Uniqueness of q-difference Polynomials of Meromorphic Functions

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Abstract—**In this paper, Applying the theory of Nevanlinna, we investigated uniqueness problem of difference polynomial of meromorphic functions and obtained uniqueness theorems of meromorphic functions , which Extended and improved the results of literature[5].**

Keywords-uniqueness of meromorphic functions; q-difference; share value; small functions

I. INTRODUCTION AND MAIN RESULTS

With the development of difference analogue of Nevanlinna theory, many authors paid their attentions to the value distribution of difference polynomials [1-5]. In particular, the difference logarithmic derivative lemma, given by Chiang and Feng [6], Halburd and Korhonen [7], plays an important part in considering the difference analogues of Nevanlinna theory.

In this paper, we assume that reader is familiar with the standard notations and results of Nevanlinna theory, see [8-11].

K.Liu, X.L.Liu, T.B.Cao in [12] got the following resuls.

Theorem A[12] Let f and g be transcendental meromorphic functions of finite order, suppose that c is nonzero constant and $n\in N$. If $n\geq 14$. $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $f\equiv tg$, or $fg=t$, where $t^{n+1}=1$.

Theorem B[12] Let f and g be transcendental meromorphic functions of finite order, suppose that c is nonzero constant and $n\in N$. If $n\geq 26$. $f^n f(z+c)$ and $g^n g(z+c)$ share 1 IM, then $f\equiv tg$, or $fg=t$, where $t^{n+1}=1$.

In this paper, we will investigate the uniqueness of q-difference polynomials and obtain the following theorems.

Theorem 1. If $f(z)$ is a transcendental meromorphic functions of zero order, If $f^n(z)\prod_{i=1}^{d} f(q_i z)$ and $g^n(z)\prod_{i=1}^{d} g(q_i z)$ share $1,\infty$,CM, n,k,m,d are positive integer and $n\geq 4d+4$, then $f=tg, t^{n+d}=1$.

Theorem 2. If $f(z)$ is a transcendental entire functions of zero order, If $f^n(z)(f^m(z)-1)\prod_{i=1}^{d} f(q_i z)$ and $g^n(z)(g^m(z)-1)\prod_{i=1}^{d} g(q_i z)$ share 1 CM, n,k,d,m are positive integer and $n\geq m+5d$, then $f=tg$, $t^{n+d}=t^m=1$.

II. PRELIMINARY LEMMAS

Lemma 1[8] Let f be a non-constant meromorphic function, $\alpha_i (i=1,2,3)$ be small functions with respect to f , then $T(r,f)\leq \sum_{i=1}^{3}\overline{N}(r,\frac{1}{f-a_i})+S(r,f)$

Lemma 2[4] Let f be transcendental meromorphic functions of zero order, $q\in C\setminus\{0\}$, then

$$T(r,f(qz))=T(r,f)+S_q(r,f)$$

Lemma 3[4] Let f be transcendental meromorphic functions of zero order, $q\in C\setminus\{0\}$, then

$$N(r,f(qz))=N(r,f)+S_q(r,f)$$

Lemma 4[11] Let f be transcendental meromorphic functions of zero order, $q\in C\setminus\{0\}$, then

$$m(r,\frac{f(qz)}{f(z)})=S_q(r,f)$$

With the same methods of Lemma 2.4 in [12],we can get the following lemma 5.

Lemma 5 Let $f^n(z)(f^m(z)-1)\prod_{i=1}^{d} f(q_i z)$ .If f be transcendental entire functions of zero order

$$T(r,F)=(n+m+d)T(r,f)+S_q(r,f).$$

If f be transcendental meromorphic functions of zero order, then

$$T(r,F)\geq(n+m-d)T(r,f)+S_q(r,f).$$
$$T(r,F)\leq(n+m+d)T(r,f)+S_q(r,f).$$

III. PROOF OF THEOREM 1

Proof of theorem 1. From the conditions of theorem 1, we know $\dfrac{f^n(z)\prod_{i=1}^{d} f(q_i z)-1}{g^n(z)\prod_{i=1}^{d} g(q_i z)-1}=c$,

c is nonzero constant,so we rewriting it as

$$f^n(z)\prod_{i=1}^{d} f(q_i z)-1+c=cg^n(z)\prod_{i=1}^{d} g(q_i z) \qquad (1)$$

First we let $F=f^n(z)\prod_{i=1}^{d} f(q_i z)$,

$$G=g^n(z)\prod_{i=1}^{d} g(q_i z).$$

If $c\neq1$, From (1) and the lemma 1, we have

$$\begin{aligned}T(r,F)&\leq\overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1+c})\\&+S(r,f)\leq(1+d)T(r,f)+(1+d)T(r,f)\\&+\overline{N}(r,\frac{1}{G})+S(r,f)\\&\leq(2+2d)T(r,f)+(1+d)T(r,g)\\&+S(r,f)\end{aligned} \qquad (2)$$

From the lemma 5, we know

$$T(r,F)\geq(n-d)T(r,f)+S_q(r,f) \qquad (3)$$

Combining (2) and (3), we have

$$\begin{aligned}(n-3d-2)T(r,f)&\leq(1+d)T(r,g)\\&+S_q(r,f)+S_q(r,g)\end{aligned} \qquad (4)$$

Applying the same methods of (4),we have

$$\begin{aligned}(n-3d-2)T(r,g)&\leq(1+d)T(r,f)\\&+S_q(r,f)+S_q(r,g)\end{aligned} \qquad (5)$$

Combining (4) and (5), we have

$$\begin{aligned}(n-4d-3)(T(r,f)+T(r,g))\\\leq S_q(r,f)+S_q(r,g)\end{aligned}$$

Which is a contradicts with $n\geq4d+4$,

Then $c=1$, from (1), we have

$$f^n(z)\prod_{i=1}^{d} f(q_i z)=g^n(z)\prod_{i=1}^{d} g(q_i z).$$

Let $h(z)=\dfrac{f(z)}{g(z)}$, then we have

$h^n(z)\prod_{i=1}^{d} h(q_i z)=1$, so $h^n(z)=\dfrac{1}{\prod_{i=1}^{d} h(q_i z)}$,

So

$$\begin{aligned}nT(r,h^n(z))&=T(r,\prod_{i=1}^{d} h(q_i z))\\&=dT(r,h(z))+S_q(r,h)\end{aligned}$$

Which is a contradicts with $n\geq4d+4$, so $h(z)$ is a constant. Let $h(z)=t$, then $t^{n+d}=1$, we complete the proof of theorem 1.

Proof of theorem 2. From the conditions of theorem 1, we know $\dfrac{f^n(z)(f^m(z)-1)\prod_{i=1}^{d} f(q_i z)-1}{g^n(z)(g^m(z)-1)\prod_{i=1}^{d} g(q_i z)-1}=c$,

c is nonzero constant, so we rewriting it as

$$\begin{aligned}f^n(z)(f^m(z)-1)\prod_{i=1}^{d} f(q_i z)-1+c\\=cg^n(z)(g^m(z)-1)\prod_{i=1}^{d} g(q_i z)\end{aligned} \qquad (6)$$

First we let $F=f^n(z)(f^m(z)-1)\prod_{i=1}^{d} f(q_i z)$,

$$G=g^n(z)(g^m(z)-1)\prod_{i=1}^{d} g(q_i z).$$

If $c\neq1$, From (6) and the lemma 1, we have

$$T(r,F)\le \overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1+c})$$
$$+S(r,f)\le (1+m+d)T(r,f)$$
$$+\overline{N}(r,\frac{1}{G})+S(r,f)$$
$$\le (1+m+d)T(r,f)$$
$$+(1+m+d)T(r,g)+S(r,f) \qquad (7)$$

From the lemma 5, we know

$$T(r,F)=(n+m+d)T(r,f)+S_q(r,f) \qquad (8)$$

Combining (7) and (8), we have

$$(n-1)T(r,f)\le (1+m+d)T(r,g)+S_q(r,f)+S_q(r,g) \qquad (9)$$

Applying the same methods of (9),we have

$$(n-1)T(r,g)\le (1+m+d)T(r,f)+S_q(r,f)+S_q(r,g) \qquad (10)$$

Combining (9) and (10), we have

$$(n-m-d)(T(r,f)+T(r,g))\le S_q(r,f)+S_q(r,g)$$

Which is a contradicts with $n\ge m+5d$.

Then $c=1$, from (6), we have

$$f^n(z)(f^m(z)-1)\prod_{i=1}^{d} f(q_i z)$$
$$=g^n(z)(g^m(z)-1)\prod_{i=1}^{d} g(q_i z)$$

Let $h(z)=\frac{f(z)}{g(z)}$, then

$$g^m(h^{n+m}(z)\prod_{i=1}^{d} h(q_i z)-1)=h^n(z)\prod_{i=1}^{d} h(q_i z)-1,$$

If $h(z)$ is not a constant, then $h(z)$ is meromorphic.

If 1 is exceptional value of $h^{n+m}(z)\prod_{i=1}^{d} h(q_i z)-1$,

Then

$$T(r,h^{n+m}(z)\prod_{i=1}^{d} h(q_i z))\le \overline{N}(r,h^{n+m}(z)\prod_{i=1}^{d} h(q_i z))$$
$$+\overline{N}(r,\frac{1}{h^{n+m}(z)\prod_{i=1}^{d} h(q_i z)})+\overline{N}(r,\frac{1}{h^{n+m}(z)\prod_{i=1}^{d} h(q_i z)-1})$$
$$\le (2+2d)T(r,h)+S_q(r,h) \qquad (11)$$

From (11) ,we have

$$(n+m)T(r,h(z))=T(r,h^{n+m}(z))\le T(r,h^{n+m}(z)\prod_{i=1}^{d} h(q_i z))$$
$$+T(r,\frac{1}{\prod_{i=1}^{d} h(q_i z)})+S_q(r,h)\le (2+3d)T(r,h)+S_q(r,h)$$

Which is a contradicts with $n\ge m+5d$,

So 1 is not exceptional value of $h^{n+m}(z)\prod_{i=1}^{d} h(q_i z)-1$.

So there exists a point z_0 such that

$$h^{n+m}(z_0)\prod_{i=1}^{d} h(q_i z_0)=1$$

Since $g(z)$ is entire, so $h^n(z_0)\prod_{i=1}^{d} h(q_i z_0)=1$,

so $h^m(z_0)=1$, then

$$T(r,h^{n+m}(z)\prod_{i=1}^{d} h(q_i z))\le \overline{N}(r,h^{n+m}(z)\prod_{i=1}^{d} h(q_i z))$$
$$+\overline{N}(r,\frac{1}{h^{n+m}(z)\prod_{i=1}^{d} h(q_i z)})+\overline{N}(r,\frac{1}{h^m(z)-1})$$
$$\le (2+2d+m)T(r,h)+S_q(r,h)$$

From (10) we have

$$(n+m)T(r,h(z))=T(r,h^{n+m}(z))\le T(r,h^{n+m}(z)\prod_{i=1}^{d} h(q_i z))$$
$$+T(r,\frac{1}{\prod_{i=1}^{d} h(q_i z)})+S_q(r,h)\le (2+3d+m)T(r,h)+S_q(r,h)$$

Which is a contradicts with $n\ge m+5d$.

So $h(z)$ is a constant. If $h(z)=t$, then $t^{n+d}=t^m=1$, we complete the proof of theorem 2.

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