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Hopf Bifurcation Analysis of a Three-stage-structured Prey-predator System with Multi-delays

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Abstract—**A three-stage-structured prey-predator model with multi-delays is considered. The characteristic equations and local stability of the equilibrium are analyzed, and the conditions for the positive equilibrium occurring Hopf bifurcation are obtained by applying the theorem of Hopf bifurcation. Finally, numerical examples and brief conclusion are given.**

Keywords-three-stage-structured; prey-predator system; multi-delays; Hopf bifurcation

I. INTRODUCTION

In the natural world, there are many species whose individual members have a life history that takes them through two stages|immature and mature. Aiello and Freedman [1] introduced a single-species stage-structured model with time delay in 1990, and they demonstrated the existence and uniqueness of positive equilibrium of the model, which is globally asymptotically stable. Two-stage-structured models have received much attention in the last 20 years [2-4]. In these papers, the authors assume that the life history of each population is divided into distinctive stages: the immature and mature members of the population, where only the mature member can reproduce themselves. However, in the nature many species go through three life stages: immature, mature and old. For example, many female animals lose reproductive ability when they are old.

A three-stage-structured single-species growth model was studied by S.J. Gao [5], the conditions for stability of equilibrium and the sufficient conditions for the existence of a globally asymptotically stable of positive equilibrium of the model are obtained. However, delays play an important role in the dynamics of populations. Naturally occurring complex dynamics are often naturally generated by well formulated delay differential equations (DDE) models. In many processes of the real word, especially, in many biological phenomena, the present dynamics, the present rate of change of the state variables depends not only on the present state of the processes but also on the history of the phenomenon, i. e. on past values of the state variables. A delayed single species with three life history stage and cannibalism model have considered by S.J. Gao [6] and it is shown that the stability of the positive equilibrium can change a finite number of times at most as time delay is increased for some parameter values.

But, there are few papers study the three-stage-structured predator-prey system with delays. A nonautonomous three-stage-structured predator-prey system with time delay have considered by S.J. Yang and B. Shi [7], and the existence of a positive periodic solution is obtained, by using the continuation theorem of coincidence degree theory. Recently, a three-stage-structured prey-predator system with predator density dependent is studied by S.Y. Li and X.G. Xue [8], the conditions for the positive equilibrium occurring Hopf bifurcation is obtained and numerical examples are given.

In this paper, we consider following three-stage-structured prey-predator model with multi-delays

$$\begin{cases} x_1'(t)=\alpha x_2(t)-(\gamma_1+\Omega)x_1(t)-\eta x_1^2(t)-Ex_1(t)y(t), \\ x_2'(t)=\Omega x_1(t)-(\theta_1+a)x_2(t), \\ x_3'(t)=ax_2(t)-bx_3(t), \\ y'(t)=kEy(t-\tau)x_1(t-\tau)-dy(t)-fy(t)y(t-\tau), \end{cases} \tag{1}$$

where $x_i(t)(i=1,2,3)$ are the densities of immature preys, mature preys and old preys at time t, $y(t)$ is the density of predator at time t, respectively. All of the parameters are positive, α is the birth rate of mature population, and γ_1,θ_1,b are the death rate of immature, mature and old prey population, respectively. Ω and a are the maturity rate and ageing rate of the prey population, respectively. η and f are the density dependent coefficients of immature prey population and predator population, respectively. $k(0<k<1)$ is the rate of conversing prey into predator and E is the predation coefficient. τ is the desinty dependent delay and gestation delay for predator population.

Note that in (1), the first and the second equations are independent of the third equation, the asymptotic behavior of $x_3(t)$ is dependent on that of $x_2(t)$. Therefore, we just need to study following subsystem

$$\begin{cases} x_1'(t)=\alpha x_2(t)-\gamma x_1(t)-\eta x_1^2(t)-Ex_1(t)y(t), \\ x_2'(t)=\Omega x_1(t)-\theta x_2(t), \\ y'(t)=kEy(t-\tau)x_1(t-\tau)-dy(t)-fy(t)y(t-\tau), \end{cases} \tag{2}$$

where $\gamma=\gamma_1+\Omega,\theta=\theta_1+a$, The initial conditions for (2) are

$x_1(t)=\phi_1(t)\ge 0, x_2(t)=\phi_2(t)\ge 0, y(t)=\phi_3(t)\ge 0, t\in[-\tau,0]$.

II. STBILITY ANALYSIS AND HOPF BIFURCATION

A. Local Stability Analysis

Obviously, (2) has two boundary equilibrium $E_0=(0,0,0)$, $E_1(x_1,x_2,0)$ (if condition C_1 holds), and a unique positive equilibrium $E_2(x_1^*,x_2^*,y^*)$ (if condition C_2 holds), where $C_1:\alpha\Omega-\gamma\theta>0$, $C_2:kEx_1^*-d>0$,

$$x_1=\frac{\alpha\Omega-\gamma\theta}{\eta\theta},\ x_2=\frac{\Omega}{\theta}x_1,$$

$$x_1^*=\frac{f(\alpha\Omega-\gamma\theta)+dE\theta}{\theta(kE^2+\eta f)},\ x_2^*=\frac{\Omega}{\theta}x_1^*,\ y^*=\frac{kEx_1^*-d}{f},$$

Let $\bar{E}=(\bar{x}_1,\bar{x}_2,\bar{y})$ be any arbitrary equilibrium. The linearized equations about $\bar{E}$ are

$$\begin{pmatrix}x_1'(t)\\x_2'(t)\\y'(t)\end{pmatrix}=A\begin{pmatrix}x_1(t)\\x_2(t)\\y(t)\end{pmatrix}+B\begin{pmatrix}x_1(t-\tau)\\x_2(t-\tau)\\y(t-\tau)\end{pmatrix} \quad (3)$$

where

$$A=\begin{pmatrix}-\gamma-2\eta\bar{x}_1-\bar{y}E & \alpha & -\bar{x}_1E\\ \Omega & -\theta & 0\\ 0 & 0 & -d-\bar{y}f\end{pmatrix},$$

$$B=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ k\bar{y}E & 0 & k\bar{x}_1E-\bar{y}f\end{pmatrix},$$

and the characteristic equation about $\bar{E}$ is given by

$$\det(A+Be^{-\lambda\tau}-\lambda I)=0 \quad (4)$$

(i) From (4), the characteristic equation about E_0 is given by

$$\det\begin{pmatrix}-\gamma-\lambda & \alpha & 0\\ \Omega & -\theta-\lambda & 0\\ 0 & 0 & -d-\lambda\end{pmatrix}=0.$$

Namely

$$(\lambda+d)[\lambda^2+(\gamma+\theta)\lambda+\gamma\theta-\alpha\Omega]=0.$$

Then $\lambda_3=-d<0$, and λ_1,λ_2 are the two other roots of $\lambda^2+(\gamma+\theta)\lambda+\gamma\theta-\alpha\Omega=0$. By Routh-Hurwitz criterion, E_0 is local stable if $\gamma\theta>\alpha\Omega$, local unstable if $\gamma\theta<\alpha\Omega$ and E_1 exist.

(ii) From (4), the characteristic equation about E_1 is given by

$$\det\begin{pmatrix}-\gamma-2\eta x_1-\lambda & \alpha & -x_1E\\ \Omega & -\theta-\lambda & 0\\ 0 & 0 & kEx_1e^{-\lambda\tau}-d-\lambda\end{pmatrix}=0,$$

Namely

$$(\lambda+d-kEx_1e^{-\lambda\tau})[\lambda^2+(\gamma+\theta+2\eta x_1)\lambda+\theta\eta x_1]=0.$$

Then, λ_1,λ_2 are the two roots of

$$\lambda^2+(\gamma+\theta+2\eta x_1)\lambda+\theta\eta x_1=0,$$

with negative real parts, and λ_3 is the root of $\lambda+d-kEx_1e^{-\lambda\tau}=0$, then E_1 is local stable if $kEx_1<d$, local unstable if $kEx_1>d$ and E_2 exist.

From (i) and (ii), we have the following result.

Theorem 1. (i) E_0 is local stable if $\gamma\theta>\alpha\Omega$, local unstable if $\gamma\theta<\alpha\Omega$ and E_1 exist.

(ii) E_1 is local stable if $kEx_1<d$, local unstable if $kEx_1>d$ and E_2 exist.

B. Existence of Hopf Bifurcation

The characteristic equation about the positive equilibrium E_2 is given by

$$\det\begin{pmatrix}-\gamma-2\eta x_1^*-Ey^*-\lambda & \alpha & -Ex_1^*\\ \Omega & -\theta-\lambda & 0\\ kEy^*e^{-\lambda\tau} & 0 & \bar{A}-\lambda\end{pmatrix}=0.$$

where $\bar{A}=(kEx_1^*-fy^*)e^{-\lambda\tau}-d-fy^*$

Namely

$$D(\lambda,\tau)=M(\lambda)+N(\lambda)e^{-\lambda\tau}=0. \quad (5)$$

where

$$M(\lambda)=\lambda^3+m_2\lambda^2+m_1\lambda+m_0,\ N(\lambda)=n_2\lambda^2+n_1\lambda+n_0,$$
$$m_2=\gamma+2\eta x_1^*+Ey^*+\theta,\ n_2=-d,$$
$$m_1=(d+fy^*)(\gamma+2\eta x_1^*+Ey^*)+\theta(\eta x_1^*+d+fy^*),$$
$$n_1=Efy^{*2}-d(\gamma+\theta+2\eta x_1^*),\ m_0=\theta\eta x_1^*(d+fy^*),$$
$$n_0=\theta x_1^*(kE^2y^*-d\eta).$$

When $\tau=0$, (5) becomes to

$$\lambda^3+(m_2+n_2)\lambda^2+(m_1+n_1)\lambda+m_0+n_0=0. \quad (6)$$

and

$$m_2+n_2=\gamma+2\eta x_1^*+Ey^*+\theta+fy^*>0,$$
$$m_1+n_1=x_1^*(\theta\eta+kE^2y^*)+fy^*(\gamma+2\eta x_1^*+Ey^*+\theta)>0,$$
$$m_0+n_0=\theta x_1^*y^*(kE^2+f\eta)>0,$$

Note that

$$(m_2+n_2)(m_1+n_1)-(m_0+n_0)$$
$$>\theta\{[2\eta fx_1^*y^*+Ey^*(d+2fy^*)]-(f\eta+kE^2)x_1^*y^*\}>0$$

By Routh-Hurwits criterion, all roots of (6) have negative real parts. Then, the equilibrium E_2 is local stable.

Suppose $\lambda=\mathrm{i}\omega(\omega>0)$ is a root of (5) and separating the real and imaginary parts, we have

$$\begin{cases}m_2\omega^2-m_0=(n_0-n_2\omega^2)\cos\omega\tau+n_1\omega\sin\omega\tau,\\ \omega^3-m_1\omega=n_1\omega\cos\omega\tau-(n_0-n_2\omega^2)\sin\omega\tau,\end{cases} \quad (7)$$

From (7), one can get that

$$(n_0-n_2\omega^2)^2+n_1^2\omega^2=(m_2\omega^2-m_0)^2+(\omega^3-m_1\omega)^2.$$

Namely

$$\omega^6+p\omega^4+q\omega^2+r=0, \quad (8)$$

where

$$p=m_2^2-2m_1-n_2^2>0,$$
$$q=m_1^2+2n_2n_0-n_1^2-2m_2m_0, \quad (9)$$

$$r = m_0^2 - n_0^2 = \theta x_1^*(m_0 + n_0)[\eta(2d + fy^*) - kE^2 y^*], \quad (10)$$

If $C_3 : \eta(2d + fy^*) < kE^2 y^*$ hold, from (10) we know that (8) has least one positive root . From (7), we have

$$\cos\omega_0\tau = \frac{(m_2\omega_0^2 - m_0)(n_0 - n_2\omega_0^2) + n_1\omega_0(\omega_0^3 - m_1\omega_0)}{(n_0 - n_2\omega_0^2)^2 + (n_1\omega_0)^2},$$

Thus

$$\tau_n = \frac{1}{\omega_0}\cos^{-1}\left[\frac{(m_2\omega_0^2 - m_0)(n_0 - n_2\omega_0^2) + n_1\omega_0(\omega_0^3 - m_1\omega_0)}{(n_0 - n_2\omega_0^2)^2 + (n_1\omega_0)^2}\right] + \frac{2n\pi}{\omega_0}, n = 0,1,2,\cdots \quad (11)$$

Let $\lambda(\tau) = v(\tau) + \mathrm{i}\omega(\tau)$ be the roots of (5) such that when $\tau = \tau_n$ satisfying $v(\tau_n) = 0$ and $\omega(\tau_n) = \omega_0$. We can claim that

$$\left.\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\tau}\right|_{\tau=\tau_0} > 0.$$

In fact, differentiating two sides of (5) with respect to τ, we get

$$[(3\lambda^2 + 2m_2\lambda + m_1) + e^{-\lambda\tau}(2n_2\lambda + n_1) - e^{-\lambda\tau}\tau(n_2\lambda^2 + n_1\lambda + n_0)] = \lambda(n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}\mathrm{d}\lambda / \mathrm{d}\tau$$

then

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{3\lambda^2 + 2m_2\lambda + m_1}{\lambda(n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}} + \frac{2n_2\lambda + n_1}{\lambda(n_2\lambda^2 + n_1\lambda + n_0)} - \frac{\tau}{\lambda}$$

Therefore

$$\mathrm{sign}\left[\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\tau}\right]_{\lambda=\mathrm{i}\omega_0} = \mathrm{sign}\left[\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right]_{\lambda=\mathrm{i}\omega_0}$$

$$= \frac{1}{\omega_0^2}\mathrm{sign}\left\{\mathrm{Re}\left[\frac{(m_0 + m_2\omega_0^2) + 2\omega_0^3\mathrm{i}}{(m_2\omega_0^2 - m_0) + (\omega_0^3 - m_1\omega_0)\mathrm{i}} + \frac{n_2\omega_0^2 + n_0}{(n_0 - n_2\omega_0^2) + n_1\omega_0\mathrm{i}}\right]\right\}$$

$$= \frac{1}{\Psi}\mathrm{sign}\left[\begin{array}{l}(m_2\omega_0^2 - m_0)(m_0 + m_2\omega_0^2) + 2\omega_0^3(\omega_0^3 - m_1\omega_0) \\ +(n_0 - n_2\omega_0^2)(n_0 + n_2\omega_0^2)\end{array}\right]$$

$$= \frac{1}{\Psi}\mathrm{sign}[2\omega_0^6 + (m_2^2 - 2m_1 - n_2^2)\omega_0^4 - (m_0^2 - n_0^2)],$$

$$= \frac{1}{\Psi}\mathrm{sign}(2\omega_0^6 + p\omega_0^4 - r)$$

where $\Psi = \omega_0^2[(n_0 - n_2^2\omega_0^2)^2 + (n_1\omega_0)^2] > 0$. Since $p > 0$ and $r < 0$, then

$$\mathrm{sign}\left[\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\tau}\right]_{\lambda=\mathrm{i}\omega_0} = 1, \quad \left.\frac{\mathrm{d}(\mathrm{Re}\,\lambda)}{\mathrm{d}\tau}\right|_{\tau=\tau_0} > 0,$$

according to the Hopf bifurcation theorem for functional differential equations [9], we have the following result.

Theorem 2. If $C_3 : \eta(2d + fy^*) < kE^2 y^*$ holds, then (i) There exists a τ_0, when $\tau \in [0, \tau_0)$ the positive equilibrium E_2 of (2) is asymptotically stable and unstable when $\tau > \tau_0$.

(ii) If $C_3' : \eta(2d + fy^*) < kE^2 y^*$ and $q > 0$ holds, system (2) can undergo a Hopf bifurcation at the positive equilibrium E_2 when $\tau = \tau_n (n = 0,1,2,\cdots)$, where τ_n is defined by (11).

Remark 1. It must be pointed out that Theorem 2 can not determine the stability and the direction of bifurcating periodic solutions, that is, the periodic solutions may exists either for $\tau > \tau_0$ or for $\tau < \tau_0$, near τ_0. To determine the stability, direction and other properties of bifurcating periodic solutions, the normal form theory and center manifold argument should be considered [10].

III. Numerical Simulation

We consider following three-stage-structured system with time delay

$$\begin{cases} x_1'(t) = 2.8x_2(t) - 1.5x_1(t) - 0.25x_1^2(t) - 2x_1(t)y(t), \\ x_2'(t) = 1.3x_1(t) - 0.85x_2(t), \\ y'(t) = 1.7y(t-\tau)x_1(t-\tau) - 0.2y(t) - 0.8y(t)y(t-\tau), \end{cases} \quad (12)$$

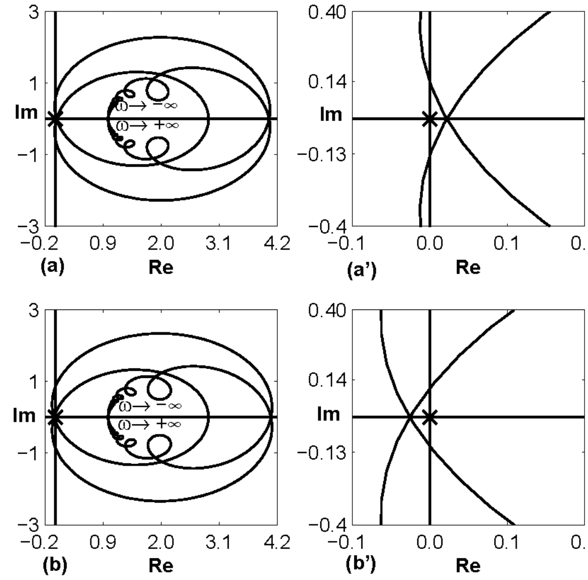

Figure 1. The Nyquist plot of $D(i\omega, 3.7)/(1+i\omega)^3$ and $D(i\omega, 3.8)/(1+i\omega)^3$, show that positive equilibrium point of (12) are asymptotically stable for $\tau = 3.7 < \tau_0$ ((a) and (a)') and unstable for $\tau = 3.8 > \tau_0$ ((b) and (b'))

where $\alpha = 2.8, \gamma_1 = 0.2, \Omega = 1.3, \eta = 0.25, E = 2, \theta_1 = 0.35,$ $a = 0.5, k = 0.85, d = 0.2, f = 0.8, X(0) = (1,1,1)$. System (12) has unique positive equilibrium point $E_2 = (0.73, 1.12, 1.3)$. We evaluate that $p = 29.4, r = \text{-}7.3, \omega_0 = 0.415, \tau_0 \approx 3.75$. The the positive equilibrium point E_2 is asymptotically stable when $\tau = 3.7 < \tau_0$. Because the Nyquist plot [11] of $D(i\omega, 3.7)/(1+i\omega)^3$ does not encircle the origin of the complex plane (Fig.1(a) and Zoom around the origin of the Nyquist plot (a')) and the time-series plot are showed (Fig.2 (a)). When $\tau = 3.8 > \tau_0$, the positive equilibrium point E_2 is unstable. Because the Nyquist plot of $D(i\omega, 3.8)/(1+i\omega)^3$

encircles the origin of the complex plane (Fig.1(b) and Zoom around the origin of the Nyquist plot (b')) and the Hopf bifurcation occurring around the positive equilibrium E_2 are shown (Fig.2 (b)). The bifurcating periodic solution (limit cycle) of (12) are stable when τ from 3.81 to 10 and the amplitudes of period oscillatory are increasing as time delays increased. But, too large time delay would make the population to be die out, because the population very close to zero (Fig.3) as time delay increase to some critical value.

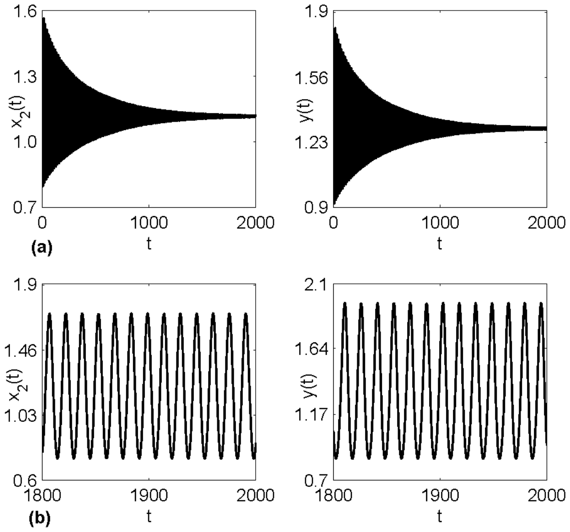

Figure 2. The time-series plot show that positive equilibrium point E_2 of (12) are: **(a)** asymptotically stable for $\tau = 3.7 < \tau_0$ and **(b)** Hopf bifurcation for $\tau = 3.8 > \tau_0$.

IV. CONCLUSION

In this paper, we consodered a multi-delayed three-stage-structured prey-predator system and analyzed the stability and the characteristic equations of the equilibrium, obtained the conditions of the positive equilibrium occurring Hopf bifurcation. Numerical examples by Nyquist plot and time-series plot, shown that the system considered local asymptotically stable when $\tau < \tau_0$, stable Hopf bifurcation periodic solutions when $\tau > \tau_0$ and τ near τ_0. That is to say, time delays can make the positive equilibrium lose stability. It is shown that populations can be coexistence with periodic fluctuating under some conditions and such fluctuation are caused by the time delays. The bifurcating periodic solution (limit cycle) are stable when τ from 3.81 to 10 and the amplitudes of period oscillatory are increasing as time delays increased. But, too large time delay would make the population to be die out, because the population arbitrary close to zero as time delay increase to some critical value. These are very interesting in mathematics and biology.

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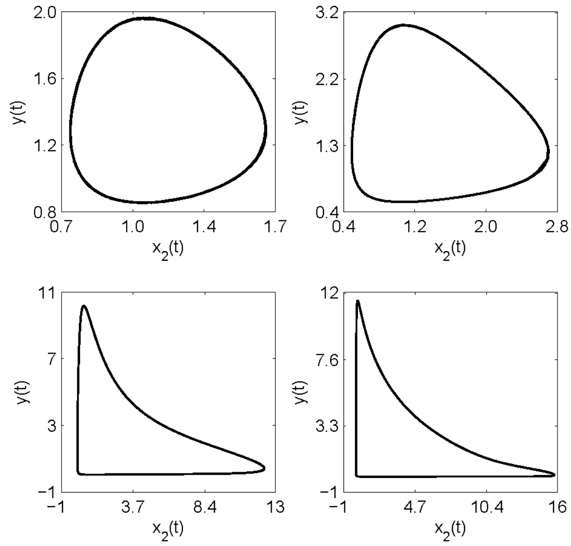

Figure 3. Limit cycles of (12) when $\tau = 3.81, 4, 7, 10$.

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