Asymptotic Scaling in a Model Class of Anomalous Reaction-Diffusion Equations

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Abstract

We analyze asymptotic scaling properties of a model class of anomalous reaction-diffusion (ARD) equations. Numerical experiments show that solutions to these have, for large t, well defined scaling properties. We suggest a general framework to analyze asymptotic symmetry properties; this provides an analytical explanation of the observed asymptotic scaling properties for the considered ARD equations.

1 Introduction

In this note we consider a class [15] of scalar partial differential equations (PDEs) of reaction-diffusion type associated to anomalous diffusion (see e.g. [4] for a recent review focusing on aspects of interest here). This is far from representing the most general anomalous reaction-diffusion (ARD) type of equation, but displays a variety of behaviors common to much more general ARD equations.

Numerical experiments on representatives of this class [15] show that for large t solutions are described by traveling fronts with a well-defined scaling behavior (see below for details); our goal is to provide an analytical explanation for this.

In order to do this we firstly recall standard notions from symmetry analysis of differential equations (Sec. 2), and then extend them to the asymptotic framework (Sec. 3). We are then able to propose a general approach to extract the asymptotic behavior of equations based on maps to equations with known asymptotic symmetry properties (Sec. 4); the basic idea is, in the language of the renormalization group, to identify an equation in the same universality class amenable to asymptotic analysis.

This approach is used to analyze the asymptotic behavior of our class of anomalous reaction-diffusion equations in terms of the known asymptotic behavior of the FKPP (Fisher-Kolmogorov-Petrovskii-Piskunov) equation. Our results provide a sound theoretical explanation of the behavior observed in numerical experiments [15] and recalled below.

2 Symmetries of differential equations

We assume the reader to be familiar with the main concepts and definitions for symmetries of differential equations (see e.g. [10, 17, 19, 21] for general treatments of these), which are to be later extended to asymptotic symmetries. In this section we fix some general notation to be freely used below, and recall general results we need later; as in this note we only consider scalar PDEs, we specialize formulas to this case.

2.1 General notation

We consider an evolution PDE of order n for a real dependent variable, $u = u(x,t)$, with x and t independent real variables. We denote by $M = \mathbf{R}^2 \times \mathbf{R}$ the total space of independent and dependent variables and by $J^{(n)}M$ the jet space of order n over M. Given a function $u = f(x,t)$ we denote its graph as $\gamma_f \subset M$, and its prolongation as $\gamma_f^{(n)} \subset J^{(n)}M$.

Consider a vector field

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} \tag{2.1}$$

in M. For ε sufficiently small the function $u = f(x,t)$ is mapped by $\exp(\varepsilon X)$ into a new function $u = \widetilde{f}(x,t)$; one obtains with standard computations that this new function is given by

$$\widetilde{f} = f + \varepsilon \left[\varphi - \left(\frac{\partial f}{\partial x}\right) \cdot \xi - \left(\frac{\partial f}{\partial t}\right) \cdot \tau \right]_{u=f(x,t)} + o(\varepsilon). \tag{2.2}$$

The action of X in $J^{(n)}M$ is described by its prolongation $X^{(n)}$. We say that X is a *symmetry* of a given equation if it maps any solution into a (generally, different) solution [11, 17, 19, 21]. If the equation is given by $\Phi = 0$, this is equivalent to $[X^{(n)}(\Phi)]_{\Phi=0} = 0$.

Geometrically Δ identifies a *solution manifold* S in $J^{(n)}M$; the function $u = f(x,t)$ is a solution to Δ if and only if $\gamma_f^{(n)} \subset S$, and X is a symmetry of Δ if and only if $X^{(n)} : S \to \mathrm{T}S$.

2.2 Invariant solutions

If Δ, which we write as $\Phi(x,t,u,...) = 0$, is a PDE for $u = u(x,t)$ admitting a vector field X as symmetry, there is a well known procedure to determine X-invariant solutions to Δ; this represents an extension of the familiar method of characteristics to solve quasilinear PDEs (for more details, see e.g. the discussion in Chap. 3 of [17]).[1]

Firstly we pass to symmetry-adapted coordinates (y,v,σ) in M; the (y,v) are X-invariant coordinates, while σ is acted upon by X. We see (y,σ) as independent variables and v as the dependent one; we can then use the chain rule to express x and t derivatives of u in terms of the σ and y derivatives of v and write Δ in terms of the (σ,y,v) coordinates, i.e. in the form $\widehat{\Phi}(\sigma,y,v,...) = 0$. As X is a symmetry of Δ, it follows that the function $\widehat{\Phi}$, when subject to the side condition $\partial v/\partial \sigma = 0$, is independent of σ.

If we are able to determine a solution $v = \widehat{f}(y)$ to the reduced equation, by writing this in terms of the (x,t,u) coordinates we get a X-invariant solution to Δ.

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[1]This method requires a transversality condition, generically satisfied for scalar equations as those we wish to consider; see [1, 13] for discussion and a more general approach.

2.3   Equivalent equations

Consider a map $\chi : M \to M$ which is not a symmetry of $\Delta := \Delta_0$; we write $\chi : (x, t, u) \mapsto (y, s, w)$ and $\chi^{(n)} : \Delta \mapsto \widehat{\Delta}$.

   If Δ is of the form $\Delta \equiv u_t - F(x, t, u, u_x, u_{xx}) = 0$ and χ is projectable [11], i.e. such that $s = s(t)$, $y = y(x, t)$, then $\widehat{\Delta}$ is of the same type: $\widehat{\Delta} \equiv w_s - G(y, s, w, w_y, w_{yy}) = 0$. Moreover, if χ is projectable, it maps $\gamma_f \subset M$ into a manifold which also is the graph of a function $w = g(y, s)$, say $\chi(\gamma_f) = \gamma_g \subset M$; this extends to prolongations, i.e. $\chi^{(2)} : \gamma_f^{(2)} \to \gamma_g^{(2)}$. Hence solutions $u = f(x, t)$ to Δ are mapped into solutions $w = g(y, s)$ to $\widehat{\Delta}$. We say therefore that χ is a *solution preserving* map [20].

   If χ is invertible (with a $\mathcal{C}^n$ inverse), we can repeat these considerations for χ^{-1}. In this case the equations Δ and $\widehat{\Delta}$ of order n are *equivalent*, in that there is a $\mathcal{C}^n$ isomorphism between solutions to Δ and solutions to $\widehat{\Delta}$.

2.4   Conditional and partial symmetries

As we mentioned above, X is a symmetry if it maps any solution to a (generally, different) solution. However, there can be cases for which this is true only for some solution; one can formulate correspondingly weaker notions of symmetry. (For a review of different "extended" notions of symmetry, see [18]; see also [5] for a recent and shorter discussion.)

   We say that X is a *partial symmetry* for Δ if there is a nonempty set $\mathcal{S}_X$ of solutions to Δ which is mapped to itself by X. The set $\mathcal{S}_X$ could be made of a single solution and more generally that there could be solutions which are left invariant by X. In this latter case we say that X is a *conditional symmetry* for Δ.

   It follows from (2.2) that $u = f(x, t)$ is invariant under X if and only if

$$\Delta_X := \varphi[x, t, f(x, t)] - u_x \xi[x, t, f(x, t)] - u_t \tau[x, t, f(x, t)] = 0; \qquad (2.3)$$

thus X-invariant solutions to Δ are solutions to the system made of Δ and Δ_X. In other words X is a conditional symmetry of Δ if and only if it is a symmetry of this system (and is not a proper symmetry of Δ).

   Partial symmetries can be seen in a similar way. As was discussed in [7], a function f in the globally invariant set of solutions to $\Delta : \Phi = 0$ ($f \in \mathcal{S}_X$ in the notation used above) is a solution to a system

$$\Phi^{(0)} = 0 , \quad \Phi^{(1)} = 0 , \quad ...... , \quad \Phi^{(p)} = 0,$$

where $\Phi^{(0)} \equiv \Phi$ and $\Phi^{(k+1)} := Y[\Phi^{(k)}]$. The integer p, i.e. the order of the system, is determined as the lowest order such that $\Phi^{(p)}$ vanishes identically on common solutions to all the previous equations. Each equation $\Phi^{(k)} = 0$ can, and should, be simplified by taking into account the previous equations; for concrete examples, see [7].

   If Δ and $\widehat{\Delta}$ are equivalent via a solution-preserving map χ, this entails a corresponding relation between their conditional and/or partial (as well as ordinary) symmetries.

3   Asymptotic symmetries of PDEs

The concept of symmetry of a differential equation can be extended to that of asymptotic symmetry; we again confine ourselves to scalar second-order evolution PDEs. Asymp-

totic symmetries (in the present sense) were introduced and discussed in quite a general framework in [10], see also [6].

As is suggested by their names, asymptotic symmetries of a differential equation Δ are vector fields X which, albeit in general not symmetries of Δ, satisfy $X^{(2)} : S_\Delta \to \mathrm{T} S_\Delta$ asymptotically (see below for the precise sense of this). Any exact symmetry is also (trivially) an asymptotic symmetry.

We write the equation Δ in the form

$$\Phi \equiv u_t - F(x, t, u, u_x, u_{xx}) = 0 \tag{3.1}$$

and denote by $\mathcal{F}$ the space of maps $F : (x, t, u, u_x, u_{xx}) \to \mathbf{R}$ which are polynomial in (u, u_x, u_{xx}) and rational in (x, t). Note that this space, which corresponds to the space of equations in the form we are considering, is invariant under scaling transformations and under translations.

Vector fields are written as in (2.1). The second prolongation of X is denoted as $Y \equiv X^{(2)}$; as seen above, if X is projectable, then Y acts in $\mathcal{F}$ [11].

We write the flow (in the space of functions $f : \mathbf{R}^2 \to \mathbf{R}$) issued by $f_0(x, t)$ as $f_\lambda(x, t) = \exp[\lambda \widehat{X}]$, with $df_\lambda/d\lambda = \widehat{X}[f_\lambda] = [\varphi - f_x \xi - f_t \tau]_{u=f(x,t)}$, see (2.2). We say that $f_0(x, t)$ is X-invariant if it is a fixed point for the flow of $\widehat{X}$, i.e. if $\widehat{X}[f_0] = 0$.

It may happen that f_0 is not X-invariant, but the flow f_λ is asymptotic to an invariant function f_*, i.e.

$$\lim_{\lambda \to \infty} |f_\lambda(x, t) - f_*(x, t)| = 0, \quad \widehat{X}[f_*] = 0. \tag{3.2}$$

When (3.2) is satisfied, we say that f is *asymptotically X-invariant* under the flow of $\widehat{X}$.

We now consider the action of X, more precisely of Y, on the space of equations in the form (3.1), i.e. in $\mathcal{F}$. We write

$$\Delta_\lambda = e^{\lambda Y} \Delta_0 := \sigma(\lambda) [u_t - F_\lambda(x, u, u_x, u_{xx})] \tag{3.3}$$

for the Y flow issued from $\Delta = \Delta_0$; by construction this satisfies $d\Delta_\lambda/d\lambda = Y(\Delta_\lambda)$. In this way X induces a vector field W in the space $\mathcal{F}$ (W is nothing else than the restriction of Y to $\mathcal{F}$); thus (3.3) is equivalent to $dF/d\lambda = W(F)$.

As was recalled above, X is a symmetry of Δ if and only if $Y : S \to \mathrm{T} S$. This condition is now rephrased in terms of $\mathcal{F}$ by saying that X is a symmetry of Δ_0 given by $\Delta_0 := u_t - F_0(x, t, u, ...)$ if and only if F_0 is a fixed point for the flow of W.

Suppose now that X is not a symmetry of Δ_0, i.e. F_0 is not a fixed point for the flow of W in $\mathcal{F}$, but that the flow F_λ issued by F_0 under $e^{\lambda W}$ satisfies

$$\lim_{\lambda \to \infty} |F_\lambda - F_*| = 0, \ W(F_*) = 0 \ .$$

In this case we say that X is an *asymptotic symmetry* for F_0, i.e. for the equation Δ_0.

By construction and by the invertibility of the map $\exp(\lambda X)$, hence of $\exp(\lambda Y)$, for λ finite, if $u = f(x, t)$ is a solution to Δ_0, then $u = f_\lambda(x, t)$ is a solution to Δ_λ; and conversely, if $u = f_\lambda(x, t)$ be a solution to Δ_λ, then $u = f(x, t)$ is a solution to Δ_0.

In the limit $\lambda \to \infty$ the invertibility of (3.3) fails if the flow goes to a limit point; we can nevertheless still say that solutions to the original equation flow into solutions to the asymptotic equation, provided both limits f_* and Δ_* exist.

If the limit Δ_* exists, it captures the behavior of Δ for large λ, i.e. for large (or small, depending on the sign of a and b) t and $|x|$.

3.1 Solution-preserving maps and asymptotic symmetries

The use of this construction, combining with solution-preserving maps, is the following.

If there is an equation $\widehat{\Delta}$, the asymptotic behavior of which is well understood, such that there exists a solution-preserving map χ with $\chi^{(2)} : \Delta \to \widehat{\Delta}$, we can study the asymptotic behavior of solutions $u(x,t)$ to Δ by means of the asymptotic behavior of solutions $w(y,s)$ to $\widehat{\Delta}$.

If χ has a smooth inverse and W is mapped by $\chi^{(2)}$ into $\widehat{W}$, we can study the flow of W using

$$e^{\lambda W} = [\chi^{(2)}]_{\mathcal{F}}^{-1} \circ e^{\lambda \widehat{W}} \circ \chi^{(2)}. \tag{3.4}$$

Note that (3.4) remains valid also in the limit $\lambda \to \infty$, i.e. for the asymptotic regime.

Needless to say this approach is particularly convenient when the asymptotic behavior of the solutions $w(y,s)$ to $\widehat{\Delta}_0$, i.e. the behavior of solutions to $\widehat{\Delta}_*$, is simple and/or in some way universal (even better if $\widehat{\Delta}_0$ is a fixed point under $\widehat{W}$).

If we denote as $w_*(y,s)$ the limit expression for the solutions to $\widehat{\Delta}_0$, the asymptotic behavior of the solution $u(x,t)$ to Δ_0 is given by

$$u_*(x,t) = \chi^{-1} \left[w_*(y,s) \right]. \tag{3.5}$$

More precisely, considering γ_f and γ_g corresponding to $u = f(x,t)$ and $w = g(y,s)$ (see Sec. 2), and going asymptotically into γ_f^* and γ_g^* respectively, we have

$$\gamma_f^* = \chi^{-1} \left(\gamma_g^* \right). \tag{3.6}$$

3.2 Conditional and partial asymptotic symmetries

It is also possible to consider asymptotic versions of conditional and partial symmetries.

Let the function $u = f_0(x,t)$ (more precisely, the corresponding graph $\gamma_0 \subset M$) flow under $e^{\lambda X}$ to a fixed point $u = f_*(x,t)$ (more precisely, a X-invariant graph $\gamma_* \subset M$) albeit Δ_0 does *not* flow to a fixed point.

In such a situation the solution manifold $S_\lambda \subset J^{(2)}M$ does not go to a limit manifold, but there is a submanifold $S_\lambda^X \subset S_\lambda$, with $\gamma_\lambda \subseteq S_\lambda^X \subset S_\lambda$, which flows to a fixed limit submanifold S_*^X, with $\gamma_* \subseteq S_*^X$. In this case we say that X is a **conditional asymptotic symmetry** for Δ.

The same construction, with suitable and rather obvious modifications, applies for what concerns partial symmetries. Suppose that $\Delta \equiv \Delta^{(0)}$ does not flow to a fixed point under W and consider the equations

$$\Delta^{(1)} := Y[\Delta^{(0)}] \ , \ \dots \ , \ \Delta^{(r)} := Y[\Delta^{(r-1)}]$$

up to an r – if it exists – such that $\Delta^{(r)}$ does admit a fixed point $\Delta_*^{(r)}$ under the W flow, while the $\Delta^{(k)}$ with $k < r$ do not. Then the manifold

$$S_\lambda^{(0)} \cap \dots \cap S_\lambda^{(r)} \ := \ \mathcal{S}_\lambda$$

(with $S_\lambda^{(k)} \subset J^{(2)}M$ the solution manifold for $\Delta_\lambda^{(k)}$) flows to a limit submanifold $\mathcal{S}_*$ and solutions $u = f_0(x,t)$ to the system $\Delta^{(k)}$ $(k = 0, ..., r)$ flow to functions $u = f_*(x,t)$ such

that the prolongation $\gamma_*^{(2)} \subset J^{(2)}M$ of the corresponding graph $\gamma_{f_*} = \{(x, t, f_*(x, t))\}$ lie in $\mathcal{S}_*$. In this case we say that X is a **partial asymptotic symmetry** for Δ.

If Δ and $\widehat{\Delta}$ are related by a solution-preserving map $\chi^{(2)}$, then χ also relates their conditional and partial symmetries. In particular, if χ is invertible and $\widehat{\Delta}$ admits X as a conditional symmetry, then Δ admits $\chi_*^{-1}(X)$ as a conditional symmetry with χ_*^{-1} the extension of χ^{-1} to vector fields. This is of use below.

4 Asymptotic symmetries as a tool to test asymptotic behavior

In many physically relevant cases one has to study nonlinear PDEs which are not amenable to an exact treatment, or at least for which such a treatment is not known, and for which a numerical study shows an asymptotic behavior which appears to be well described by some kind of invariance. Usually the latter corresponds to a scale invariance (self-similar solutions), a translation invariance (travelling waves) or a combination of both of these.

The discussion conducted so far can be implemented into a procedure allowing on the one hand to test if the observed asymptotic behavior is a characteristic of the equation (rather than an artifact of the numerical experiments conducted on it) and on the other hand to formulate simpler equations extracting the asymptotic behavior.

We now describe the procedure in operational terms; as a (rather simple, but relevant) example to illustrate our procedure we consider here the heat equation and then anomalous diffusion equations, while in later sections we apply the procedure on anomalous reaction-diffusion equations.

We denote by X the vector field describing the observed invariance and consider a second-order equation for $u = u(x, t)$ of the form $\Delta := u_t - F(x, t, u, u_x, u_{xx}) = 0$ (hence $M = \{(x, t; u)\}$).

- **Step 1.** Pass to symmetry-adapted coordinates in M, i.e. coordinates $(\sigma, y; v)$ such that $X(y) = X(v) = 0$; thus in these coordinates $X = f(\sigma, y, v)(\partial/\partial\sigma)$. [2]

- **Step 2.** Identify v as the new dependent variable, i.e. $v = v(\sigma, y)$. This allows one to write x and t derivatives of u as σ and y derivatives of v, hence to write the differential equation $\Delta(x, t; u^{(2)})$ as $\widehat{\Delta}(\sigma, y; v^{(2)})$.

- **Step 3.** Reduce the equation $\widehat{\Delta}(\sigma, y; v^{(2)})$ to the space of X-invariant functions, i.e. to $v(\sigma, y)$ satisfying $v_\sigma = 0$. For X an exact symmetry, the reduced equation $\widehat{\Delta}_X$ does not depend on σ at all. For X a conditional or partial symmetry, σ still appears parametrically in the reduced equation.

- **Step 4.** Study the asymptotic behavior of the solutions to the reduced equation $\widehat{\Delta}_X$ for $\sigma \to \infty$.

- **Step 5.** Return to the original variables.

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[2]We stress that we are not requiring $f = 1$; actually when we deal with scaling symmetries it is appropriate to require $f(\sigma, y, v) = \sigma$.

4.1   Elementary example: the heat equation

We illustrate our procedure by applying it to the heat equation $u_t = u_{xx}$. Its asymptotic solutions are of the form

$$u(x,t) = \frac{A}{\sqrt{2t}} \; \exp\left[-\frac{4x^2}{t}\right] \tag{4.1}$$

and are invariant under the scaling vector field

$$X = x\partial_x \; + \; 2t\partial_t \; - \; u\partial_u \; , \tag{4.2}$$

which is also a symmetry of the equation.

The symmetry-adapted coordinates are $\sigma = t$, $y = x^2/t$ and $v = xu$. In these variables, the heat equation is

$$\sigma \, v_\sigma = 4\, y\, v_{yy} + (2+y)\, v_y + v \; . \tag{4.3}$$

Note that (4.3) necessarily requires that for $\sigma \to \infty$, $v_\sigma = 0$: this shows in a simple way that the full asymptotic behavior of (4.3) is captured by (4.1).

When we impose $v_\sigma = 0$ in (4.3), σ disappears completely. Needless to say, the equation obtained in this way has solutions $v(y) = \widehat{A}\sqrt{y}\exp[-y/4]$, which when mapped back to the original coordinates produce the gaussian (4.1).

4.2   Example: anomalous diffusion equations

The procedure described above can also be applied to what is our model class of anomalous diffusion equations and was studied numerically in [15]; these are written as

$$u_t = \frac{x^{1-\alpha/2}}{t^{1-\nu\alpha}} \; \frac{\partial}{\partial x}\left[x^{1-\alpha/2}u_x\right] \; . \tag{4.4}$$

To focus ideas we mention two examples of equations in this class: **(i)** For $\alpha = 2$ we have the generalized gaussian process; **(ii)** For $\alpha = 2/3$, $\nu = 3/2$ we deal with the Richardson equation describing the evolution of the distance between two particles in developed turbulent regime; we refer to [15] for a discussion of the interest of the class of anomalous RD equations (4.4).

One can check that the map

$$s = t^{\alpha\nu} \; , \; y = x^{\alpha/2} \; , \; w = t^{(2-\alpha)(\nu/2)}u \tag{4.5}$$

is solution preserving from (4.4) to the heat equation $w_s = w_{yy}$. Using the inverse change of coordinates, the universal asymptotic solution $w(y,s) \simeq s^{-1/2}e^{-y^2/s}$ of the heat equation is mapped back into

$$u(x,t) \; \simeq \; \frac{1}{t^\nu} \; \exp\left[-x^\alpha/t^\nu\right], \tag{4.6}$$

which represents therefore the universal asymptotic solution to (4.4). This result is confirmed by numerical experiments [15].

5 The FKPP equation: asymptotic solutions, and symmetries

In the same way as in the previous example the heat equation played the role of target for solution-preserving maps applied to anomalous diffusion equations, in the case of (our model class of) anomalous reaction-diffusion (ARD) equations we look for a solution-preserving map to the well known Fisher-Kolmogorov-Petrovskii-Piskunov (FKPP) equation [9, 14, 16]. This is

$$u_t = D u_{xx} + \varepsilon u(1 - u),$$

with ε and D real positive constants, and one requires that $u(x,t) \geq 0$ for all x and t. There are two stationary homogeneous states, i.e. $u = 0$ and $u = 1$; the latter is stable while the former is unstable against small perturbations. In this section we study asymptotic symmetries of the equation and of its solutions.

It is well known [14] that, if the initial datum is suitably concentrated, e.g. $u(x,0) = 0$ for $|x - x_0| > L$ or more generally $u(x,0) < A \exp[-x/L]$, then asymptotically for $t \to \infty$ and $x \to \infty$ the solution is of the form $u = f(x,t) \simeq \exp[-(x - vt)/\lambda]$, with $\lambda = \sqrt{D/\varepsilon}$ and $v = \sqrt{4\varepsilon D}$. This represents a *front* of width λ traveling with speed v; it connects the stable state $u = 1$ and the unstable state $u = 0$.

In the discussion of the FKPP equation it is convenient to pass to rescaled coordinates $\widetilde{t} = \varepsilon t$, $\widetilde{x} = (\sqrt{\varepsilon/D})x$. ¿From now on we use these coordinates and omit the tildas for ease of notation. In these coordinates the FKPP equation is

$$u_t = u_{xx} + u(1 - u). \tag{5.1}$$

As for the asymptotic solution described above, the asymptotic solution is now

$$u = f_0(x,t) \simeq A \exp\left[-(x - 2t)\right]; \tag{5.2}$$

note the front has speed $v = 2$ and width $\lambda = 1$.

It should be stressed that the $f(x,t)$, or $f_0(x,t)$, given above provides the solution for $x \to \infty$, i.e. in the region of small u; in this region (5.1) is well approximated by its linearization around $u = 0$, i.e.

$$u_t = u_{xx} + u; \tag{5.3}$$

the ansatz $u(x,t) = w(z) := w(x - 2t)$ takes this into the ODE

$$w'' + 2w' + w = 0 \tag{5.4}$$

for $w = w(z)$, with solution

$$w(z) = c_1 e^{-z} + c_2 z e^{-z}. \tag{5.5}$$

We denote by $\mathcal{W}$ the set of solutions described by (5.5); note that $\mathcal{W} = \mathbf{R}^2$, and (c_1, c_2) provide coordinates in $\mathcal{W}$.

The f_0 given above, see (5.2), corresponds to $c_2 = 0$. This can be characterized in terms of symmetry properties as discussed below. It is convenient to consider linear combinations of the shifts, given by $X_\pm = X_1 \mp (1/2)X_2$; note that $z = x - 2t$ is invariant under X_- and that $X_+ = \partial_z$. We also have $X_0 = w\partial_w$. Needless to say $[X_0, X_+] = 0$.

Lemma 1. *The symmetry algebra of the linearized equation (5.3) is generated by the scaling $X_0 = u\partial_u$ and by the translations $X_1 = \partial_x$ and $X_2 = \partial_t$. The quotient equation (5.4) admits only X_0 as scaling symmetry; it also admits the translation symmetry generated by X_+, while X_- has been removed by passing to the z variable.*

Proof. This follows from standard (and elementary) computations.   ∎

Lemma 2. *The propagating front solutions correspond to a subspace of $\mathcal{W}$ invariant under the action of the group generated by the vector fields X_0 and X_+.*

Proof. A general element of the group generated by X_0 and X_+ is written as $g(\alpha, \beta) := \exp[\alpha X_0 + \beta X_+]$ and acts on $\mathcal{W}$ by

$$g(\alpha, \beta) : (c_1, c_2) \;\to\; \left(e^{\alpha + \beta}(c_1 + \beta c_2), \; e^{\alpha + \beta} c_2 \right).$$

The subspace $c_2 = 0$ is invariant under this action. As was remarked above, the propagating front solutions (5.2) correspond to $c_2 = 0$.   ∎

It is immediately seen that the only scaling or shift symmetries of the full FKPP equation (5.1) are those, with generators $X_1 = \partial_x$ and $X_2 = \partial_t$, corresponding to translations in x and t; these reflect the fact that (5.1) is a homogeneous equation.

The situation is different for what concerns asymptotic symmetries, and in particular scaling ones, as we now discuss.

Lemma 3. *Let X be a scaling vector field such that $\lim_{\lambda \to \infty} \exp(\lambda X)$ extracts the behavior for large $|x|$ and t. Let Δ_0 be the FKPP equation and $\Delta_\lambda = e^{\lambda Y}\Delta_0$ with Y the prolongation of X. Then $\lim_{\lambda \to \infty} \Delta_\lambda = \Delta_*$ is the heat equation $u_t - u_{xx} = 0$.*

Proof. We consider the most general scaling generator, i.e. a vector field in the form (2.3), $X = ax\partial_x + bt\partial_t + cu\partial_u$. We can always set one of the constants (a, b, c) equal to unity (provided it is nonzero); this amounts to a redefinition of the scaling group parameter.

Applying the procedure described in previous sections, with of course $\Delta_0 := u_t - u_{xx} - u(1 - u) = 0$ the FKPP equation, we obtain at once that

$$\Delta_\lambda = \lambda^{c-b} \left[u_t \,-\, \lambda^{b-2a} u_{xx} \,-\, \lambda^b u \,+\, \lambda^{b+c} u^2 \right]. \tag{5.6}$$

We choose $c = b$ and $a = b/2$. In order for $\lim_{\lambda \to \infty} \exp(\lambda X)$ to extract the behavior for large $|x|$ and t, we must choose $a < 0$, $b < 0$. We can set the modulus of one of the constants, say b for definiteness, equal to unity; i.e. $b = -1$. With these choices we have

$$\Delta_\lambda = u_t \,-\, u_{xx} \,-\, \lambda^{-1} u \,+\, \lambda^{-2} u^2 \;. \tag{5.7}$$

The limit, $\Delta_* := \lim_{\lambda \to \infty} \Delta_\lambda$, is the heat equation $u_t - u_{xx} = 0$, as claimed.   ∎

6 Anomalous reaction-diffusion equations

The FKPP is a (relevant) representative of a more general class of anomalous reaction-diffusion equations which we write as

$$u_t = \widehat{L}[u] + h(u); \tag{6.1}$$

here $\widehat{L}$ is the linear operator describing passive transport of the field u, hence anomalous diffusion, while $h(u)$ describes its growth. Logistic growth, which we consider, corresponds to the choice $h(u) := u(1-u)$. With this choice and considering the $\widehat{L}$ associated to anomalous diffusion in our model class, see the r.h.s. of (4.4), we get the equation

$$u_t = \frac{x^{2-\alpha}}{t^{1-\nu\alpha}} \left[u_{xx} + \frac{(2-\alpha)}{2x} u_x \right] + u(1-u). \tag{6.2}$$

One is usually interested in solutions with initial data $u(x,0)$ which are suitably regular and with compact support. A detailed numerical study of systems of the form (6.2) with such initial data was conducted in [15].

We summarize the findings of these numerical experiments as follows:

- (i) asymptotically for large x and t the solution is described by a travelling front with varying speed $c(t)$ and width $\lambda(t)$; the form of this front for small u is well described by

$$u(x,t) \simeq A \exp\left[-\frac{x - c(t) \cdot t}{\lambda(t)} \right]; \tag{6.3}$$

- (ii) the (asymptotic) time dependencies of $c(t)$ and $\lambda(t)$ are described by

$$c(t) \simeq c_0 \cdot t^\delta, \quad \lambda(t) \simeq \lambda_0 \cdot t^\delta, \tag{6.4}$$

where c_0 and λ_0 are dimensional constants;

- (iii) the scaling exponent δ is given by

$$\delta := \nu + (1/\alpha) - 1. \tag{6.5}$$

Thus we rewrite (6.3) in the form

$$u(x,t) \simeq A \exp\left[-\frac{x - (c_0 t^\delta)t}{\lambda_0 t^\delta} \right]. \tag{6.6}$$

Note that for $\delta = 0$, i.e. for $\nu = 1 - (1/\alpha)$, the front travels with constant speed and width as for the FKPP equation.

We want now to describe precisely the invariance properties of the observed asymptotic solution (6.6), in particular for what concerns scaling transformations.

Lemma 4. *The scaling invariance of (6.6) is described by the generalized scaling group*

$$x \to (\lambda^\delta)\, x\, , \quad t \to \lambda\, t\, , \quad u \to [\exp((\lambda - 1)\, K\, t)]\, u, \tag{6.7}$$

with λ the group parameter.

Proof. The generator of the one-parameter group described by (6.7) is

$$X = \delta\, x\, \partial_x + t\, \partial_t - K t\, u\, \partial_u. \tag{6.8}$$

The invariance of (6.6) under this can be easily checked using (2.2). It can be shown that this is the only scaling type symmetry of (6.6). ∎

Theorem 1. *The vector field (6.8) is not a symmetry of the equation (6.2), but it is an asymptotic symmetry of the same equation.*

Proof. Denote (6.2) by Δ_0 and (6.8) by X; applying $Y \equiv X^{(2)}$ on Δ_0 and restricting to the solution manifold S_0 of Δ_0 (this amounts to substituting for w_σ according to Δ_0 itself), we obtain

$$\Delta_1 := [Y(\Delta_0)]_{S_0} = [(1-\alpha)t^{\alpha\delta}\,(x/t)^{2-\alpha}]\, u_{xx} +$$
$$+ \left[(1/2)(\alpha-1)(\alpha-2)t^{\alpha\delta-1}\,(x/t)^{1-\alpha} \right]\, u_x -$$
$$- u(1 + K - u + Kut).$$

This is not zero, i.e. X is not a symmetry of (6.2).

Going further with our procedure we have to compute $\Delta_2 := [Y(\Delta_1)]_{S_0 \cap S_1}$. It gives

$$\Delta_2 := \left[2 - 4Kt + K^2 t^2 - \alpha(1 - Kt) \right] u^2 - [(1+K)(\alpha-2)]\, u = 0.$$

This has the trivial solution $u = 0$, which is also solution to Δ_0 and Δ_1, and the nontrivial solution

$$u(t) = \frac{(2-\alpha)(1+K)}{(2-\alpha) - (4-\alpha)Kt + K^2 t^2}.$$

The latter, as easily checked by explicit computation, is in general *not* a solution to Δ_0 and Δ_1: inserting this into Δ_0 and Δ_1 we have respectively

$$\widetilde{\Delta_0} = [(\alpha-2)K(1+K)] \left(\frac{Kt^2 + (2K - 4 + \alpha)t + (2\alpha - 6)}{(K^2 t^2 + (\alpha - 4K)t + (2-\alpha))^2} \right),$$
$$\widetilde{\Delta_1} = [(\alpha-2)K(1+K)^2] \left(\frac{Kt^2 - 2t}{(K^2 t^2 + (\alpha - 4K)t + (2-\alpha))^2} \right).$$

For $K \neq 0, -1$, both of these expressions are not zero (unless $\alpha = 2$, corresponding to gaussian processes). However, both of these go to zero (like $1/t^2$) for all α and K in the limit $t \to \infty$. ∎

Having determined that X is an asymptotic (partial) symmetry for our equation Δ_0, we now apply our general procedure. The first step consists in passing to symmetry adapted coordinates; these are

$$\sigma = t\,, \quad y = x/t^\delta\,, \quad w = u e^{Kt}. \tag{6.9a}$$

In these coordinates the vector field (6.8) is simply $X = \sigma \partial_\sigma$ and the (obviously X-invariant) asymptotic solution (6.6) is $w = A \exp[y/\lambda_0]$. With standard computations we obtain

$$
u_t = [w_\sigma - \delta(y/\sigma)w_y - Kw] \ e^{-K\sigma},
$$
$$
u_x = \left[(1/\sigma^\delta) \, w_y\right] \ e^{-K\sigma}, \tag{6.9b}
$$
$$
u_{xx} = \left[(1/\sigma^\delta)^2 \, w_{yy}\right] \ e^{-K\sigma}.
$$

Using (6.9a) and (6.9) we express Δ_0 in the new coordinates; this gives

$$
w_\sigma = \left[\frac{y^{2-\alpha}}{\sigma^\chi}\right] \ w_{yy} + \left[\left(\frac{2-\alpha}{2}\right)\left(\frac{y^{1-\alpha}}{\sigma^\mu}\right) + \alpha\left(\frac{y}{\sigma}\right)\right] \ w_y +
$$
$$
+ (K+1)w - e^{-K\sigma}w^2, \tag{6.10}
$$

where we have defined $\mu = \alpha(\delta - \nu + 1/\alpha)$ for ease of writing.

The expression (6.10) holds for the general map (6.9); however, we are specially interested in the choice $\delta = (\nu - 1 + 1/\alpha)$, see (6.5). With this we have $\mu = (2 - \alpha)$ and finally (6.2) is

$$
w_\sigma = \left(\frac{y}{\sigma}\right)^{2-\alpha} w_{yy} + \left[\left(\frac{2-\alpha}{2\sigma}\right)\left(\frac{y}{\sigma}\right)^{1-\alpha} + \alpha\frac{y}{\sigma}\right] w_y +
$$
$$
+ (K+1)\,w - e^{-K\sigma}\,w^2. \tag{6.11}
$$

In the limit $\sigma \to \infty$ the last term disappears (faster than any power in σ) and (6.11) reduces to a linear equation.

Theorem 2. *The equation (6.11) has no nontrivial X-invariant solutions, but admits nontrivial asymptotically X-invariant solutions.*

Proof. The X-invariant solutions to (6.11) are obtained by requiring that $w_\sigma = 0$; with this the equation reduces to

$$
\left(\frac{y}{\sigma}\right)^{2-\alpha} w_{yy} + \left[\left(\frac{2-\alpha}{2\sigma}\right)\left(\frac{y}{\sigma}\right)^{1-\alpha} + \alpha\frac{y}{\sigma}\right] w_y +
$$
$$
+ (K+1)\,w - e^{-K\sigma}\,w^2 = 0. \tag{6.12}
$$

Note that σ appears parametrically here and (6.12) splits into the equations corresponding to the vanishing of coefficients of different powers of σ (this is a general feature of partial or "weak" symmetries, see [5]). The only common solution to these is $w = 0$, which proves the first part of the statement.

We return to the consideration of (6.11). In order to study its asymptotic behavior for $\sigma \to \infty$, we disregard the term which is exponentially small for large σ. The resulting linear equation for $w = w(y)$, i.e.

$$
\left(\frac{y}{\sigma}\right)^{2-\alpha} w_{yy} + \left[\left(\frac{2-\alpha}{2\sigma}\right)\left(\frac{y}{\sigma}\right)^{1-\alpha} + \alpha\frac{y}{\sigma}\right] w_y + (K+1)\,w \ = \ 0,
$$

yields as solution

$$w(y) = C_1 \, \mathcal{K}[0,0] + \left(\sqrt{2(K+1)\sigma^2(y/\sigma)^\alpha} \right) C_2 \, \mathcal{K}[1/2,1], \tag{6.13}$$

where C_1, C_2 are arbitrary constants and

$$\mathcal{K}[x,y] := F_{11} \left[(K+1)\sigma/\alpha^2 + x; 1/2 + y; -\sigma(y/\sigma)^\alpha \right]$$

with $F_{11} \equiv {}_1F_1$ being the Kummer confluent hypergeometric function

$$F_{11}[a; b; z] := {}_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(b-a)\,\Gamma(a)} \int_0^1 e^{zt}\, t^{a-1}\, (1-t)^{b-a-1}\, \mathrm{d}t.$$

The asymptotic solution could now be expressed in terms of the original variables using (6.9); this yields an explicit but involved and not specially illuminating expression. ∎

7 Other asymptotic partial symmetries of ARD equations

The scaling symmetry (6.8) is not the only symmetry of the observed asymptotic solution (6.6) to (6.2). In this section we identify different symmetries and apply our approach on the basis of these.

Lemma 5. *The vector field $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a symmetry of the asymptotic solution (6.6) to the equation (6.2) if and only if it belongs to the two dimensional module (over smooth real functions $\mathcal{C}^\infty(\mathbf{R}^3, \mathbf{R})$ of x, t, u) generated by*

$$X_1 = \partial_x + \left(\frac{1}{\lambda_0 t^\delta} \right) \partial_u \; ; \; X_2 = \left(\lambda_0 t^{1+\delta} \right) \partial_t - \left(x\delta + c_0 t^{1+\delta} \right) \partial_u. \tag{7.1}$$

Proof. This follows easily by using (2.2) and the explicit expression (6.6) of the asymptotic solution $u = f_*(x,t)$. Indeed applying (2.2) we get

$$\frac{(\widetilde{f} - f)}{\varepsilon} = \varphi - A \exp\left[(c_0 t - xt^{-\delta})/\lambda_0 \right] \frac{(x\delta + c_0 t^{1+\delta})\tau - t\xi}{\lambda_0 t^{1+\delta}},$$

and the result follows immediately. ∎

In the following we consider in particular

$$X_0 := \left(x\delta + c_0 t^{1+\delta} \right) \partial_x + t\partial_t, \tag{7.2}$$

as well as X_1 and X_2 themselves. Note that X_0 operates in the space of independent variables alone.

We write second-prolonged vector fields in the form

$$Y \equiv X^{(2)} = X + \Psi_x \frac{\partial}{\partial u_x} + \Psi_t \frac{\partial}{\partial u_t} + \Psi_{xx} \frac{\partial}{\partial u_{xx}} + \Psi_{xt} \frac{\partial}{\partial u_{xt}} + \Psi_{tt} \frac{\partial}{\partial u_{tt}}; \tag{7.3}$$

in view of (6.2) we actually need only the coefficients $\Psi_x, \Psi_t, \Psi_{xx}$. We will denote (6.2) as Δ_0, as in the previous discussion.

Theorem 3. *The vector fields X_0, X_1 and X_2 are partial symmetries of Δ_0.*

Proof. We denote by Y_i the second prolongations of X_i. We start by considering X_1. In this case the coefficients of the second-prolonged vector field Y_1 are:

$$\Psi_x = 0 \ , \ \Psi_t = -(\delta/\lambda_0)t^{-(1+\delta)}) \ , \ \Psi_{xx} = 0 \ , \ \Psi_{xy} = 0, \Psi_{tt} = (\delta/\lambda_0)(1+\delta)t^{-(2+\delta)}.$$

We define $\Delta_1 = [Y_1(\Delta_0)]_{S_0}$; $\Delta_2 = [Y_1(\Delta_1)]_{S_0 \cap S_1}$. By explicit computation we obtain

$$[Y_1(\Delta_2)]_{S_0 \cap S_1 \cap S_2} = 0;$$

this shows that X_1 is a partial symmetry for Δ_0.

For the other vector fields note that the relevant coefficients of Y_2 are

$$\Psi_x = -\frac{\delta}{\lambda_0 t^{1+\delta}} \ , \ \Psi_t = \frac{(1+\delta)\delta x}{\lambda_0 t^{2+\delta}} \ , \ \Psi_{xx} = 0,$$

and those of Y_0 are

$$\Psi_x = -\delta u_x \ , \ \Psi_t = -(1+\delta)c_0 t^\delta u_x - u_t \ , \ \Psi_{xx} = -2\delta u_{xx}.$$

With the use of these, the theorem follows by explicit computation. ∎

It turns out that reduction and invariant solutions under the vector field X_0 are of special interest. This is due to the following theorem, which provides an analytic explanation of the numerically observed behavior.

Theorem 4. *The equation (6.2) admits an asymptotically X_0-invariant solution described by (6.6).*

Proof. In this case the symmetry-adapted coordinates are

$$\sigma = t \ , \ y = (x/t^\delta) - c_0 t \ , \ w = u;$$

the relevant u derivatives are expressed in the new coordinates as

$$u_t = w_\sigma - \left(\frac{\delta y + c_0(1+\delta)\sigma}{\sigma}\right) w_y \ , \ u_x = \frac{1}{t^\delta} w_y \ , \ u_{xx} = \frac{1}{t^{2\delta}} w_{yy}.$$

In these coordinates the equation (6.2) is

$$w_\sigma = A\, w_{yy} + B\, w_y + f(w), \tag{7.4}$$

with

$$A = \left(\frac{y + c_0\sigma}{\sigma}\right)^{2-\alpha} \ ; \ B = \left(c_0 + \delta\frac{y + c_0\sigma}{\sigma} + \epsilon\left(\frac{y + c_0\sigma}{\sigma}\right)^{2-\alpha}\frac{1}{y + c_0\sigma}\right).$$

The vector field X_0 is simply $X_0 = \sigma\partial_\sigma$; its second prolongation is

$$Y_0 = \sigma\frac{\partial}{\partial\sigma} - w_\sigma\frac{\partial}{\partial w_\sigma} - w_{\sigma y}\frac{\partial}{\partial w_{\sigma y}} - 2w_{\sigma\sigma}\frac{\partial}{\partial w_{\sigma\sigma}} \ .$$

For $\sigma \to \infty$ (7.4) is

$$w_\sigma = c_0^{2-\alpha}\, w_{yy} + c_0(1+\delta)\, w_y + f(w).$$

The X_0-invariant solutions satisfy $w_\sigma = 0$ and are thus obtained as solution to

$$c_0^{2-\alpha} w_{yy} + c_0(1+\delta) w_y + w = 0, \tag{7.5}$$

where we have used $w \ll 1$ in the region we are investigating (i.e. for $\sigma \to \infty$) so that $f(w) \simeq w$.

Solutions to (7.5) are of the form

$$w(y) = c_1 e^{-\omega_+ y} + c_2 e^{-\omega_- y},$$

where

$$\omega_\pm = \frac{(1+\delta)}{2c_0^{1-\alpha}} \left[1 \pm \sqrt{1 - \frac{4}{c_0^\alpha (1+\delta)^2}} \right].$$

If we require the solutions to be non-oscillating, this implies a lower bound on the parameter c_0, i.e. $c_0 \geq (2/(1+\delta))^{2/\alpha}$.

The solution $e^{-\omega_+ z}$ is unstable against small perturbations, while $e^{-\omega_- z}$ is stable [14]. As was proved by Kolmogorov, the asymptotic solution is the stable one with the lowest speed giving nonoscillating behavior, i.e. $c_0 = [2/(1+\delta)]^{2/\alpha}$. This means $w(y) \simeq e^{-\omega_0 y}$ with $\omega_0 = [2/(1+\delta)]^{1-2/\alpha}$. Going back to the original variables we get

$$u(x,t) \;\simeq\; A\, \exp\left[-\frac{x - v(t)t}{\lambda(t)} \right] = A\, \exp\left[-\omega_0 \frac{x - c_0 t^{1+\delta}}{t^\delta} \right].$$

This is precisely the numerically observed asymptotic behavior (6.6).   ∎

8 Conclusions and discussion

We provided suitable definitions of asymptotic symmetries – both in proper and in conditional or partial sense – and proposed a method for the analysis of asymptotic symmetry properties of PDEs and their solutions.

We applied our general method to a model class of anomalous reaction-diffusion (ARD) equations, discussed and studied numerically in [15]. We firstly considered the standard FKPP equation and described in detail its asymptotic symmetry properties; we have also shown that our approach recovers the well known asymptotic properties of FKPP solutions.

We have then tackled general ARD equations in our model class, i.e. with anomalous diffusion associated to (4.5). We recalled the features of asymptotic solutions as observed in numerical experiments and identified the Lie generator X of the observed asymptotic generalized scaling invariance; in Theorem 1 we showed that this is *not* a symmetry, rather an asymptotic symmetry, of the ARD equation. We have then considered the solution-preserving maps associated to this asymptotic scaling symmetry, focusing on the physical value of the parameter δ; in Theorem 2 we have shown that in this case the ARD equation has no solution invariant under X (which therefore is not a conditional symmetry of the

equation), but admits solutions which are *asymptotically* invariant under it, that is, X is an asymptotic conditional symmetry of the ARD equation.

Finally in Sec. 7 we passed to consider in more detail the numerically observed asymptotic solution (6.6). We identified the full symmetry algebra $\mathcal{G}$ of it; vector fields in this algebra are asymptotic conditional symmetries of the ARD equation. We focused in particular on certain vector fields in $\mathcal{G}$ and showed that they are partial symmetries for the ARD equation. Among these vector field is the scaling vector field X_0 given by (7.2) and not depending, nor acting, on the dependent variable u. We proved that the ARD equation does admit an asymptotically X_0-invariant solution, which is precisely the numerically observed one (6.6).

We have thus provided an analytic explanation for the numerically observed behavior based on our general method.

We conclude by presenting some brief final remarks on our method.

(a) This method represents an evolution of the classical method to determine partially invariant solutions for symmetric PDEs [17, 21] and a blend of it with the method of conditional and partial symmetries [7] in order to analyze equations which do not have complete (as opposed to asymptotic) symmetries. Our method is based on the abstract approach developed in [10], based itself on ideas and previous work by several authors [2, 3, 8, 12].

(b) The application of the method to (generalized) scaling symmetries and our model class of ARD equations was greatly facilitated by the form of the vector fields and of the initial equations Δ_0: indeed, the W-accessible part of $\mathcal{F}$ was finite-dimensional.

(c) Our method deals, strictly speaking, with properties which are asymptotic in the group parameter; these correspond to properties asymptotic in time and space only if the considered vector field has favourable properties itself. Also our method does not intend to tackle intermediate asymptotics [2].

(d) This work was concerned only with scale invariance (at infinity or near a travelling front). We trust, however, that suitable generalizations of our approach can also deal with more general asymptotic invariance properties and more general differential equations, and that it is potentially capable to provide a sound explanation – or prediction – of the asymptotic invariance of their solutions.

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