\mathbb{C}[t_{i\alpha}])$. It is the generator of the $LT(R)$-module $M^{(\infty)}$ of *oscillating matrices at infinity.* This module consists of formal products

$$\{\sum_{j=-\infty}^N d_j \Lambda^{kj}\} \exp(\sum_{i=0}^\infty t_{i\alpha} i_k(E_\alpha)\Lambda^{ki}), \text{ where } d_j \in \mathcal{D}_k(R).$$

The action of $LT(R)$ on $M^{(\infty)}$ is defined as follows: for all p_1 and $p_2 \in LT(R)$ one puts

$$p_1\{p_2\} \exp(\sum_{i=0}^\infty t_{i\alpha} i_k(E_\alpha)\Lambda^{ki}) = \{p_1 p_2\} \exp(\sum_{i=0}^\infty t_{i\alpha} i_k(E_\alpha)\Lambda^{ki}).$$

Clearly, it is a free $LT(R)$-module with ψ_0 as generator. Since the matrix ψ_0 commutes with Λ^k and $i_k(E_\alpha)$, multiplication from the right with these elements is well-defined on $M^{(\infty)}$. Also the action of the derivations $\{\partial_{P_{i\alpha}}\}$ can be extended to $M^{(\infty)}$ as if the product was real

$$\partial_{P_{i\alpha}}\{\sum_{j=-\infty}^N d_j \Lambda^j\}\psi_0 = \{\sum_{j=-\infty}^N \partial_{P_{i\alpha}}(d_j)\Lambda^{kj} + \sum_{j=-\infty}^N d_j \Lambda^{k(i+j)} i_k(E_\alpha)\}\psi_0.$$

For matrices $\mathcal{L}$ and the U_α of the form (3.1) one can consider inside $M^{(\infty)}$ the equations

$$\mathcal{L}\psi = \psi\Lambda^k, U_\beta\psi = \psi i_k(E_\beta) \text{ and } \partial_{P_{i\alpha}}(\psi) = B_{i\alpha}\psi, \tag{3.3}$$

for all $i \geq 0$, $1 \leq \alpha \leq m_L$. This is the *linearization* of the lower triangular h_L-hierarchy. Under a mild condition on $\psi \in M^{(\infty)}$ they imply that $\mathcal{L}$ and the U_α satisfy the Lax equations (3.2). Assume namely that $\psi = \hat{\psi}\delta_L\psi_0$, where δ_L is an invertible element of $LT(\mathbb{C})$ that commutes with Λ^k and all the $i_k(E_\alpha)$ and where $\hat{\psi} \in LT(R)$ is such that its leading coefficient is invertible. In particular $\hat{\psi}$ is invertible in $LT(R)$ and this ψ is also a free generator of $M^{(\infty)}$. An oscillating matrix at infinity of this form is called *of type* δ_L.

The first two equations of the linearization imply then that $\mathcal{L}$ and the U_β are totally determined by $\hat{\psi}$. Namely, they are equivalent to

$$\mathcal{L} := \hat{\psi}\delta_L\Lambda^k\delta_L^{-1}\hat{\psi}^{-1} \text{ and } U_\beta = \hat{\psi}\delta_L i_k(E_\beta)\delta_L^{-1}\hat{\psi}^{-1} \tag{3.4}$$

To get the Lax equations for $\mathcal{L}$ one applies $\partial_{P_{i\alpha}}$ to the first equation in (3.3) and substitute the last one. This yields

$$\partial_{P_{i\alpha}}(\mathcal{L}\psi - \psi\Lambda^k) = \partial_{P_{i\alpha}}(\mathcal{L})\psi + \mathcal{L}(\partial_{P_{i\alpha}}(\psi)) - (\partial_{P_{i\alpha}}(\psi))\Lambda^k = \tag{3.5}$$

$$\partial_{P_{i\alpha}}(\mathcal{L})\psi + \mathcal{L}B_{i\alpha}\psi - B_{i\alpha}\psi\Lambda^k = \{\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}\,]\}\psi = 0. \tag{3.6}$$

As ψ is a free generator of the $LT(R)$-module $M^{(\infty)}$,

$$\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}\,] = 0.$$

Similarly, applying $\{\partial_{P_{i\alpha}}\}$ to the second equation of the linearization and substituting the last one, renders

$$\partial_{P_{i\alpha}}(U_\beta\psi - \psi i_k(E_\alpha)) = \partial_{P_{i\alpha}}(U_\beta)\psi + U_\beta(\partial_{P_{i\alpha}}(\psi)) - (\partial_{P_{i\alpha}}(\psi))E_\beta =$$
$$\partial_{P_{i\alpha}}(U_\beta)\psi + U_\beta B_{i\alpha}\psi - B_{i\alpha}\psi E_\beta = \{\partial_{P_{i\alpha}}(U_\beta) - [B_{i\alpha}, U_\beta\,]\}\psi = 0.$$

This gives then the Lax equations for the U_β. An oscillating matrix at infinity of type δ_L, $\psi = \hat{\psi}\delta_L\psi_0$, is also called *a wavematrix at infinity of type* δ_L for the operators $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ and $U_\beta = \hat{\psi}i_k(E_\beta)\hat{\psi}^{-1}$, if it satisfies the equations (3.3). To prove the equations (3.3) for an oscillating matrix at infinity ψ of the right form, one uses the fact that it suffices to prove a weaker result, namely

Proposition 1. *Let $\psi = \hat{\psi}\delta_L\psi_0$, with $\hat{\psi} - \Lambda^{kj} \in LT_{j-1}$, be an oscillating matrix at infinity of type δ_L. If it satisfies for all $i \geq 0$ and all $\alpha, 1 \leq \alpha \leq m_L$,*

$$\partial_{P_{i\alpha}}(\psi) = F_{i\alpha}\psi, \text{ with } F_{i\alpha} \in LT(R) \cap UT_0(R),$$

then $F_{i\alpha} = (\mathcal{L}^i U_\alpha)_+$, where $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ and $U_\beta = \hat{\psi}i_k(E_\beta)\hat{\psi}^{-1}$. In particular the $\mathcal{L}$ and U_β form a solution to the lower triangular h_L-hierarchy.

Proof. From the definition of the action of $\{\partial_{P_{i\alpha}}\}$ on $M^{(\infty)}$ and the fact that $M^{(\infty)}$ is a free $LT(R)$-module with generator ψ_0, one obtains inside $LT(R)$ the matrix equality

$$\partial_{P_{i\alpha}}(\hat{\psi}) + \hat{\psi}(\Lambda^k)^i i_k(E_\alpha) = F_{i\alpha}\hat{\psi}.$$

Since $\partial_{P_{i\alpha}}(\hat{\psi})\hat{\psi}^{-1} \in LT_{-1}$, multiplying this equation from the right with $\hat{\psi}^{-1}$ and taking the uppertriangular part gives the desired result. ■

The second hierarchy is concerned with perturbations inside $UT(R)$ of the commutative algebra generated by Λ^{-k} and $i_k(h_U)$ with h_U a commutative algebra of $M_k(\mathbb{C})$. Let $\{F_\alpha \mid 1 \leq \alpha \leq m_U\}$ be a basis of h_U. Concretely, one searches then for matrices $\mathcal{M}$ and V_α of the form

$$\mathcal{M} := \sum_{i \geq -1} m_i \Lambda^{ki} \text{ and } V_\alpha := w_0 i_k(F_\alpha) w_0^{-1} + \sum_{i>0} \tilde{v}_{i,\alpha} \Lambda^{ki}, \tag{3.7}$$

where the element m_{-1} is invertible and $w_0 = diag(w_0(ks))$ is the gauge corresponding to

$$m_{-1} = w_0 \Lambda^{-k} w_0^{-1}$$

as given in equation (2.2). If the multiplication inside h_U is given by

$$F_\alpha F_\beta = \sum_\gamma D^\gamma_{\alpha\beta} F_\gamma,$$

then the matrices $\mathcal{M}$ and V_α should first of all satisfy the algebraic relations of their unperturbed counterparts

$$[\mathcal{M}, V_\alpha] = 0 \text{ and } V_\alpha V_\beta = \sum_\gamma D^\gamma_{\alpha\beta} V_\gamma. \tag{3.8}$$

Note that these equations are automatically satisfied if one takes $\mathcal{M}$ and V_α of the form

$$\mathcal{M} = W \Lambda^{-k} W^{-1} \text{ and } V_\alpha = W i_k(F_\alpha) W^{-1},$$

with $W = \sum_{j \geq m} w_j \Lambda^{kj}, w_j \in \mathcal{D}_k(R)$ and w_m invertible. For all $j \geq 1$ and all $\beta, 1 \leq \beta \leq m_U$, one writes $Q_{j\beta} := \mathcal{M}^j V_\beta$ and $C_{j\alpha} = (Q_{j\alpha})_-$. The search is for a $\mathbb{C}$-algebra R equiped with a collection of $\mathbb{C}$-linear commuting derivations $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$. The nonlinear differential equations one wants $\mathcal{M}$ and the V_β to satisfy are

$$\partial_{Q_{j\alpha}}(\mathcal{M}) = [C_{j\alpha}, \mathcal{M}] \text{ and } \partial_{Q_{j\alpha}}(V_\beta) = [C_{j\alpha}, V_\beta] \tag{3.9}$$

and are called the *Lax equations of the upper triangular* h_U-hierarchy. They follow from a linear system that requires the introduction of a suitable left $UT(R)$-module. Again the actual form of the elements in the module is guided by the trivial solution $\mathcal{M} = \Lambda^{-k}$ and $V_\alpha = i_k(F_\alpha)$ of the hierarchy. Thinking of $\partial_{Q_{j\beta}}$ as taking the partial derivative $\partial_{s_{j\beta}}$ w.r.t. the parameter $s_{j\beta}$ along the direction $i_k(F_\alpha)\Lambda^{-jk}$, consider the $\mathbb{Z} \times \mathbb{Z}$-matrix

$$\phi_0 := \exp(\sum_{j=1}^{\infty} \sum_{\beta=1}^{m_U} s_{j\beta} i_k(F_\alpha) \Lambda^{-jk})$$

This matrix belongs to $LT(\mathbb{C}[s_{j\alpha}])$. The module for the linearization will consist of perturbations in $UT(R)$ of this matrix ϕ_0. Consider namely the collection $M^{(0)}$ consisting of formal products

$$\{\sum_{j=N}^{\infty} d_j \Lambda^{kj}\} \exp(\sum_{j=1}^{\infty} \sum_{\beta=1}^{m_U} s_{j\beta} i_k(F_\beta) \Lambda^{-kj}), \text{ where } d_j \in \mathcal{D}_k(R).$$

The elements of $M^{(0)}$ are called *oscillating matrices at zero*. In general these formal products do not give a well-defined $\mathbb{Z} \times \mathbb{Z}$-matrix. Nevertheless there is a well-defined left action of $UT(R)$ on it. For all u_1 and $u_2 \in UT(R)$ one puts namely

$$u_1\{u_2\}\phi_0 = \{u_1u_2\}\phi_0.$$

Note that $M^{(0)}$ is a free $UT(R)$-module with generator ϕ_0. Other generators are elements of the form $\phi = \hat{\phi}\delta_U\phi_0$ in $M^{(0)}$, where δ_U is an invertible element of $UT(\mathbb{C})$ that commutes with Λ^{-k} and all the $i_k(F_\beta)$ and $\hat{\phi} \in UT(R)$ is such that

$$\hat{\phi} = \sum_{i=m}^{\infty} d_i\Lambda^{ki}, \text{ with } d_m \text{ invertible.} \tag{3.10}$$

An element of $M^{(0)}$ of this form is called an *oscillating matrix at zero of type* δ_U. Note that the right multiplication with Λ^{-k} and $i_k(F_\beta)$ is well-defined on elements of $M^{(0)}$, since both matrices commute with the generator ϕ_0. An action of the derivations $\partial_{Q_{j\beta}}$ on $M^{(0)}$ can be defined as follows

$$\partial_{Q_{j\beta}}\{\sum_{j=N}^{\infty} d_j\Lambda^j\}\phi_0 = \{\sum_{j=N}^{\infty} \partial_{Q_{j\beta}}(d_j)\Lambda^j + \sum_{j=N}^{\infty} d_j\Lambda^j(\Lambda)^{-kj}i_k(F_\beta)\}\phi_0.$$

For matrices $\mathcal{M}$ and V_α of the required form and satisfying the conditions (3.8) the *linearization of the h_U-hierarchy* consists of the following equations inside $M^{(0)}$

$$\mathcal{M}\phi = \phi\Lambda^{-k}, V_\alpha\phi = \phi i_k(F_\alpha) \text{ and } \partial_{Q_{j\beta}}(\phi) = C_{j\beta}\phi. \tag{3.11}$$

Note that if $\phi = \hat{\phi}\delta_U\phi_0$ in these equations is of the form (3.10), then the first two equations imply that the matrices $\mathcal{M}$ and V_α are given by

$$\mathcal{M} = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1} \text{ and } V_\alpha = \hat{\phi}i_k(F_\alpha)\hat{\phi}^{-1}. \tag{3.12}$$

This is silently assumed from now on. To get the Lax equations for $\mathcal{M}$ one applies the derivation $\partial_{Q_{j\beta}}$ to the first equation in (3.11) and subtitutes the last one. This leads to the following manipulations

$$\partial_{Q_{j\beta}}(\mathcal{M}\phi - \phi\Lambda^{-k}) = \partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))\Lambda^{-k} =$$
$$\partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}C_{j\beta}\phi - C_{j\beta}\phi\Lambda^{-k} = \{\partial_{Q_{j\beta}}(\mathcal{M}) - [C_{j\beta}, \mathcal{M}\ ]\}\phi = 0.$$

Since ϕ may be scratched from this equation, one obtains in this way the Lax equations for $\mathcal{M}$. For the operator V_α one applies $\partial_{Q_{j\beta}}$ to the second equation in (3.11) and substitutes the last one. Thus one gets

$$\partial_{Q_{j\beta}}(V_\alpha\phi - \psi F_\alpha) = \partial_{Q_{j\beta}}(V_\alpha)\psi + V_\alpha(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))F_\alpha =$$
$$\partial_{Q_{j\beta}}(V_\alpha)\phi + V_\alpha C_{j\beta}\phi - C_{j\beta}\phi F_\alpha = \{\partial_{Q_{j\beta}}(V_\alpha) - [C_{j\beta}, V_\alpha\ ]\}\phi = 0.$$

and, since one can leave out ϕ again, this yields the Lax equations for the V_α.

An oscillating matrix at zero of type δ_U, $\phi = \hat{\phi}\delta_U\phi_0$, is called *a wavematrix at zero of type* δ_U for the matrices $\mathcal{M} = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\alpha = \hat{\phi}i_k(F_\alpha)\hat{\phi}^{-1}$, if it satisfies the equations (3.11). Since the manipulations to get the Lax equations are well-defined on such a ϕ, the set of matrices $(\mathcal{M}, V_\alpha)$ forms a solution of the upper triangular h_U-hierarchy.

If one wants to prove the equations (3.11) for an oscillating matrix at zero ϕ of the right form, it suffices to prove a weaker result, for there holds

Proposition 2. *Let $\phi = \hat{\phi}\delta_U\phi_0$ be an oscillating matrix at zero of type δ_U. If it satisfies for all $j \geq 1$ and all $\beta, 1 \leq \beta \leq m_U$*

$$\partial_{Q_{j\beta}}(\phi) = G_{j\beta}\psi, \text{ with } G_{j\beta} \in UT(R) \cap LT_0(R),$$

then $G_{j\beta} = (\mathcal{M}^j V_\beta)_-$, where $\mathcal{M} := \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\beta = \hat{\phi} i_k(F_\beta)\hat{\phi}^{-1}$. In particular $\mathcal{M}$ and the V_β form a solution to the upper triangular h_U-hierarchy

Proof. From the definition of the action of $\partial_{Q_{j\beta}}$ on $M^{(0)}$ and the fact that $M^{(0)}$ is a free $UT(R)$-module with generator ϕ_0, we get the operator equation

$$\partial_{s_{j\beta}}(\hat{\phi}) + \hat{\phi}(\Lambda)^{-kj} i_k(F_\beta) = G_{j\beta}\hat{\phi}.$$

Multiplying this equation from the right with $\hat{\phi}^{-1}$ and taking the lowertriangular part gives the desired result. ■

The third hierarchy is a combination of the two foregoing ones and is called the (h_L, h_U)-hierarchy. First of all one has the corresponding potential solutions, namely the matrices $\mathcal{L}$ and U_α in $LT(R)$ of the form (3.1) and the matrices $\mathcal{M}$ and V_α in $UT(R)$ of the form (3.7). Further one assumes the $\mathbb{C}$-algebra R to be equiped with two collections of $\mathbb{C}$-linear commuting derivations namely the $\{\partial_{P_{i\alpha}}, i \geq 0, 1 \leq \alpha \leq m_L\}$ and the $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$. The Lax equations of this hierarchy consist not only of those in (3.2) and (3.9), but also include the following evolution of $\mathcal{L}$ and U_α w.r.t. $\partial_{Q_{j\beta}}$

$$\partial_{Q_{j\beta}}(\mathcal{L}) = [C_{j\beta}, \mathcal{L}] \text{ and } \partial_{Q_{j\beta}}(U_\alpha) = [C_{j\beta}, U_\alpha]$$

and that of $\mathcal{M}$ and V_σ w.r.t. $\partial_{P_{i\alpha}}$

$$\partial_{P_{i\alpha}}(\mathcal{M}) = [B_{i\alpha}, \mathcal{M}] \text{ and } \partial_{P_{i\alpha}}(V_\sigma) = [B_{i\alpha}, V_\sigma].$$

From these last two sets of equations one sees that the unperturbed choice

$$\mathcal{L} = \Lambda^k, U_\alpha = i_k(E_\alpha), \mathcal{M} = \Lambda^{-k} \text{ and } V_\beta = i_k(F_\beta),$$

is a solution of these Lax equations if and only if the algebras h_L and h_U commute. This will be assumed from now on, without further mentioning.

Again there exists a *linearization of the (h_L, h_U)-hierarchy* from which these Lax equations can be deduced. It consists of the equations

$$\mathcal{L}\psi = \psi\Lambda^k, U_\gamma\psi = \psi i_k(E_\gamma), \tag{3.13}$$

$$\partial_{Q_{j\beta}}(\psi) = C_{j\beta}(\psi), \text{ and } \partial_{P_{i\alpha}}(\psi) = B_{i\alpha}\psi, \tag{3.14}$$

$$\mathcal{M}\phi = \phi\Lambda^{-k}, V_\sigma\phi = \phi i_k(F_\sigma), \tag{3.15}$$

$$\partial_{P_{i\alpha}}(\phi) = B_{i\alpha}\phi, \text{ and } \partial_{Q_{j\beta}}(\phi) = C_{j\beta}\phi. \tag{3.16}$$

Here the action of the $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$ on the elements of $M^{(\infty)}$ is defined by

$$\partial_{Q_{j\beta}}(\{\sum_{r=-\infty}^{N} d_r\Lambda^{kr}\}\psi_0) = \{\sum_{r=-\infty}^{N} \partial_{Q_{j\beta}}(d_r)\Lambda^{kr}\}\psi_0$$

and that of the $\{\partial_{P_{i\alpha}}, i \geq 0, 1 \leq \alpha \leq m_L\}$ on $M^{(0)}$

$$\partial_{P_{i\alpha}}(\{\sum_{s=N}^{\infty} d_s \Lambda^{ks}\}\phi_0) = \{\sum_{s=N}^{\infty} \partial_{P_{i\alpha}}(d_s)\Lambda^{ks}\}\phi_0.$$

Let as before δ_L be an invertible element of $UT(\mathbb{C}) \cap LT(\mathbb{C})$ that commutes with Λ^k and all the $i_k(E_\alpha)$ and let δ_U be an invertible element of $UT(\mathbb{C}) \cap LT(\mathbb{C})$ that commutes with Λ^{-k} and all the $i_k(F_\beta)$. Assume that δ_L and δ_U commute. For the unperturbed solution the oscillating functions

$$\psi = \{\exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_U} s_{j\beta} i_k(F_\alpha)\Lambda^{-jk})\delta_U\}\delta_L\psi_0$$

and

$$\phi = \{\exp(\sum_{i=0}^{\infty} t_{i\alpha} i_k(E_\alpha)\Lambda^{ki})\delta_L\}\delta_U\phi_0$$

satisfy the linearization for the derivations $\partial_{P_{i\alpha}} = \frac{\partial}{\partial t_{i\alpha}}$ and $\partial_{Q_{j\beta}} = \frac{\partial}{\partial s_{j\beta}}$. Assume $\psi = \hat{\psi}\delta_L\psi_0 \in M^{(\infty)}$ in the equations (3.13) and (3.14) is an oscillating matrix at infinity of type δ_L and let $\phi = \hat{\phi}\delta_U\phi_0 \in M^{(0)}$ in the equations (3.15) and (3.16) be an oscillating matrix at zero of type δ_U. It follows from (3.13) that the matrices $\mathcal{L}$ and U_α are given by (3.4) and the matrices $\mathcal{M}$ and V_α by (3.12). By applying again both sets of derivations to the equations of (3.13) resp. (3.15) and by substituting those of (3.14) resp. (3.16) and scratching the function ψ resp. ϕ one obtains the Lax equations for $\mathcal{L}$ and the U_α and those for $\mathcal{M}$ and the V_β. A pair $(\psi, \phi) = (\hat{\psi}\delta_L\psi_0, \hat{\phi}\delta_U\phi_0)$ in $M^{(\infty)} \times M^{(0)}$ consisting of an oscillating matrix ψ at infinity of type δ_L and an oscillating matrix ϕ at zero of type δ_U is called a *pair of wavematrices of the (h_L, h_U)-hierarchy of type (δ_L,δ_U)*, if they satisfy the equations in (3.13) ,(3.14),(3.15) and (3.16) for the $\mathbb{Z} \times \mathbb{Z}$-matrices $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$, $U_\alpha := \hat{\psi}E_\alpha\hat{\psi}^{-1}$, $\mathcal{M} := \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\alpha := \hat{\phi}E_\alpha\hat{\phi}^{-1}$. As one has seen this collection of matrices forms then a solution of the Lax equations of the (h_L, h_U)-hierarchy.

Also in the coupled case, it suffices that an apparently weaker version of the equations (3.14) resp. (3.16) holds for a candidate pair (ψ, ϕ). By combining the propositions (1) and (2) one gets namely

Proposition 3. *Consider a pair $(\psi, \phi) = (\hat{\psi}\delta_L\psi_0, \hat{\phi}\delta_U\phi_0)$ in the space $M^{(\infty)} \times M^{(0)}$ consisting of an oscillating matrix ψ at infinity of type δ_L and an oscillating matrix ϕ at zero of type δ_U. If they satisfy the equations*

$$\partial_{P_{i\alpha}}(\psi) = F_{i\alpha}\psi \text{ and } \partial_{P_{i\alpha}}(\phi) = F_{i\alpha}\phi, \text{ with } F_{i\alpha} \in LT(R) \cap UT_0(R),$$

$$\partial_{Q_{j\beta}}(\psi) = G_{j\beta}\psi \text{ and } \partial_{Q_{j\beta}}(\phi) = G_n\phi, \text{ with } G_{j\beta} \in UT(R) \cap LT_0(R),$$

then $F_{i\alpha} = (\mathcal{L}^i U_\alpha)_+$, where $\mathcal{L} := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ and $U_\alpha = \hat{\psi}i_k(E_\alpha)\hat{\psi}^{-1}$, and $G_{j\beta} = (\mathcal{M}^j V_\beta)_-$, with $\mathcal{M} := \hat{\phi}\Lambda^{-1}\hat{\phi}^{-1}$ and $V_\beta = \hat{\phi}i_k(F_\beta)\hat{\phi}^{-1}$. In particular the set $(\mathcal{L}, U_\alpha, \mathcal{M}, V_\beta)$ is a solution of the (h_L, h_U)-hierarchy.

Remark 1. Note that if one chooses the algebra $h_U = \{0\}$, then the $(h_L, \{0\})$-hierarchy is the lower triangular h_L-hierarchy and the choice $h_L = \{0\}$ yields the upper triangular h_U-hierarchy.

Under suitable convergence conditions the formal products occurring in as well the space $M^{(\infty)}$ as $M^{(0)}$ turn into real products. Such an analytic setting is described in the following section. It will allow you to generate families of solutions of the (h_L, h_U)-hierarchy.

4 An analytic setting

One starts with a complex Banach space $(H, ||\dot{|}||_H)$ equiped with a topological basis $\{e_i \mid i \in \mathbb{Z}\}$. That is to say every $h \in H$ decomposes uniquely as

$$h = \sum_{i \in \mathbb{Z}} \alpha_i e_i \text{ and } h = \lim_{N \mapsto \infty} \sum_{i=-N}^{N} \alpha_i e_i.$$

To each bounded linear operator $A : H \mapsto H$ one can associate the $\mathbb{Z}\times\mathbb{Z}$-matrix $[A]; = (\alpha_{ji})$ w.r.t. this basis defined by

$$A(e_i) = \sum_{j \in \mathbb{Z}} \alpha_{ji} e_j.$$

In view of the character of the flows of the hierarchies it is convenient to realize H as a space of vector-valued series. More concretely, let $\{f_i \mid 0 \leq i \leq k-1\}$ be the standard basis of $\mathbb{C}^k$, where f_i has a one as the $i+1$-th entry and zeros elsewhere. Then we make for all $j \in \mathbb{Z}$ and all $s, 0 \leq s \leq k-1$, the identification

$$e_{s+kj} := f_s z^j.$$

and thus get that any element $h \in H$ can be uniquely written as

$$h = \sum_{j \in \mathbb{Z}} h(j) z^j, \text{ with } h(j) \in \mathbb{C}^k.$$

In order that we can carry out the construction of the solutions of the (h_L, h_U)-hierarchy the space H has to satisfy a number of assumptions. First of all multiplying with z

$$\sum_{j \in \mathbb{Z}} h(j) z^j \mapsto \sum_{j \in \mathbb{Z}} h(j) z^{j+1}$$

should be a bounded invertible operator $M_z : H \mapsto H$ whose operator norm is equal to $||M_z||$. Its matrix $[M_z]$ is the matrix Λ^k. Also for each complex $k \times k-$matrix A multiplication with A,

$$\sum_{j \in \mathbb{Z}} h(j) z^j \mapsto \sum_{j \in \mathbb{Z}} A(h(j)) z^j$$

should be a bounded operator $M_A : H \mapsto H$. Its matrix $[M_A]$ is clearly $i_k(A)$. Examples of spaces satisfying these conditions are the $L^p(S^1, \mathbb{C}^k)$.
For each $i \in \mathbb{Z}$, let $H^{(i)}$ be the complex subspace of H spanned by the

$$\{f_s z^i \mid 0 \leq s \leq k-1\}.$$

The projection $H \mapsto H^{(i)}$ given by $\sum_{j\in\mathbb{Z}} h(j)z^j \mapsto h(i)z^i$ is denoted by $p^{(i)}$. The space H decomposes as the direct sum

$$H = \oplus_{i\in\mathbb{Z}} H^{(i)}$$

and this determines for each bounded linear operator $B \in B(H)$ the corresponding block decomposition $(p^{(i)} \circ B \mid H^{(j)})$. Inside $\mathrm{GL}(H)$ we introduce two fundamental groups that occur in the basic decomposition. First there is the parabolic group

$$P = \left\{ g \mid g \in \mathrm{GL}(H), \begin{array}{c} p^{(i)} \circ g \mid H^{(j)} = p^{(i)} \circ g^{-1} \mid H^{(j)} = 0 \\ \text{for all } i, j \in \mathbb{Z}, i < j \end{array} \right\}$$

with its Lie algebra

$$L(P) = \left\{ g \mid g \in B(H), p^{(i)} \circ g \mid H^{(j)} = 0 \text{ for all } i, j \in \mathbb{Z}, i < j \right\}.$$

Further there is the unipotent part of the opposite parabolic

$$U_- = \left\{ g \mid g \in \mathrm{GL}(H), \begin{array}{c} p^{(i)} \circ g \mid H^{(i)} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \\ p^{(i)} \circ g \mid H^{(j)} = 0 \text{ for all } i, j \in \mathbb{Z}, i > j \end{array} \right\}$$

with its Lie algebra

$$L(U_-) = \left\{ g \in B(H), p^{(i)} \circ g \mid H^{(j)} = 0 \text{ for all } i, j \in \mathbb{Z}, i \geq j \right\}.$$

Clearly $B(H) = L(P) \oplus L(U_-)$ and the map $\chi : L(P) \oplus L(U_-) \mapsto \mathrm{GL}(H)$ defined by $\chi(u,p) = \exp(u)\exp(p)$ is a local diffeomorphism at $(0,0)$. Hence the set $\Omega := U_-.P$ is open in $\mathrm{GL}(H)$. The Birkhoff-type decomposition of the elements of Ω enables one to construct solutions of the hierarchies.

Next the commuting flows relevant for the (h_L, h_U)-hierarchy will be discussed. Let U be an open connected neighbourhood in the complex plane of the circle

$$S(||M_z||) = \{z \mid z \in \mathbb{C}, |z| = ||M_z||\}.$$

For any commutative subalgebra $\mathbf{h}$ of $\mathrm{gl}_k(\mathbb{C})$ let $\Gamma(U, \mathbf{h})$ be the set of holomorpic maps $\gamma : U \mapsto \mathbf{h}$ such that $det(\gamma(u)) \neq 0$ for all $u \in U$. It is a group for the pointwise multiplication in $\mathrm{GL}_k(\mathbb{C})$. If two such neighbourhoods U_1 and U_2 satisfy $U_2 \subset U_1$ then one has a natural embedding of $\Gamma(U_1, \mathbf{h})$ into $\Gamma(U_2, \mathbf{h})$ and the inductive limit is denoted by $\Gamma(\mathbf{h})$. Each $\gamma \in \Gamma(\mathbf{h})$ has a Fourier series

$$\sum_{i\in\mathbb{Z}} \gamma_i z^i, \text{ with } \gamma_i \in \mathbf{h}.$$

and the multiplication with this series defines a bounded operator $M_\gamma : H \mapsto H$. This determines an embedding of $\Gamma(\mathbf{h})$ into $\mathrm{GL}(H)$. Let $\mathbf{h}_{ss}$ denote the subset of semi simple elements in $\mathbf{h}$ and let $\mathbf{h}_n$ be the collection of nilpotent elements in $\mathbf{h}$. From the fact that $\mathbf{h}$ is the direct sum of these subspaces one deduces that the group $\Gamma(\mathbf{h})$ is the direct product of the groups

$$\Gamma(\mathbf{h})_{ss} := \{\gamma \mid \gamma(u) \in \mathbf{h}_{ss} \text{ for all } u\}$$

and

$$\Gamma(\mathbf{h})_u := \{\gamma \mid \gamma(u) \text{ is unipotent for all } u\}.$$

Now it is easy to see that any $\gamma \in \Gamma(\mathbf{h})_u$ can be written as

$$\begin{aligned}\gamma &= \exp(\sum_{s\in\mathbb{Z}} k_s z^s), \text{ where } k_s \in \mathbf{h}_n \text{ for all } s \in \mathbb{Z},\\ &= \exp(\sum_{s\geq 0} k_s z^s)\exp(\sum_{s<0} k_s z^s)\end{aligned}$$

This shows that the elements of $\Gamma(\mathbf{h})_u$ split up perfectly in those that have an analytic continuation to the interior of $S(||M_z||)$ and those that extend holomorphically around "infinity".

As for the group $\Gamma(\mathbf{h})_{ss}$, recall, see e.g. [4], that, if U is an open connected neighbourhood of $S(||M_z||)$, any holomorphic $f : U \mapsto \mathbb{C}^*$ decomposes as

$$f(z) = \{1 + \sum_{i<0} b_i z^i\} z^m \{\sum_{j\geq 0} c_j z^j\}, \text{ with } c_0 \neq 0 \text{ and } m \in \mathbb{Z}.$$

By applying this to the group $\Gamma(\mathbf{h})_{ss}$, one arrives at the following decomposition of $\Gamma(\mathbf{h})$

Proposition 4. *There is a subgroup* $\Delta(\mathbf{h})$ *of* $\Gamma(\mathbf{h})_{ss}$ *isomorphic to* $\mathbb{Z}^r$*, where* r *is the dimension of* $\mathbf{h}_{ss}$ *such that* $\Gamma(\mathbf{h})$=$\Gamma(\mathbf{h})_+$ $\Delta(\mathbf{h})$ $\Gamma(\mathbf{h})_-$*, where*

$$\Gamma(\mathbf{h})_+ = \{\gamma \mid \gamma = \exp(\sum_{s\geq 0} \gamma_s z^s), \textit{ with } \gamma_s \in \mathbf{h} \textit{ for all } s \geq 0\}$$

and

$$\Gamma(\mathbf{h})_- = \{\gamma \mid \gamma = \exp(\sum_{s<0} \gamma_s z^s), \textit{ with } \gamma_s \in \mathbf{h} \textit{ for all } s < 0\}.$$

In the case that $\mathbf{h}$ equals the diagonal matrices, one can take

$$\Delta(\mathbf{h}) = \left\{ \begin{pmatrix} z^{m_1} & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & z^{m_k} \end{pmatrix} \mid m_i \in \mathbb{Z} \right\}$$

If $\{H_\sigma \mid 1 \leq \sigma \leq m\}$ is a basis of $\mathbf{h}$, then there is for each element γ_+ of $\Gamma(\mathbf{h})_+$ an $N > ||M_z||$ such that

$$\gamma_+ = \exp(\sum_{i=0}^{\infty}\sum_{\sigma=1}^{m} h_{i\sigma} H_\sigma z^i), h_{i\sigma} \in \mathbb{C}, \sum_{i,\sigma} |h_{i\sigma}| N^i < \infty$$

and if $\gamma_- \in \Gamma(\mathbf{h})_-$, then there is a $M < ||M_z||$ such that

$$\gamma_- = \exp(\sum_{j=1}^{\infty}\sum_{\sigma=1}^{m} h_{j\sigma} H_\sigma z^{-j}), h_{j\sigma} \in \mathbb{C}, \sum_{j,\sigma} |s_{j\sigma}| M^i < \infty.$$

In other words, the $\underline{t} = \{t_{i\alpha}\}$ are the coordinates on $\Gamma(h_U)_+$ w.r.t. the basis $\{E_\alpha\}$ and the $\underline{s} = \{s_{j\beta}\}$ are the coordinates on $\Gamma(h_L)_-$ w.r.t. the basis $\{F_\beta\}$. The next step will be the construction of pairs of wavematrices in this analytic context.

5 The construction of the solutions

Let g belong to $\mathrm{GL}(H)$. Consider the map G_g from the product of the flows $\Gamma(h_L)_+ \times \Delta(h_L) \times \Gamma(h_U)_- \times \Delta(h_U)$ to group $\mathrm{GL}(H)$ defined by

$$G_g(\gamma_+(\underline{t}), \Delta_L, \gamma_-(\underline{s}), \Delta_U) = \gamma_+(\underline{t})\Delta_L g \gamma_-(\underline{s})^{-1}\Delta_U^{-1}.$$

Let $\Omega(h_L, h_U)$ be the inverse image under G_g of the open set Ω in $\mathrm{GL}(H)$. For the ring of functions R one chooses now those that are holomorphic on $\Omega(h_L, h_U)$ and for the two sets of commuting derivations of R one takes the

$$\{\partial_{P_{i\alpha}} = \partial_{t_{i\alpha}} := \frac{\partial}{\partial t_{i\alpha}}, i \geq 0, 1 \leq \alpha \leq m_L\}$$

and the

$$\{\partial_{Q_{j\beta}} = \partial_{s_{j\beta}} := \frac{\partial}{\partial s_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}.$$

By definition, the operator $G_g(\gamma_+(\underline{t}), \Delta_L, \gamma_-(\underline{s}), \Delta_U)$ splits for each point in $\Omega(h_L, h_U)$ as

$$G_g(\gamma_+(\underline{t}), \Delta_L, \gamma_-(\underline{s}), \Delta_U) = \hat{\Phi}^{(\infty)}(\underline{t}, \Delta_L, \underline{s}, \Delta_U)^{-1}\hat{\Phi}^{(0)}(\underline{t}), \Delta_L.\underline{s}, \Delta_U),$$

where one operator $\hat{\Phi}^{(\infty)} := \hat{\Phi}^{(\infty)}(\underline{t}), \Delta_L, \underline{s}), \Delta_U)$ belongs to U_- and the other one $\hat{\Phi}^{(0)} := \hat{\Phi}^{(0)}(\underline{t}, \Delta_L, \underline{s}, \Delta_U)$ to P. Note that if one introduces the operator

$$\Phi^{(\infty)}_{\Delta_L} = \hat{\Phi}^{(\infty)}\Delta_L\gamma_+(\underline{t}),$$

then its $\mathbb{Z} \times \mathbb{Z}$-matrix $[\Phi^{(\infty)}_{\Delta_L}]$ is an oscillating matrix at infinity of type $\delta_L := [\Delta_L]$ and likewise if one considers the operator

$$\Phi^{(0)}_{\Delta_U} := \hat{\Phi}^{(0)}\Delta_U\gamma_-(\underline{s}),$$

then its matrix $[\Phi^{(0)}_{\delta_U}]$ is an oscillating matrix at zero of type $\delta_U := [\Delta_U]$. Moreover one has $\Phi^{(\infty)}_{\Delta_L} g = \Phi^{(0)}_{\Delta_U}$ and the same identity holds for the corresponding matrices. The present construction works for all g in the open set $\Gamma(h_L)\Omega\Gamma(h_U)$ because

$$\begin{aligned}\Gamma(h_L)\Omega\Gamma(h_U) &= \Gamma(h_L)_+\Delta(h_L)\Gamma(h_L)_- U_- P\Gamma(h_U)_+\Gamma(h_U)_-\Delta(h_U)\\ &= \Gamma(h_L)_+\Delta(h_L)\Omega\Gamma(h_U)_-\Delta(h_U).\end{aligned}$$

This set however does not have to equal $\mathrm{GL}(H)$. The final result is now

Theorem 1. *1. Let the element g belong to the open set $\Gamma(h_L)\Omega\Gamma(h_U)$. Then the pair $([\Phi^{(\infty)}_{\delta_L}], [\Phi^{(0)}_{\delta_U}])$, as constructed above, is a pair of wavematrices of the (h_U, h_L)-hierarchy of type (δ_L, δ_U). In particular, the matrices*

$$L = [\hat{\Phi}^{(\infty)}]\Lambda^k[\hat{\Phi}^{(\infty)}]^{-1}, U_\alpha = [\hat{\Phi}^{(\infty)}]E_\alpha[\hat{\Phi}^{(\infty)}]^{-1}$$

$$M = [\hat{\Phi}^{(0)}]\Lambda^{-k}[\hat{\Phi}^{(0)}]^{-1} \text{ and } V_\beta = [\hat{\Phi}^{(0)}]F_\beta[\hat{\Phi}^{(0)}]^{-1}$$

are a solution to the (h_U, h_L)-hierarchy.

2. *For each $\gamma(L) \in \Gamma(h_L)_-$ and each $\gamma(U) \in \Gamma(h_U)_+$, the solutions of the (h_U, h_L)-hierarchy corresponding to g and $\gamma(L)g\gamma(U)$ are the same.*

Proof. To prove the first part of the theorem, we make use of proposition 3. We begin with the derivative of $[\Phi^{(\infty)}_{\delta_L}]$ w.r.t. the parameter $t_{i\alpha}$

$$\begin{aligned}\partial_{t_{i\alpha}}([\Phi^{(\infty)}_{\delta_L}]) &= \{\partial_{t_{i\alpha}}([\hat{\Phi}^{(\infty)}]) + [\hat{\Phi}^{(\infty)}]\Lambda^{ki}E_\alpha\}\delta_L[\gamma_+(\underline{t})] \\ &= \{\partial_{t_{i\alpha}}([\hat{\Phi}^{(\infty)}])[\hat{\Phi}^{(\infty)}]^{-1} + L^iU_\alpha\}[\Phi^{(\infty)}_{\Delta_L}] \\ &= F_{i\alpha}[\Phi^{(\infty)}_{\Delta_L}]\end{aligned}$$

with $F_{i\alpha}$ lower k-block triangular of level $\leq i$. On the other hand one knows that

$$[\Phi^{(\infty)}_{\Delta_L}] = [\Phi^{(0)}_{\delta_U}][g]^{-1} \tag{5.1}$$

and substituting this relation gives

$$\begin{aligned}\partial_{t_{i\alpha}}([\Phi^{(\infty)}_{\delta_L}]) &= \partial_{t_{i\alpha}}([\Phi^{(0)}_{\delta_U}])[g]^{-1} \\ &= \partial_{t_{i\alpha}}([\hat{\Phi}^{(0)}])\delta_U[\gamma_-(\underline{s})][g]^{-1} \\ &= \{\partial_{t_{i\alpha}}([\hat{\Phi}^{(0)}])[\hat{\Phi}^{(0)}]^{-1}\}[\Phi^{(\infty)}_{\delta_L}].\end{aligned}$$

Since $\partial_{t_{i\alpha}}([\hat{\Phi}^{(0)}])[\hat{\Phi}^{(0)}]^{-1}$ is upper k-block triangular of level ≥ 0 and $[\Phi^{(\infty)}_{\delta_L}]$ is a generator of $M^{(\infty)}$, this shows that $F_{i\alpha}$ has a k-block decomposition in a finite sum of positive powers of Λ^k. From the relation (5.1) follows

$$\partial_{t_{i\alpha}}([\Phi^{(0)}_{\delta_U}]) = \partial_{t_{i\alpha}}([\Phi^{(\infty)}_{\delta_L}])[g] = F_{i\alpha}[\Phi^{(0)}_{\delta_U}].$$

This concludes the proof that the first set of equations from proposition 3 is satisfied. Consider now the derivative of $[\Phi^{(0)}_{\delta_U}]$ w.r.t. the parameter $s_{j\beta}$

$$\begin{aligned}\partial_{s_{j\beta}}([\Phi^{(0)}_{\delta_U}]) &= \{\partial_{s_{j\beta}}([\hat{\Phi}^{(0)}]) + [\hat{\Phi}^{(0)}]\Lambda^{-jk}F_\beta\}\delta_U[\gamma_-(\underline{s})] \\ &= \{\partial_{s_{j\beta}}([\hat{\Phi}^{(0)}])[\hat{\Phi}^{(0)}]]^{-1} + M^jV_\beta\}[\Phi^{(0)}_{\delta_U}] \\ &= G_{j\beta}[\Phi^{(0)}_{\delta_U}],\end{aligned}$$

with $G_{j\beta}$ upper k-block triangular of level $\geq -j$. By using the relation (5.1) one sees on the other hand

$$\begin{aligned}\partial_{s_{j\beta}}([\Phi^{(0)}_{\delta_U}]) &= \partial_{s_{j\beta}}([\Phi^{(\infty)}_{\delta_L}])[g] \\ &= \partial_{s_{j\beta}}([\hat{\Phi}^{(\infty)}])\delta_L[\gamma_+(\underline{t})] \\ &= \{\partial_{s_{j\beta}}([\hat{\Phi}^{(\infty)}])[\hat{\Phi}^{(\infty)}]^{-1}\}[\Phi^{(0)}_{\delta_U}].\end{aligned}$$

The matrix $\partial_{s_{j\beta}}([\hat{\Phi}^{(\infty)}])[\hat{\Phi}^{(\infty)}]^{-1}$ is however lower k-block triangular of level < 0. Since $[\Phi^{(0)}_{\delta_U}]$ is a generator of the module $M^{(0)}$, one may conclude now that the matrix $G_{j\beta}$ has a k-block decomposition in a finite number of negative powers of Λ^k just as required in proposition 3. Thus we have proved part (1) of the theorem.

As for the second part, one has by definition the identity

$$\begin{aligned} G_{\gamma(L)g\gamma(U)}(\gamma_+, \Delta_L, \gamma_-, \Delta_U) &= \gamma_+ \Delta_L \gamma(L) g \gamma(U) \gamma_-^{-1} \Delta_U^{-1} \\ &= \gamma(L)(\hat{\Phi}^{(\infty)})^{-1} \hat{\Phi}^{(0)} \gamma(U) \end{aligned}$$

and this gives directly the decomposition of $G_{\gamma(L)g\gamma(U)}(\gamma_+, \Delta_L, \gamma_-, \Delta_U)$. Hence the solutions corresponding to $\gamma(L)g\gamma(U)$ are

$$\begin{aligned} L &= [\hat{\Phi}^{(\infty)}]\gamma(L)^{-1}\Lambda^k \gamma(L)[\hat{\Phi}^{(\infty)}]^{-1} = [\hat{\Phi}^{(\infty)}]\Lambda^k[\hat{\Phi}^{(\infty)}]^{-1}, \\ U_\alpha &= [\hat{\Phi}^{(\infty)}]\gamma(L)^{-1}E_\alpha \gamma(L)[\hat{\Phi}^{(\infty)}]^{-1} = [\hat{\Phi}^{(\infty)}]E_\alpha[\hat{\Phi}^{(\infty)}]^{-1}, \\ M &= [\hat{\Phi}^{(0)}]\gamma(U)\Lambda^{-k}\gamma(U)^{-1}[\hat{\Phi}^{(0)}]^{-1} = [\hat{\Phi}^{(0)}]\Lambda^{-k}[\hat{\Phi}^{(0)}]^{-1}, \\ V_\beta &= [\hat{\Phi}^{(0)}]\gamma(U)F_\beta \gamma(U)^{-1}[\hat{\Phi}^{(0)}]^{-1} = [\hat{\Phi}^{(0)}]F_\beta[\hat{\Phi}^{(0)}]^{-1}. \end{aligned}$$

This proves the claims in part (2) of the theorem. ■

References

[1] Adler M and van Moerbeke P, Matrix integrals, Toda symmetries, Virasoro constraints, and orthogonal polynomials, *Duke Math. J.* **80** (1995), 863–911.

[2] Flaschka H, On the Toda Lattice. II, *Prog. Theor. Phys.* **5** (1974), 703–706.

[3] Gerasimov A, Marshakov A, Mironov A, Morozov A, and Orlov A, Matrix models of Two-dimensional gravity and Toda theory, *Nucl. Phys. B* **357** (1991), 565–618.

[4] Grothendieck A, Sur la classification des fibrés holomorphes sur la sphère de Riemann, *Amer. J. Math.* **79** (1957), 121–138.

[5] Haine L. and Horozov E, Toda Orbits of Laguerre Polynomials and Representations of the Virasoro Algebra, *Bulletin des Sciences Math.(2)* **117** (1993), 485–518.

[6] Helminck G F, The algebraic structure of Lax equations for $\mathbb{Z} \times \mathbb{Z}$-matrices. *Int. J. Differ. Eq. Appl.* **8** (2003), 151–188.

[7] Kharchev S, Marshakov A, Mironov A, and Morozov A, Generalized Kontsevich model versus Toda hierarchy and discrete matrix models. *Nucl. Phys. B* **397** (1993), 339–378.

[8] Lax P D, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467–490.

[9] van Moerbeke P, Integrable foundations of string theory, Proceedings of the CIMPA-school, World Scientific, Singapore, 1994, 163–267.

[10] Dijkgraaf R, Integrable hierarchies and quantum gravity. Geometric and quantum aspects of integrable systems (Scheveningen, 1992), *Lect. Notes Phys.* **424**, Springer, Berlin, 1993, 67–89.

[11] Toda M, Nonlinear waves and solitons, *Mathematics and Its Applications* **5**, Kluwer Academic Publishers Group, Dordrecht; KTK Scientific Publishers, Tokyo, 1989.

[12] Ueno K and Takasaki K, Toda lattice hierarchy, Group representations and systems of differential equations (Tokyo, 1982), *Adv. Stud. Pure Math.* **4**, North-Holland, Amsterdam-New York, 1984, 1–95.