Quantum-behaved Particle Swarm Optimization Algorithm for Solving Nonlinear Equations

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Abstract—**A quantum-behaved particle swarm optimization algorithm is presented in this paper for solving nonlinear equations. The positions of particle are coded by probability amplitudes of qubits that are updated by quantum rotation gates in this method. The corresponding real number solution at specified interval can be extracted by this algorithm for solving nonlinear equations. Compared to real traditional method, the simulation results show that this algorithm is more accurate and effective.**

Keywords-Quantum Particle Swarm Optimization, Quantum rotation gates, Nonlinear equations, Function optimization

I. INTRODUCTION

Most of engineering problems can be described by nonlinear equations. Therefore, there is a necessity to investigate such numerical methods in order to provide approximate solutions as well as meeting certain precision. Worldwide researchers have been dedicating to developing more effective methods to solve nonlinear equations. However, these methods can only be used by meeting some rigorous conditions, which also have some drawbacks for the case of solving multi-dimensional complex equations. For instance, Newton method having the second order convergence is an important method for solving nonlinear equations, perhaps you can sentence with red color which demands that initial vector is selected in the adjacent solutions. If the initial vector selection is inaccurate, Newton method may be non-convergence and time costing, causing the local extremism.

Particle Swarm Optimization(PSO) is a new global optimized evolutionary algorithm, which is suggested by American psychologist Kennedy and electrical engineer Eberhart who were inspired by the predatory behavior of birds in 1995[1]. However, in practical application, PSO algorithm has some shortcomings, such as the ease to generate premature convergence, long computational time, and low global optimization capability. In order to overcome such drawbacks, a lot of efforts have been made to improve PSO algorithm, but there is premature convergence phenomenon [2]. Quantum-inspired evolutionary algorithm(QEA) based on the concept and theory of quantum computing [3] is a new probability evolutionary algorithm, obtained from observing the quantum state of chromosomes to generate the needed binary solution. However, it is a probabilistic procedure with a great deal of randomness. In order to avoid the observation of the randomness and frequent decoding from binary to decimal, probability amplitude of qubit is directly expressed as solutions of optimization problems in Ref.[4]. Therefore, a Quantum Particle Swarm Optimization(QPSO) algorithm is presented here by incorporating PSO with QEA. This algorithm directly codes position of particle by probability amplitudes of qubits and updates probability amplitude of qubits by quantum rotation gates to realize update of the particle location, search of the optimal position and the variation of qubits by quantum NOT gate, which can increase the diversity of the population and avoid premature convergence. Because each qubit has two probability amplitudes, each particle occupies both locations of the optimization space. Meanwhile, due to the use of double probability amplitude encoding mechanism, the chance for particles to locate their optimal positions is increased significantly. This algorithm has a strong search capability and optimizing efficiency, especially for complex functions optimization and high-dimensional optimization problems. In this case, solving nonlinear equations is transformed into the function extreme value optimization problems in this paper, and subsequently quantum-behaved particle swarm optimization algorithm is proposed for solving nonlinear equations. This method improves algorithmic optimization speed, local search ability and global optimization ability.llow.

II. QUANTUM CALCULATION

A. Qubits

A qubit is defined as the smallest unit of information in QEA, which can be described by a pair of numbers (α, β) as $[\alpha, \beta]^T$. A qubit may be in the $|0\rangle$ state, in the $|1\rangle$ state, or in any linear superposition of the two. The state of a qubit can be represented as $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$, $|\alpha|^2$ gives the probability of the qubit that will be found in the $|0\rangle$ state and $|\beta\rangle$ gives the probability of the qubit that will be found in the $|1\rangle$ state. Therefore, by probability amplitude qubits can be expressed as $|\phi\rangle = [\alpha, \beta]^T$.

B. Quantum Gates

In quantum calculation, we can achieve some logic functions by carrying out a series of unitary transformation with quantum state. Since the role of the transformation is equivalent to the role of logic gates, remove so quantum gate is quantum device of logical transformation within a certain time interval. Quantum gate is the basis of physics implement quantum computing, and contains the characteristics of quantum computing. There is existing proof [5] that the quantum gates contain universal quantum gates groups, which is similar to the commonality of NAND gate of classic bit. Consequently, the arbitrary quantum gates can be composed of these quantum gates groups. It should be noted that the most basic universal quantum gates group is composed of one phase shift gate and two controlled NOT gate.

First, one phase shift gate is defined as

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$

the quantum state

$$|\phi\rangle = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} \tag{2}$$

When $R(\theta)$ act on $|\phi\rangle$, then

$$R(\theta)|\phi\rangle = \begin{bmatrix} \cos(\theta_0 + \theta) \\ \sin(\theta_0 + \theta) \end{bmatrix} \tag{3}$$

Obviously $R(\theta)$ plays the role of phase shift to realize the search of optimal location of the particles in solving equations.

The role of quantum NOT gate is to exchange two qubits probability amplitudes, which is defined as

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

Then

$$U|\phi\rangle = \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix} = \begin{bmatrix} \cos\left(\dfrac{\pi}{2} - \theta_0\right) \\ \sin\left(\dfrac{\pi}{2} - \theta_0\right) \end{bmatrix} \tag{5}$$

$$= R\left(\frac{\pi}{2} - 2\theta_0\right)|\phi\rangle$$

Quantum NOT gate plays a role as a phase rotation in solving equations, which realizes particle position variation to avoid premature convergence. Therefore controlled NOT gate [6] can be constructed as

$$C(k) = \begin{bmatrix} \cos\left(\dfrac{k\pi}{2} - 2\theta_0\right) & -\sin\left(\dfrac{k\pi}{2} - 2\theta_0\right) \\ \sin\left(\dfrac{k\pi}{2} - 2\theta_0\right) & \cos\left(\dfrac{k\pi}{2} - 2\theta_0\right) \end{bmatrix} \tag{6}$$

According to the value of controlled parameter k, its controlled effect can be expressed as

$$c(k) = \begin{cases} [\sin\theta_0, \cos\theta_0]^T & k = 1 \\ [\cos\theta_0, \sin\theta_0]^T & k = 0 \\ \quad else & 0 < k < 1 \end{cases} \tag{7}$$

III. QUANTUM PARTICLE SWARM ALGORITHM

A. Initializing Population

In QPSO, due to the code randomness in population initialization, we code the quantum current position using the probability amplitude of qubits, giving rise to:

$$|P_i\rangle = \begin{bmatrix} \cos\theta_{i1} & \cos\theta_{i2} & \cdots & \cos\theta_{in} \\ \sin\theta_{i1} & \sin\theta_{i2} & \cdots & \sin\theta_{in} \end{bmatrix} \tag{8}$$

where $\theta_{ij} = 2\pi \times rnd$; rnd is the random number between 0 and 1, where i=1,2,…,m; j=1,2,… ,n; m is the population size; n is the number of spatial dimensions.

B. Solution Space Transformation

In QPSO, due to the calculation of particle current advantages and disadvantages, we need to transform the solution space. Using linear transformation, remove 'the' every probability amplitude of qubits can be mapped from the unit space into the solution space of optimization problem. However, given that each probability amplitude of particle corresponds to an optimized variable of the solution space, we define the domain of the variable X_j of optimization problem and the the j-th qubit of particle $|P_i\rangle$ to be $[a_i, b_i]$ the j-th qubit of particleis and $[\alpha_j^i, \beta_j^i]^T$, respectively. The corresponding solution space variable is given by:

$$\begin{bmatrix} X_{j1}^i \\ X_{j2}^i \end{bmatrix} = \begin{bmatrix} \dfrac{b_i - a_i}{2} & 0 \\ 0 & \dfrac{b_i - a_i}{2} \end{bmatrix} \begin{bmatrix} \alpha_j^i \\ \beta_j^i \end{bmatrix} + \begin{bmatrix} \dfrac{b_i + a_i}{2} \\ \dfrac{b_i + a_i}{2} \end{bmatrix} \tag{9}$$

Namely

$$\begin{cases} X_{j1}^i = \dfrac{1}{2}\left[b_j(1 + \alpha_j^i) + a_j(1 - \alpha_j^i)\right] \\ X_{j2}^i = \dfrac{1}{2}\left[b_j(1 + \beta_j^i) + a_j(1 - \beta_j^i)\right] \end{cases} \tag{10}$$

where i=1,2, ··· ,m; j=1,2, ··· ,n.

C. Updating Particle State

In QPSO, updates of particles state can be described as the updates of qubit argument angle incremental in each particle and the updates of qubit probability amplitude of quantum rotation gates. Such a strategy is shown as follows:

We let the particles $|P_i\rangle$ search for the optimal location $|P_l\rangle$, and the entire population search for the optimal location $|P_g\rangle$. The update of qubit argument angle incremental in particle P_i is

$$\Delta\theta_{ij}(t+1) = w\Delta\theta_{ij}(t) + c_1 r_1(\Delta\theta_l) + c_2 r_2(\Delta\theta_g) \quad (11)$$

definition

$$\Delta\theta_k = \begin{cases} 2\pi + \theta_{kj} - \theta_{ij} & \theta_{kj} - \theta_{ij} < -\pi \\ \theta_{kj} - \theta_{ij} & -\pi < \theta_{kj} - \theta_{ij} < \pi \\ \theta_{kj} - \theta_{ij} - 2\pi & \theta_{kj} - \theta_{ij} > \pi \end{cases} \quad (12)$$

where i =1,2, $\cdots$, m; j=1,2, $\cdots$,n; k=l, g; w is the inertia factor, c_1, c_2 are constant, c_1 is own factor and c_2 is global factor; r_1, r_2 are the random numbers between 0 and 1, $\Delta\theta_l$ is qubit argument angle incremental of the particle that search for the optimal position, $\Delta\theta_g$ is qubit argument angle incremental of the entire population that search for the optimal position. The update of qubit probability amplitude of quantum rotation gates in particles $|P_i\rangle$ is

$$\begin{bmatrix} \cos(\theta_{ij}(t+1)) \\ \sin(\theta_{ij}(t+1)) \end{bmatrix} = R(\Delta\theta_{ij}(t+1)) \begin{bmatrix} \cos(\theta_{ij}(t)) \\ \sin(\theta_{ij}(t)) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta_{ij}(t) + \Delta\theta_{ij}(t+1)) \\ \sin(\theta_{ij}(t) + \Delta\theta_{ij}(t+1)) \end{bmatrix} \quad (13)$$

The angular size and orientation of quantum rotation gates are very important and affect directly the convergence speed and optimization direction. Definition is shown below

$$A_{ij} = \begin{vmatrix} \cos\theta_{lj} & \cos\theta_{gj} \\ \sin\theta_{lj} & \sin\theta_{gj} \end{vmatrix} \quad (14)$$

the relationship of angle is

$$\Delta\theta_{ij} = -\text{sgn}(A_{ij})\theta_0 e^{-t} \quad (15)$$

where i=1,2, $\cdots$,m; j=1,2, $\cdots$,n; θ_0 is the initial value, t is optimization step. The determinating rules of angular orientation[7]: the orientation is $-\text{sgn}(A_{ij})$ when $A_{ij} \neq 0$, the orientation can be positive and negative when $A_{ij} = 0$. The coding after the particle update is

$$|\tilde{P}_i\rangle = \begin{bmatrix} \cos(\theta_{i1}(t) + \Delta\theta_{i1}(t+1)) \\ \sin(\theta_{i1}(t) + \Delta\theta_{i1}(t+1)) \end{bmatrix} \cdots \begin{bmatrix} \cos(\theta_{in}(t) + \Delta\theta_{in}(t+1)) \\ \sin(\theta_{in}(t) + \Delta\theta_{in}(t+1)) \end{bmatrix} \quad (16)$$

D. Variation Processing

In QPSO, The quantum NOT gate is adopted to realize particle location variations, to change the probability amplitude of particle location, and to flip particle optimization direction. Meanwhile, by introducing the mutation operator, this method not only avoids the loss of the population diversity in the search process, but also prevents premature convergence and increases the opportunities of particles search for the optimal location. In particle $|P_i\rangle$, the variation of qubit probability amplitude of quantum NOT gate is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta_{ij} \\ \sin\theta_{ij} \end{bmatrix} = \begin{bmatrix} \sin\theta_{ij} \\ \cos\theta_{ij} \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{2} - \theta_{ij}\right) \\ \sin\left(\frac{\pi}{2} - \theta_{ij}\right) \end{bmatrix} \quad (17)$$

IV. QPSO FOR SOLVING NONLINEAR EQUATIONS

A. Problem Description

The general mathematical model of nonlinear equations is

$$\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \quad \vdots \quad \vdots \\ f_m(x_1, x_2, \cdots, x_n) = 0 \end{cases} \quad (18)$$

Considering problem: $\min_{x \in R^n}\|f(x)\|_p$, the $\|\bullet\|_p$ is p-norm of vector space, and the $\min_{x \in R^n}\|f(x)\|_p$ can be taken as the minimum value 0 if and only if $f_i(x_1, x_2 \cdots x_n) = 0$. So the problem of solving arbitrary multivariate nonlinear equations is transformed into solving multivariate nonlinear minimum value. The solution of (the) minimal value is 0 or close to 0, which is the solution of the equation $f_i(x_1, x_2 \cdots x_n) = 0$. We suppose that p=1, then

$$\min_{x \in R^n}\|f(x)\| = \min_{x \in R^n}\left(\sum_{i=1}^m |f_i(x_1, x_2, \cdots x_n)|\right) \quad (19)$$

B. Numerical Experiments

In order to examine the performance of quantum particle swarm algorithm, the numerical simulation is performed using the examples which is given in Ref. [8]. Simulation parameter parameters are given as follows: population size is 50; quantum bits is 2; limited algebra is 500; inertia factor is 0.5; autogenous factor is 2; global factor is 2; mutation probability is 0.05 and the nonlinear equations of simulation is

$$\begin{cases} e^{4\sin x_1} + e^{2\cos x_2} = 2 \\ e^{x_1} + \dfrac{1}{2}e^{x_2} = 1 \end{cases} \quad (20)$$

TABLE I.　　NUMERICAL SIMULATION RESULTS

Algorithm	initial point	approximate solution	Iterations times	converging conditions

Literature [8]	non	-0.131 789 994 367 941 -1.39 856 980 462 69	770	Convergence
Fsolve function	(-3,-3)	-0.131 789 994 380 944 -1.39 856 980 442 676	22	Convergence (concern initial point)
Newton method	(-0.1,-0.5)	-0.131 789 991 803 407 -1.39 856 984 393 425	5	convergence (concern initial point)
QPSO	non	-0.131 789 993 092 176 -1.39 856 982 453 578	371	Convergence (operation10t imes convergence 8times)

C. Results Analysis

Experimental results show that the traditional methods require the rigorous initial value condition for solving nonlinear equations. If the initial value choice is inaccurate, it will affect function convergence to produce optimal solutions of target function. Although Ref.[8] has no initial value, it limits the search interval of the optimal value. However, since QPSO algorithm does not limit the search interval of optimal solutions, and it is the global search in range of all real numbers field, their approximate solutions are more accurate. Therefore, QPSQ can be regarded as a more effective and accurate method for solving nonlinear equations compared to the traditional methods.

V. CONCLUSIONS

A quantum particle swarm optimization algorithm is applied to solve nonlinear equations. The capability of such an algorithm to generate faster convergence speed, better objective function and more accurate solution precision than the traditional methods is demonstrated. Particle swarm algorithm as a kind of more effective evolution algorithm is proposed for solving nonlinear equations.

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