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Orthogonal matrix polynomials satisfying first order differential equations: a collection of instructive examples

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Abstract

We describe a few families of orthogonal matrix polynomials of size $N \times N$ satisfying *first order differential equations*. This problem differs from the recent efforts reported for instance in [7] (*Orthogonal matrix polynomials satisfying second order differential equations*, Internat. Math. Research Notices, **2004 : 10** (2004), 461–484) and [15] (*Matrix valued orthogonal polynomials of the Jacobi type*, Indag. Math. **14** nrs. 3, 4 (2003), 353–366). While we restrict ourselves to considering only first order operators, we do not make any assumption as to their *symmetry*. For simplicity we restrict to the case $N = 2$. We draw a few lessons from these examples; many of them serve to illustrate the fundamental difference between the scalar and the matrix valued case.

1 Introduction

The subject of matrix valued orthogonal polynomials was introduced by M.G. Krein, see [16, 17] more than fifty years ago. If one thinks of their scalar valued analogs it is clear that these polynomials could play a very important role in many areas of mathematics and its applications. If history is to be any guide it is natural to focus on those matrix valued orthogonal polynomials that satisfy some extra property, an issue raised in [6] following similar considerations in the scalar case in [2]. The results in [2, 6] deal with the case of operators of order two, and a general approach to the case of higher order stems from [5, 12].

In this paper we give examples of orthogonal matrix polynomials $(P_n)_n$ satisfying right hand side **first order** differential equations of the form:

$$P_n'(t)A_1(t) + P_n(t)A_0(t) = \Lambda_n P_n(t), \quad n \geq 0, \tag{1.1}$$

where A_1 and A_0 are matrix valued coefficients (independent of n) and Λ_n are appropriate matrices, i.e. each matrix polynomial P_n is an eigenvector of the right hand side first order differential operator

$$\ell_{1,R} = D^1 A_1(t) + D^0 A_0(t). \tag{1.2}$$

All matrices have a common size $N \times N$ and D stands for the usual differentiation operator. For simplicity our selection of examples is restricted to the case of 2×2 matrices with real entries , but the method that we use to find these examples is not restricted to this case. The examples are chosen to illustrate a number of issues that are **completely absent** in the scalar case.

It is clear that we could consider left hand side operators

$$\ell_{1,L} = A_1 D^1 + A_0 D^0.$$

instead of right hand side operators as above. In that case (1.1) gets replaced by the equation obtained from it by taking adjoints on both sides.

Given a selfadjoint positive definite matrix valued weight function $W(t)$ we can consider the skew symmetric bilinear form defined for any pair of matrix valued functions $P(t)$ and $Q(t)$ by the numerical matrix

$$\langle P, Q \rangle = \langle P, Q \rangle_W = \int_{\mathbb{R}} P(t) W(t) Q^*(t) dt,$$

where $Q^*(t)$ denotes the conjugate transpose of $Q(t)$. It can be shown (see e.g. [15, Section 2]) that there exists a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W, with P_n of degree n and monic. Just as in the scalar case, a sequence of monic orthogonal matrix polynomials $(P_n)_n$ satisfies a three term recurrence relation

$$A_n P_{n-1}(t) + B_n P_n(t) + P_{n+1}(t) = t P_n(t), \tag{1.3}$$

where P_{-1} is the zero matrix and P_0 is the identity matrix. We stress that we will take (1.3) as our **starting point**: we will consider a family of (monic) polynomials defined by (1.3) that also satisfies (1.1). Issues such as the positivity of an orthogonality matrix valued weight function going with these polynomials will be a secondary one, some of our examples will go with a positive measure and some will not.

We recall that if $\tilde{P}_n(t)$ is a sequence of orthonormal polynomials with respect to a positive definite weight matrix then the recursion relation reads

$$\tilde{A}_n^* \tilde{P}_{n-1}(t) + \tilde{B}_n \tilde{P}_n(t) + \tilde{A}_{n+1} \tilde{P}_{n+1}(t) = t \tilde{P}_n(t), \tag{1.4}$$

with $\tilde{A}_n$ non singular matrices and $\tilde{B}_n$ Hermitian. Conversely, this relation characterizes the orthonormality of a sequence of matrix polynomials with respect to a positive definite matrix of measures, see [9, Section 2] and [10, Theorem 6.1]. Using this one can see that for a monic sequence given by (1.3), the existence of a positive definite weight of matrices is equivalent to the existence of a sequence $(D_n)_{n\geq 0}$ of nonsingular matrices, such that, for $S_n = D_n D_n^*$ the following conditions are fulfilled,

$$A_n = S_n S_{n-1}^{-1}, \qquad B_n S_n \quad \text{is Hermitian.} \tag{1.5}$$

Indeed, writing $P_n(t) = D_n \tilde{P}_n(t)$ in (1.3) while assuming (1.5), one obtains the new recurrence relation (1.4) with $\tilde{A}_n = D_{n-1}^{-1} D_n$ and $\tilde{B}_n = D_n^{-1} B_n D_n$ Hermitian.

The problem considered in this paper is related to one that has received quite a bit of attention recently, namely the study of **second order differential operators** under the extra assumption that the corresponding operator

$$\ell_{2,R} = D^2 A_2(t) + D^1 A_1(t) + D^0 A_0(t). \tag{1.6}$$

is *symmetric* with respect to the orthogonality weight matrix $W(t)$ going along with the family of polynomials $(P_n)_n$. This is to say that

$$\langle \ell_{2,R} P, Q \rangle_W = \langle P, \ell_{2,R} Q \rangle_W$$

for all matrix valued polynomial functions P and Q. For a number of recent results along this line see [7, 8, 14, 15].

It is well known that in the case of second order difference or differential operators with real coefficients acting on **scalar** functions it is always possible to *make them symmetric.* The difficulties brought about by situations when this is not originally the case are the main concern of, for instance, [3] and [1]. In the **matrix valued case** this reduction is not possible, and symmetry is an extra condition which we will not insist on here.

This lack of any symmetry assumption on the part of $\ell_{1,R}$ makes the use of the tools in [7, 14, 15] inappropriate. We also observe that in those papers one assumes *from the beginning* that the coefficients of our second order differential operator are matrix polynomials satisfying a degree condition that insures that the space of matrix polynomials of a given degree is invariant under the action of the differential operator.

We resort here to the tools originally developed for the purely differential scalar valued case in [5], adapted to the case of scalar valued orthogonal polynomials in [12] and recently used in the matrix valued case in [13]. In this last paper one *does not assume* that one is dealing with orthogonal polynomials, rather one deals with a bispectral situation consisting of a *doubly infinite* second order difference operator of the form $\mathcal{L} = E + B_n + A_n E^{-1}$, where E is the customary shift operator $Ef(n) = f(n+1)$, and a differential operator having a common set of eigenfunctions $f(n,t)$.

We are now in a position to state the purpose and method of this paper: using the extra assumption that we are dealing with orthogonal polynomials we adapt the tools in [13] to produce some nontrivial examples of matrix valued orthogonal polynomials that are common eigenfunctions of a first order differential operator.

In closing this section it may be appropriate to observe that the **bispectral problem** considered here is the **matrix or noncommutative version** of a situation that does not arise in the scalar case, where the first interesting example already involves differential operators of order two. In that case the characters in the plot are the well known polynomials going with the names of Hermite, Laguerre and Jacobi, as well as Bessel. Within the confines of the scalar case, as soon as one relaxes the constraint on the order of the differential operator and allows for higher order ones, the plot thickens and the description of the solutions involves the Toda flows as well as their master symmetries, the Darboux process, and several other apparently unrelated pieces of mathematics. Notice that here we are going in the opposite direction, by assuming that the order of the differential operator is one.

It is only natural to conjecture that an equally rich situation arises in the matrix case, and that a huge collection of examples could play a useful role in future developments.

2 The ad-conditions

The method discussed in [5, 12, 13] shows that one needs to look at solutions of the equation

$$\mathrm{ad}^2(\mathcal{L})(\Lambda) = 0 \tag{2.1}$$

where $\mathrm{ad}(X)(Y) = [X, Y] = XY - YX$ is the usual commutator of the operators X and Y. Specifically, we have $\mathrm{ad}^2(\mathcal{L})(\Lambda) = [\mathcal{L}, [\mathcal{L}, \Lambda]]$.

Here $\mathcal{L}$ is the block tridiagonal semi-infinite matrix

$$\mathcal{L} = \begin{pmatrix} B_0 & I & 0 & \dots & \\ A_1 & B_1 & I & & \\ 0 & A_2 & B_2 & I & \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.2}$$

going with the three term recurrence relation (1.3) satisfied by the matrix valued orthogonal polynomials $(P_n)_n$ and $\Lambda = \mathrm{diag}\{\Lambda_0, \Lambda_1, \Lambda_2 \dots\}$ stands for the block diagonal matrix of eigenvalues associated to the differential operator (when written as a right-hand-side operator).

This leads to a set of five equations, one for each diagonal of the matrix

$$\mathrm{ad}^2(\mathcal{L})(\Lambda).$$

The first one is very simple,

$$\Lambda_{n+1} - 2\Lambda_n + \Lambda_{n-1} = 0$$

with solution

$$\Lambda_n = nP + Q.$$

Here P and Q are arbitrary matrices.

Using this, the second one can be rewritten as

$$\Lambda_{n+1}B_n - B_n\Lambda_n = \Lambda_{n-1}B_{n-1} - B_{n-1}\Lambda_{n-2}.$$

The third equation, coming from the main diagonal in $\mathrm{ad}^2(\mathcal{L})(\Lambda) = 0$ can be rewritten as

$$[B_{n-1}, [B_{n-1}, \Lambda_{n-1}]] = (\Lambda_{n+1}A_n - A_n\Lambda_{n-1}) - (\Lambda_{n-1}A_{n-1} - A_{n-1}\Lambda_{n-3})$$

The fourth one is

$$\Lambda_n B_n A_n + \Lambda_n A_n B_{n-1} - 2B_n\Lambda_n A_n + B_n A_n \Lambda_{n-1} - 2A_n\Lambda_{n-1}B_{n-1} + A_n B_{n-1}\Lambda_{n-1} = 0.$$

Finally the fifth equation is

$$\Lambda_{n+1}A_{n+1}A_n - 2A_{n+1}\Lambda_n A_n + A_{n+1}A_n\Lambda_{n-1} = 0.$$

Note that if all the matrices involved here commute with each other then the last two equations are consequences of the previous ones, namely (when written in its original form) the fourth one follows from the second one by factoring A_n out of it, and the fifth one is just the first one multiplied by $A_{n+1}A_n$. In this case, if one assumes that P is non-singular, the second equation implies $B_n = B_{n-1}$, the third one gives that all A_n vanish, and the fourth equation becomes trivial. Incidentally, this explains why in the scalar case one does not encounter examples of the bispectral problem involving semi-infinite three term recursions and first order differential operators.

The last paper referenced above, [13], poses these equations and finds some particular solutions that do not correspond to polynomials, since the matrix A_0 in the doubly-infinite matrix $\mathcal{L}$ in [13] does not vanish.

The examples that we display below are obtained by solving the equations given above. The examples are chosen to illustrate a variety of possible behaviors.

3 A Chebyshev example

In this example we have, using the same notation as above,

$$B_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; n \geq 1$$

and

$$A_n = \frac{1}{4}I, \quad n \geq 1.$$

The monic orthogonal polynomials given by (1.3) satisfy the differential equation written in the form of a left-hand-side operator acting on $P_n^*(t)$

$$\left[\begin{pmatrix} 1 & t \\ -t & -1 \end{pmatrix}\frac{d}{dt} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right]P_n^*(t) = P_n^*(t)\begin{pmatrix} 0 & n \\ -n-1 & 0 \end{pmatrix}. \tag{3.1}$$

This is not the only first order differential equation satisfied by our P_n^*. One also has

$$\left[\begin{pmatrix} t & 1 \\ -1 & -t \end{pmatrix}\frac{d}{dt} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]P_n^*(t) = P_n^*(t)\begin{pmatrix} n+1 & 0 \\ 0 & -n \end{pmatrix}. \tag{3.2}$$

The differential operators appearing on the left hand side of (3.1) and (3.2) do not commute with each other. However, the operators

$$D_a = \begin{pmatrix} 1 & t \\ -t & -1 \end{pmatrix}\frac{d}{dt} \quad \text{and} \quad D_b = \begin{pmatrix} t & 1 \\ -1 & -t \end{pmatrix}\frac{d}{dt}$$

anti-commute, this is, $D_a D_b = -D_b D_a$.

It is easy to express these polynomials in terms of Chebyshev polynomials and it is also easy to see that the orthogonality matrix is given by

$$W(t) = \frac{1}{\sqrt{1-t^2}} \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \qquad -1 < t < 1.$$

Let us denote by $U_n(t)$ the Chebyshev polynomials of the second kind, which satisfy

$$U_{n+1}(t) + U_{n-1}(t) = 2tU_n(t), \quad \text{with} \quad U_{-1}(t) = 0 \quad \text{and} \quad U_0(t) = 1.$$

These matrix valued polynomials were introduced for a different purpose in [4, Page 585]. The relation between $P_n(t)$ and the Chebyshev polynomials $U_n(t)$ is given by

$$P_n(t) = \frac{1}{2^n} \begin{pmatrix} U_n(t) & -U_{n-1}(t) \\ -U_{n-1}(t) & U_n(t) \end{pmatrix}.$$

Remark. To bring these expressions more in line with those that appear in the examples below in section 6, notice that for instance in (3.1)

$$\begin{aligned} \Lambda_n^* &= nP^* + Q^* \\ &\equiv n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

and the differential operator is

$$\ell_{1,L} = (tP^* + R)\frac{d}{dt} + Q^*$$

with

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark. The polynomials $P_n(t)$ also satisfy a zero-th order equation, a phenomenon that would be devoid of any interest in the scalar case. Here we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P_n^*(t) = P_n^*(t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.3}$$

4 A Jacobi type example

The next example is inspired by a remark contained in [15]. In that paper one obtains an example of a sequence of matrix valued orthogonal polynomials satisfying a one parameter family second order differential equations. Here we exploit this free parameter to produce a situation that solves the *ad-conditions* described above.

In this case the block tridiagonal matrix is given by (2.2) with

$$B_n = \frac{1}{2}I - c_n \begin{pmatrix} (\beta-\alpha)(\alpha+\beta+3) & 2(\alpha+1) \\ 2(\beta+1) & (\beta-\alpha)(\alpha+\beta+1) \end{pmatrix},$$

where

$$c_n = \frac{\alpha+\beta+1}{2(\alpha+\beta+2)(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}.$$

The matrices A_n have a more complicated n dependence, namely

$$A_n = d_n \begin{pmatrix} A_n^{11} & A_n^{12} \\ A_n^{21} & A_n^{22} \end{pmatrix},$$

with

$$d_n = \frac{n(n+\alpha+\beta+1)}{(\alpha+\beta+2)(2n+\alpha+\beta)(2n+\alpha+\beta+1)^2(2n+\alpha+\beta+2)}$$

and

$$\begin{aligned} A_n^{11} &= (\alpha+\beta+2)n^2 + (\alpha+\beta+1)(\alpha+\beta+2)n + (\alpha\beta+1)(\alpha+\beta) + 4\alpha\beta, \\ A_n^{12} &= (\alpha-\beta)(\alpha+1), \\ A_n^{21} &= (\alpha-\beta)(\beta+1), \\ A_n^{22} &= (\alpha+\beta+2)n^2 + (\alpha+\beta+1)(\alpha+\beta+2)n + (\alpha+1)(\beta+1)(\alpha+\beta). \end{aligned}$$

An easy consequence of these formulas is that as $n \to \infty$ we have $A_n \to I/16$, $B_n \to I/2$.
The monic orthogonal polynomials P_n, corresponding to the weight matrix

$$W(t) = t^\alpha(1-t)^\beta(tF_1 + F_0),$$

with

$$F_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$F_1 = \frac{\alpha+\beta+2}{\alpha+1} \begin{pmatrix} 0 & -1 \\ -1 & \frac{\beta-\alpha}{\alpha+1} \end{pmatrix},$$

satisfy not only the recurrence relation (1.3) but they also satisfy

$$\ell_{1,L} P_n^*(t) = P_n^*(t) \begin{pmatrix} n & \frac{\alpha-\beta}{\alpha+1}n \\ 0 & -n-\alpha-\beta-2 \end{pmatrix}$$

with $\ell_{1,L}$ given by

$$\ell_{1,L} = \tfrac{1}{\alpha+\beta+2} \begin{pmatrix} (\alpha+\beta+2)t-(\alpha+1) & \frac{(\alpha^2-\beta^2+2(\alpha-\beta))t-(\alpha+1)^2}{\alpha+1} \\ \alpha+1 & \alpha+1-(\alpha+\beta+2)t \end{pmatrix} D^1 + \begin{pmatrix} 0 & 0 \\ 0 & -(\alpha+\beta+2) \end{pmatrix} D^0.$$

The equation above can be trivially converted into a right hand side differential operator acting on $P_n(t)$.

Remark. To bring these expressions more in line with those that appear in the examples below in section 6, notice that

$$\begin{aligned} \Lambda_n^* &= nP^* + Q^* \\ &\equiv n \begin{pmatrix} 1 & \frac{\alpha-\beta}{\alpha+1} \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha-\beta-2 \end{pmatrix} \end{aligned}$$

and the differential operator is

$$\ell_{1,L} = (tP^* + R)\frac{d}{dt} + Q^*$$

with

$$R = \frac{\alpha+1}{\alpha+\beta+2} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Remark. By working directly on the situation discussed in [15] one arrives at the same differential operator as above and a *diagonal* eigenvalue matrix, namely

$$\Lambda_n = \begin{pmatrix} n & 0 \\ 0 & -n-\alpha-\beta-2 \end{pmatrix}$$

The relation between these two results is a consequence of the following general analysis.

Assume that we have

$$\ell_{1,L} P_n^*(t) = P_n^*(t)\Lambda_n \tag{4.1}$$

and that the family of polynomials $P_n(t)$ are orthogonal with respect to a weight matrix $W(t)$.

If there exists a sequence of nonsingular matrices $(T_n)_{n\geq 0}$ such that for each $n = 0, 1, \ldots$ we have

$$T_n^{-1}\Lambda_n T_n = \Theta_n,$$

then the polynomials

$$Q_n(t) \equiv T_n^* P_n(t)$$

are also orthogonal with respect to $W(t)$ and satisfy

$$\ell_{1,L} Q_n^*(t) = Q_n^*(t)\Theta_n \quad n = 0, 1, 2, \ldots \tag{4.2}$$

It follows that if the sequence of "eigenvalue matrices" Λ_n is made of diagonalizable matrices, we can replace (4.1) by (4.2) without changing the operator $\ell_{1,L}$ or the orthogonality matrix $W(t)$. The new eigenvalue matrices are now **diagonal**.

5 Squaring the first order operator

It is natural to consider squaring the operator $\ell_{1,L}$ in the examples above as well in some extra examples that will appear in the next section. In this fashion we are guaranteed to have a family of orthogonal polynomials satisfying a second order differential equation. The only nontrivial point here is that since we are dealing with operators with matrix valued coefficients, we may encounter some unexpected phenomena. Notice that in the examples above the leading coefficient of the first order differential operator is not a scalar.

For instance, in the Chebyshev example above the leading coefficient of the square of $\ell_{1,L}$ is the scalar $1-t^2$. In the Jacobi type example above, the square of $\ell_{1,L}$ is not the operator that appears in [15], which has a scalar leading coefficient given by $t(1-t)$ but rather it has as leading coefficient the scalar $t(t-(\alpha+1)(3\beta+\alpha))/(\alpha+\beta)(\alpha+\beta+2)$.

In the next section we will display some extra examples where the behavior of the coefficients B_n and A_n is simpler than in the examples above. These examples show an even more surprising behavior when one goes from the appropriate first order differential operator $\ell_{1,L}$ to its square.

6 Some further examples

In this section we consider three examples that are different from the ones above and from each other.

Once again these examples illustrate phenomena that are not present in the scalar case. For instance, as we will see below, the square of the respective first order linear differential operators $\ell_{1,L}$ has a very different behavior in each one of these three cases.

Example 1. In this example we have

$$\begin{aligned} B_n &= r_1 I + r_2 \begin{pmatrix} 0 & -p_{12}^2/p_{11}^2 \\ \\ 1 & 2p_{12}/p_{11} \end{pmatrix}, \\ \\ A_n &= n\zeta I, \end{aligned}$$

with

$$\zeta = \frac{p_{12} r_2^2}{p_{11}^4}\left(p_{11}p_{12}(q_{22}-q_{11}) + p_{11}^2 q_{12} - p_{12}^2 q_{21}\right).$$

The "eigenvalue matrix" Λ_n^* is given by the adjoint of

$$\begin{aligned} \Lambda_n &= nP + Q \\ \\ &\equiv n\begin{pmatrix} p_{11} & p_{12} \\ -p_{11}^2/p_{12} & -p_{11} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \end{aligned}$$

and the monic orthogonal polynomials satisfy the equation

$$\ell_{1,L} P_n^*(t) = P_n^*(t)\Lambda_n^*$$

with

$$\ell_{1,L} = (tP^* + R)\,\frac{d}{dt} + Q^*$$

and

$$R = r_1 \begin{pmatrix} -p_{11} & p_{11}^2/p_{12} \\ -p_{12} & p_{11} \end{pmatrix} + \frac{r_2}{p_{11}^2} \begin{pmatrix} -(p_{12}^2 q_{21} + p_{11}^2 q_{12} + p_{11}^2 p_{12}) & -p_{11}^2(q_{22} - q_{11} - p_{11}) + 2p_{11}p_{12}q_{21} \\ -p_{12}(p_{12}q_{22} - q_{11} + p_{11}) - 2p_{11}q_{12} & p_{12}^2 q_{21} + p_{11}^2 q_{12} + p_{11}^2 p_{12} \end{pmatrix}.$$

In this case the square of $\ell_{1,L}$ is an operator of order two, but its leading coefficient is scalar and independent of the variable t.

We will return to this example in the last section of the paper.

Example 2. In this example we have

$$\begin{aligned} B_n &= r_1 I + r_2 \begin{pmatrix} -2a/b & -a^2/b^2 \\ 1 & 0 \end{pmatrix}, \\ A_n &= (nc + a) \begin{pmatrix} -1 & -a/b \\ b/a & 1 \end{pmatrix}, \end{aligned}$$

Note that the parameter c does not appear in the expression of the first order differential operator $\ell_{1,L}$ below. Note also that A_n is singular.

The eigenvalue matrix is given by the transpose of

$$\begin{aligned} \Lambda_n &= nP + Q \\ &\equiv n \begin{pmatrix} p_{11} & p_{12} \\ -p_{12}\frac{b^2}{a^2} & p_{11} - 2p_{12}\frac{b}{a} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ \frac{b}{a^2}(2p_{11}a - q_{12}b - 2p_{12}b) & \frac{1}{a}((q_{11} + 2p_{11})a - 2(q_{12} + p_{12})b) \end{pmatrix} \end{aligned}$$

The monic orthogonal polynomials satisfy

$$\ell_{1,L} P_n^*(t) = P_n^*(t)\Lambda_n^*$$

with

$$\ell_{1,L} = (tP^* + R)\,\frac{d}{dt} + Q^*$$

and R given by

$$R = r_1 \begin{pmatrix} -p_{11} & p_{12}\dfrac{b^2}{a^2} \\ -p_{12} & -p_{11} + 2p_{12}\dfrac{b}{a} \end{pmatrix} + r_2 \begin{pmatrix} p_{12} & p_{11} - 2p_{12}\dfrac{b}{a} \\ 2p_{12}\dfrac{a}{b} - p_{11}\dfrac{a^2}{b^2} & 2p_{11}\dfrac{a}{b} - 3p_{12} \end{pmatrix}.$$

In this case the square of $\ell_{1,L}$ is a second order differential operator with a quadratic matrix polynomial as a leading coefficient. We observe that the search considered in [6], [7] deals mainly with cases where the second order differential operator has a scalar as a leading coefficient. On the other hand, the family of examples that can be obtained from [14] shows that there are naturally arising examples when the leading coefficient of the second order differential operator can be **nonscalar**, as in this example.

Example 3. In this example we have

$$\begin{aligned} B_n &= r_1 I + r_2 \begin{pmatrix} -2p_{12}/p_{11} & -p_{12}^2/p_{11}^2 \\ 1 & 0 \end{pmatrix}, \\ A_n &= c \begin{pmatrix} -p_{12}/p_{11} & -p_{12}^2/p_{11}^2 \\ 1 & p_{12}/p_{11} \end{pmatrix}, \end{aligned}$$

Note that the parameter c does not appear in the expression of the first order differential operator $\ell_{1,L}$ below. Note also that A_n is singular.

The eigenvalue matrix is given by the transpose of

$$\begin{aligned} \Lambda_n &= nP + Q \\ &\equiv n \begin{pmatrix} p_{11} & p_{12} \\ -\dfrac{p_{11}^2}{p_{12}} & -p_{11} \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ -\dfrac{p_{11}^2 q_{12}}{p_{12}^2} & q_{11} - 2\dfrac{p_{11}q_{12}}{p_{12}} \end{pmatrix} \end{aligned}$$

The monic orthogonal polynomials satisfy

$$\ell_{1,L} P_n^*(t) = P_n^*(t)\Lambda_n^*$$

where $\ell_{1,L}$ is given by

$$\ell_{1,L} = (tP^* + R)\frac{d}{dt} + Q^*$$

and the expression for the matrix R is

$$R = r_1 \begin{pmatrix} -p_{11} & p_{11}^2/p_{12} \\ -p_{12} & p_{11} \end{pmatrix} + r_2 \begin{pmatrix} p_{12} & -p_{11} \\ p_{12}^2/p_{11} & -p_{12} \end{pmatrix}.$$

In this case the square of $\ell_{1,L}$ is not an operator of order two.

7 A further look at example 1

In this last section we make a few comments regarding the search for an orthogonality matrix going with example 1 above. One could tackle the same situation for examples 2 and 3.

First we exhibit a matrix valued weight function $W(t)$ with the following orthogonality property: if $P_i(t)$ and $P_j(t)$ are monic polynomials corresponding to example 1 in the previous section, with $i \neq j$, then their inner product

$$\langle P_i, P_j \rangle = \int P_i(t) W(t) P_j^*(t) dt, \tag{7.1}$$

gives the zero matrix. This is just the way things should be.

The weight matrix $W(t)$ is given by

$$W(t) = e^{-t^2} \begin{pmatrix} a_1 + a_2 t & b_1 + b_2 t \\ b_1 + b_2 t & c_1 + c_2 t \end{pmatrix},$$

where $a_1,\ a_2,\ b_1,\ b_2,\ c_1$ and c_2 are for the time being, arbitrary constants. Notice that this matrix can not be positive definite for all real t.

To obtain the "orthogonality" alluded to above, we need to make the following choices

$$\begin{aligned}
p_{12} &= -p_{11} r_1 / r_2 \\
q_{21} &= \frac{r_2}{2r_1^3}(2q_{12} r_1 r_2 - 2q_{22} r_1^2 + 2q_{11} r_1^2 + p_{11}) \\
a_2 &= \frac{2r_1^2}{r_2^2}(b_1 r_2 - c_1 r_1) \\
b_2 &= \frac{2r_1}{r_2}(b_1 r_2 - c_1 r_1) \\
c_2 &= 2(b_1 r_2 - c_1 r_1) \\
a_1 &= \frac{r_1}{r_2^2}(2b_1 r_2 - c_1 r_1)
\end{aligned}$$

The only trouble here is that if $i = j$ we do not get a positive definite matrix for the corresponding "length" of P_i. This is not surprising since, as we noticed earlier, the weight matrix $W(t)$ is not positive definite for all t.

This leaves open the possibility that one could find some positive definite orthogonality matrix in this example. We will see that this is not the case.

By (1.5) in the Introduction, this positivity would require the existence of a nonsingular matrix D, such that

$$B_0 D D^* \quad \text{is Hermitian} \tag{7.2}$$

in our case this reduces to finding a matrix $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ such that the product

$$\begin{pmatrix} 0 & -p_{12}^2/p_{11}^2 \\ 1 & 2p_{12}/p_{11} \end{pmatrix} DD^*$$

should be Hermitian.

An explicit computation shows that this amounts to the requirement that

$$(d_{11}, d_{12}) + \frac{p_{12}}{p_{11}}(d_{21}, d_{22}) = (0, 0),$$

where (d_{11}, d_{12}) and (d_{21}, d_{22}) are the rows of D. This makes D singular and shows that (7.2) can not be satisfied.

Now that we know that example 1 can not go with a positive definite matrix valued measure we close this paper by mentioning a general method that should give nice results when such a positive measure exists. We then observe how this general method fails in this case.

A possible strategy for determining the orthogonality measure in a case when the orthogonal polynomials satisfy a first order differential equation as above is as follows: square the differential operator $\ell_{1,L}$, and hope that the resulting second order differential operator is **symmetric** with respect to some positive definite matrix valued $W(t)$. If this is the case then the differential equations (and appropriate boundary conditions) derived in [7, Section 3] and [15, Section 4] could be useful to determine $W(t)$. Notice that the coefficient matrices $A_2(t)$, $A_1(t)$ and $A_0(t)$ are already determined from $\ell_{1,L}$. So much for the general method; in the next example we see this method can not handle some of the examples discussed above.

We can see that in example 1 if, for instance, we have $q_{12} = q_{21}$ and $q_{11} = -q_{22}$ then the solution to the differential equations mentioned earlier is given by the matrix

$$W(t) = \begin{pmatrix} w_{11}(t) & w_{21}(t) \\ & \\ w_{21}(t) & -\dfrac{p_{11}(2p_{12}w_{21} + p_{11}w_{11})}{p_{12}^2} \end{pmatrix},$$

where $w_{11}(t)$ and $w_{21}(t)$ are so far arbitrary.

Since the determinant of this matrix is $-(p_{11}w_{11}+p_{12}w_{21})^2/p_{12}^2$ it **fails to be positive definite**.

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References

[1] Aptekarev A, Kaliaguine V and Van Iseghem J, The genetic sums' representation for the moments of a sum of Stieltjes functions and its application, *Constr. Approx.* **16** (2000), 487–524.

[2] Bochner S, Über Sturm–Liouvillesche Polynomsysteme, *Math. Z.* **29** (1929), 730–736.

[3] Beckermann B and Castro M, On the determinacy of complex Jacobi matrices, *Publication ANO 446 Université de Lille 1* (2002). Math. Scand., to appear.

[4] Berezans'kii Ju M, Expansions in Eigenfunctions of Selfadjoint Operators, *Translations of Mathematical Monographs* **17**, American Mathematical Society, Rhode Island, 1968.

[5] Duistermaat J J and Grünbaum F A, Differential equations in the spectral parameter, *Comm. Math. Phys.* **103** (1986), 177–240.

[6] Durán A J, Matrix inner product having a matrix symmetric second order differential opeator, *Rocky Mountain J. Math.* **27** (1997), 585–600.

[7] Durán A J and Grünbaum F A, Orthogonal matrix polynomials satisfying second order differential equations, *Internat. Math. Research Notices*, **2004 : 10** (2004), 461–484.

[8] Durán A J and Grünbaum F A, A survey on orthogonal matrix polynomials satisfying second order differential equations, *J. Comput. Appl. Math.*, to appear.

[9] Durán A J and López P, Orthogonal matrix polynomials: zeros and Blumenthal's theorem, *J. Approx. Theory* **84** (1996), 96–118.

[10] Durán A J and López P, Orthogonal matrix polynomials, in Advances in the Theory of Special Functions and Orthogonal Polynomials, Laredo's SIAG Lecture Notes, Editors: R. Álvarez-Nodarse et al., Nova Science Publishers, **1**, 2003, 11–43.

[11] Durán A J and Van Assche W, Orthogonal matrix polynomials and higher order recurrence relations, *Linear Algebra Appl.* **219** (1995), 261–280.

[12] Grünbaum F A and Haine L, A theorem of Bochner revisited, in Algebraic Aspects of Integrable Systems, Editors: Fokas A S and Gelfand I M, *Progr. Nonlinear Differential Equations* **26**, Birkhäuser, Boston, 1997, 143–172.

[13] Grünbaum F A and Iliev P, A noncommutative version of the bispectral problem, *J. Comput. and Appl. Math.* **161** (2003), 99–118.

[14] Grünbaum F A, Pacharoni I and Tirao J A, Matrix valued spherical functions associated to the complex projective plane, *J. Funct. Anal.* **188** (2002), 350–441.

[15] Grünbaum F A, Pacharoni I and Tirao J A, Matrix valued orthogonal polynomials of the Jacobi type, *Indag. Math.* **14** (2003), 353–366.

[16] Krein M G, Fundamental aspects of the representation theory of hermitian operators with deficiency index (m, m), AMS Translations, Series 2, **97**, Providence, Rhode Island (1971), 75–143.

[17] Krein M G, Infinite J-matrices and a matrix moment problem, *Dokl. Akad. Nauk SSSR* **69** (1949), 125–128.

[18] Sinap A and Van Assche W, Orthogonal matrix polynomials and applications, *J. Comput. Appl. Math.* **66** (1996), 27–52.