Functional Representation of the Volterra Hierarchy

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Abstract

In this paper I study the functional representation of the Volterra hierarchy (VH). Using the Miwa's shifts I rewrite the infinite set of Volterra equations as one functional equation. This result is used to derive a formal solution of the associated linear problem, a generating function for the conservation laws and to obtain a new form of the Miura and Backlund transformations. I also discuss some relations between the VH and KP hierarchy.

1   Introduction.

In this paper I want to discuss an application of the so-called functional equation approach to one of the oldest integrable discrete systems, namely the Volterra model,

$$\dot{u}_n = u_n \left(u_{n+1} - u_{n-1} \right). \tag{1.1}$$

where the dot stands for the differentiating with respect to time. It was proposed many years ago for the description of the population dynamics [1, 2, 3]. Later it was applied to many physical phenomena, such as, e.g., collapse of Langmuir waves, nonlinear LC nets, Liouville field theory. It is known to be integrable since the paper by Manakov [4] (see also [5]) who developed the corresponding version of the inverse scattering transform (IST).

The IST is a method which drastically changed the theory of PDEs, of nonlinear systems as well as many other fields of nonlinear mathematics and physics. However the practical implementation of its algorithms is not so easy as one might expect. The results which can be obtained using the IST are usually formulated in terms of some auxiliary objects (Jost functions, which are solutions of some auxiliary linear problems, and scattering data, which link different Jost functions) and the main difficulty of this approach is that one cannot find them explicitly or to express them in terms of the u_n. As a result, for example, one cannot get closed expression for the generating function for the constants of motion, but only an algorithm how to derive it. That is why during all the years of the modern theory

of integrable systems people were looking for some other tools to deal with these particular equations which are called integrable. One of such approaches is the topic of this work. In a few words it can be described as follows: instead of your equation (Volterra equation in our case) you consider an infinite family of similar equations (Volterra hierarchy in our case) and then instead of an infinite number of differential equations you deal with one (or a few) equation of other kind, functional or difference one.

Starting from the standard IST representation of the VH, I derive in section 2 the functional equations for the tau-functions of the VH. After discussing the superposition formulae for different Miwa's shifts (section 3) I obtain in section 4 a formal solution of the auxiliary linear problem. Using these results I derive the generating function for the conservation laws of the VH (section 5), address the problem of Miura and Backlund transformations (section 6) and obtain in section 7 dark-soliton solutions of the Volterra equations. Finally, I present some results on interrelations between the Volterra and some other hierarchies (section 8).

2   Zero-curvature representation and Miwa's shifts.

The inverse scattering approach for integrable systems is based on the zero-curvature representation (ZCR), when our nonlinear equations are presented as a compatibility condition for some linear system. For the VH this system can be written as [6]

$$\begin{cases} \Psi_{n+1} & = & U_n\Psi_n \\ \dot{\Psi}_n & = & V_n\Psi_n \end{cases} \tag{2.1}$$

where Ψ_n is a 2-column (or 2×2 matrix),

$$U_n = \begin{pmatrix} \lambda & u_n \\ -1 & 0 \end{pmatrix} \tag{2.2}$$

and V_n is some 2×2 matrix depending on the quantities u_n, $u_{n\pm1}$, ... as well as on the auxiliary parameter λ

Traditionally Volterra chains are considered more often in the framework of the 'big' Lax representation, when the system is finite ($n = 1, ..., N$) and U and V are $N \times N$ matrices. In this paper we will use the 2×2 U-V pair. This approach is equivalent to the $N \times N$ representation and sometimes even more convenient (say, it can be more easily modified to the cases of different boundary conditions, including the soliton case when $N = \infty$).

To provide the self-consistency of the system (2.1) the matrices V_n have to satisfy the following equation

$$\dot{U}_n = V_{n+1}U_n - U_nV_n. \tag{2.3}$$

The choice of the V-matrix for the given U-matrix, is not unique. One can find an infinite number of matrices V, which are polynomials of different order in λ, such that ZCR (2.3) will be satisfied for all values of λ provided the functions u_n solve some nonlinear evolutionary equations. These equations are called 'higher Volterra equations'. Taken together they constitute the VH.

Using the notation

$$V_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{2.4}$$

one can rewrite (2.3) as

$$0 = \lambda (a_{n+1} - a_n) - b_{n+1} - u_n c_n \tag{2.5}$$
$$0 = b_n + u_n c_{n+1} \tag{2.6}$$
$$0 = \lambda c_{n+1} + a_n - d_{n+1} \tag{2.7}$$

and

$$\dot{u}_n = u_n (a_{n+1} - d_n) - \lambda b_n \tag{2.8}$$

or, after eliminating b_n and d_n,

$$0 = \lambda (a_{n+1} - a_n) + u_{n+1} c_{n+2} - u_n c_n, \tag{2.9}$$

and

$$\dot{u}_n = u_n [a_{n+1} - a_{n-1} + \lambda (c_{n+1} - c_n)] . \tag{2.10}$$

It is easy to show that (2.9) and (2.10) possess solutions where a_n's are polynomials of the $(2j-2)$th order while c_n's are polynomials of the $(2j)$th order in λ for $j = 1, 2, ...$ In what follows I indicate different polynomials with the upper index, $a_n^{(j)}, c_n^{(j)}$, and introduce an infinite set of times, t_j, to distinguish the resulting nonlinear equations.

By simple algebra one can establish the following relations between different polynomials:

$$a_n^{(j+1)} = \lambda^2 a_n^{(j)} + \alpha_n^{(j)}, \tag{2.11}$$
$$c_n^{(j+1)} = \lambda^2 c_n^{(j)} + \lambda \gamma_n^{(j)} \tag{2.12}$$

where $\alpha_n^{(j)}$ and $\gamma_n^{(j)}$ do not depend on λ. Substituting (2.11) and (2.12) into (2.9) and (2.10) one can convert our system into

$$\alpha_n^{(j)} + \alpha_{n-1}^{(j)} + \gamma_n^{(j+1)} = 0, \tag{2.13}$$
$$\alpha_{n+1}^{(j)} - \alpha_n^{(j)} + u_{n+1} \gamma_{n+2}^{(j)} - u_n \gamma_n^{(j)} = 0 \tag{2.14}$$

and

$$\partial_j u_n = u_n \left[\gamma_n^{(j)} - \gamma_{n+1}^{(j)} \right] = u_n \left[\alpha_{n+1}^{(j-1)} - \alpha_{n-1}^{(j-1)} \right] \tag{2.15}$$

with $\partial_j = \partial/\partial t_j$

The simplest solution of (2.13) and (2.14),

$$\alpha_n^{(0)} = u_n, \qquad \gamma_n^{(0)} = -1 \tag{2.16}$$

leads to the classical Volterra equation:

$$\partial_1 u_n = u_n (u_{n+1} - u_{n-1}) \tag{2.17}$$

which can be rewritten as

$$\tau_{n-1}\partial_1\tau_n - \tau_n\partial_1\tau_{n-1} = \tau_{n+1}\tau_{n-2} \qquad (2.18)$$

where the tau-functions of the VH, τ_n, are defined by

$$u_n = \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}}. \qquad (2.19)$$

In terms of the tau-functions the quantities $\alpha_n^{(j)}$ and $\gamma_n^{(j)}$ can be presented as

$$\alpha_n^{(j)} = \partial_{j+1}\ln\frac{\tau_n}{\tau_{n-1}} \qquad (2.20)$$

$$\gamma_n^{(j)} = \partial_j\ln\frac{\tau_{n-2}}{\tau_n} \qquad (2.21)$$

which makes (2.15) and (2.13) to be satisfied automatically while equation (2.14) (together with (2.18)) becomes a recurrent formula for the Volterra flows:

$$\partial_{j+1}\ln\frac{\tau_n}{\tau_{n-1}} = u_n\partial_j\ln\frac{\tau_n}{\tau_{n-2}} + \partial_j\partial_1\ln\tau_n. \qquad (2.22)$$

Till now we were following the standard zero-curvature scheme, but hereafter, since our purpose is to discuss the VH as a whole, I will deal not with the quantities $\alpha_n^{(j)}$ and $\gamma_n^{(j)}$ (which describe the jth flow) but with series defined by

$$\alpha_n(\zeta) = \sum_{j=0}^{\infty}\alpha_n^{(j)}\zeta^j, \qquad\qquad \gamma_n(\zeta) = \sum_{j=0}^{\infty}\gamma_n^{(j)}\zeta^j \qquad (2.23)$$

and will consider the quantities u_n (and τ_n) as functions of an infinite set of times $\{t_j\}_{j=1,2,...}$:

$$u_n = u_n(\mathrm{t}) = u_n(t_1, t_2, t_3, ...) \qquad (2.24)$$

which is possible due to the fact that all flows of the Volterra hierarchy $\partial/\partial t_j$ commute.

Using $\alpha_n(\zeta)$ and $\gamma_n(\zeta)$ one can present the elements of the V-matrices describing different Volterra flows in a typical for the theory of integrable systems 'multiply+project' way:

$$a_n^{(j)}(\lambda) = \left[\lambda^{2j-2}\alpha_n\left(\lambda^{-2}\right)\right]_{\geq 0}, \qquad\qquad c_n^{(j)}(\lambda) = \left[\lambda^{2j-1}\gamma_n\left(\lambda^{-2}\right)\right]_{\geq 0} \qquad (2.25)$$

with projection $[...]_{\geq 0}$ being defined by

$$\left[\sum_{j=-\infty}^{\infty}f_j\lambda^j\right]_{\geq 0} = \sum_{j=0}^{\infty}f_j\lambda^j. \qquad (2.26)$$

In terms of $\alpha_n(\zeta)$ and $\gamma_n(\zeta)$ and the differential operator

$$\partial(\zeta) = \sum_{j=1}^{\infty}\zeta^j\partial_j \qquad (2.27)$$

equations (2.20), (2.21) become

$$\alpha_n(\zeta) = \frac{1}{\zeta} \partial(\zeta) \ln \frac{\tau_n}{\tau_{n-1}} \tag{2.28}$$

$$\gamma_n(\zeta) = -1 + \partial(\zeta) \ln \frac{\tau_{n-2}}{\tau_n} \tag{2.29}$$

while equations (2.13), (2.14) and (2.22) lead to

$$1 + \zeta \alpha_n(\zeta) + \zeta \alpha_{n-1}(\zeta) + \gamma_n(\zeta) = 0 \tag{2.30}$$

$$\alpha_{n+1}(\zeta) - \alpha_n(\zeta) + u_{n+1}\gamma_{n+2}(\zeta) - u_n\gamma_n(\zeta) = 0 \tag{2.31}$$

and

$$\alpha_n(\zeta) = -u_n\gamma_n(\zeta) + \partial_1\partial(\zeta) \ln \tau_n. \tag{2.32}$$

By multiplying (2.31) by $\gamma_{n+1}(\zeta)$ and using

$$\partial(\zeta)u_n = u_n \left[\gamma_n(\zeta) - \gamma_{n+1}(\zeta)\right] \tag{2.33}$$

one can derive an invariant of the map (2.30) and (2.31):

$$\alpha_n(\zeta) + \zeta\alpha_n^2(\zeta) - u_n\gamma_n(\zeta)\gamma_{n+1}(\zeta) = \text{constant}. \tag{2.34}$$

Noting that $\alpha_n^{(j)}$ and $\gamma_n^{(j)}$ are homogeneous polynomials in u_n of $(j+1)$th and jth order correspondingly one can conclude that the constant in the right-hand side of the last formula is zero,

$$\alpha_n(\zeta) + \zeta\alpha_n^2(\zeta) - u_n\gamma_n(\zeta)\gamma_{n+1}(\zeta) = 0. \tag{2.35}$$

To proceed further we have to derive some differential identities which are satisfied by $\alpha_n(\zeta)$ and $\gamma_n(\zeta)$ as functions of the infinite set of times t_j. It is easy to note that the simplest of the Volterra equations, (2.17), gives for the derivative $\partial_1\alpha_n^{(0)}$, $\alpha_n^{(0)} = u_n$, the following expression:

$$\partial_1\alpha_n^{(0)} = u_n \left[\gamma_n^{(0)}\gamma_{n+1}^{(1)} - \gamma_n^{(1)}\gamma_n^{(0)}\right]. \tag{2.36}$$

By simple algebra one can derive the similar expression for the first derivative of $\alpha_n^{(1)} = u_n(u_{n-1} + u_n + u_{n-1})$,

$$\partial_1\alpha_n^{(1)} = u_n \left[\gamma_n^{(0)}\gamma_{n+1}^{(2)} - \gamma_n^{(2)}\gamma_n^{(0)}\right]. \tag{2.37}$$

It turns out that these formulae are the simplest cases of a more general identity which in terms of the series $\alpha_n(\zeta)$ and $\gamma_n(\zeta)$ reads

$$\partial(\eta)\alpha_n(\xi) = \frac{\eta}{\xi - \eta} u_n \left[\gamma_{n+1}(\xi)\gamma_n(\eta) - \gamma_n(\xi)\gamma_{n+1}(\eta)\right]. \tag{2.38}$$

A proof of (2.38) can be given as follows. First let us note that

$$\eta\,\partial(\xi)\alpha_n(\eta) = \xi\,\partial(\eta)\alpha_n(\xi) \tag{2.39}$$

(as follows from (2.28)). Then, application of $\partial(\xi)$ to (2.32) leads to

$$\partial(\xi)\alpha_n(\eta) = u_n\gamma_{n+1}(\xi)\gamma_n(\eta) + S_n(\xi,\eta) \tag{2.40}$$

where $S_n(\xi,\eta)$ is a symmetric function of ξ and η, $S_n(\xi,\eta) = S_n(\eta,\xi)$. Subtracting from this equation a similar one with ξ and η being interchanged one can get

$$\partial(\xi)\alpha_n(\eta) - \partial(\eta)\alpha_n(\xi) = u_n\left[\gamma_{n+1}(\xi)\gamma_n(\eta) - \gamma_n(\xi)\gamma_{n+1}(\eta)\right]. \tag{2.41}$$

Now, using (2.39) one can rewrite the left-hand side of this equation as $(\xi/\eta - 1)\partial(\eta)\alpha_n(\xi)$ which ends the proof of (2.38).

Formula (2.38) is crucial for the derivation of the functional representation of the VH. By taking the $\xi \to \eta$ limit and using some simple algebra one can come to the very important fact: the quantity f_n given by

$$f_n = f_n(\zeta, \mathrm{t}) = f_n(\zeta, t_1, t_2, ...) = \frac{\tau_{n-1}}{\tau_{n-2}}\frac{\alpha_n(\zeta)}{\gamma_n(\zeta)} \tag{2.42}$$

satisfies equation

$$\frac{1}{\zeta}\partial(\zeta)f_n = \frac{\partial}{\partial\zeta}f_n \tag{2.43}$$

which means that

$$f_n(\zeta, t_1, t_2, ...) = F_n\left(t_1 + \zeta, t_2 + \frac{\zeta^2}{2}, t_3 + \frac{\zeta^3}{3}, ...\right) \tag{2.44}$$

or,

$$f_n(\zeta, \mathrm{t}) = F_n(\mathrm{t} + [\zeta]) \tag{2.45}$$

where square brackets indicate the so-called Miwa's shifts:

$$F(\mathrm{t} + \epsilon[\zeta]) = F\left(..., t_k + \epsilon\frac{\zeta^k}{k}, ...\right). \tag{2.46}$$

Setting ζ equal to zero in (2.44) one can get that $F_n(\mathrm{t}) = -\tau_{n+1}(\mathrm{t})/\tau_n(\mathrm{t})$ which gives

$$\frac{\alpha_n(\zeta, \mathrm{t})}{\gamma_n(\zeta, \mathrm{t})} = -\frac{\tau_{n-2}(\mathrm{t})}{\tau_{n-1}(\mathrm{t})}\frac{\tau_{n+1}(\mathrm{t} + [\zeta])}{\tau_n(\mathrm{t} + [\zeta])}. \tag{2.47}$$

Noting that (2.35) leads to

$$\frac{1 + \zeta\alpha_{n-1}}{\gamma_n} = \frac{u_{n-1}\gamma_{n-1}}{\alpha_{n-1}} \tag{2.48}$$

equation (2.47) can be rewritten as

$$\frac{1 + \zeta\alpha_{n-1}(\zeta, \mathrm{t})}{\gamma_n(\zeta, \mathrm{t})} = -\frac{\tau_n(\mathrm{t})}{\tau_{n-1}(\mathrm{t})}\frac{\tau_{n-1}(\mathrm{t} + [\zeta])}{\tau_n(\mathrm{t} + [\zeta])}. \tag{2.49}$$

Substitution of (2.47) and (2.49) into relation (2.30) leads to the following bilinear equation for the tau-functions:

$$\zeta\tau_{n-2}(\mathrm{t})\tau_{n+1}(\mathrm{t} + [\zeta]) - \tau_{n-1}(\mathrm{t})\tau_n(\mathrm{t} + [\zeta]) + \tau_n(\mathrm{t})\tau_{n-1}(\mathrm{t} + [\zeta]) = 0. \tag{2.50}$$

This is the central formula of this paper. This functional equation contains all differential equations of the VH: expanding (2.50) in the Taylor series in ζ

$$f(\mathrm{t} + [\zeta]) = f(\mathrm{t}) + \zeta\partial_1 f(\mathrm{t}) + \frac{\zeta^2}{2}\left(\partial_2 + \partial_{11}\right)f(\mathrm{t}) + ... \tag{2.51}$$

and gathering terms with different powers of ζ one can obtain all Volterra flows. For example, ζ^1 terms give

$$\tau_{n-2}\tau_{n+1} - \tau_{n-1}\partial_1\tau_n + \tau_n\partial_1\tau_{n-1} = 0 \tag{2.52}$$

which is nothing but equation (2.18).

To simplify the following formulae hereafter I will also use another designation for the Miwa's shifts:

$$\left(\mathbb{E}_\zeta f\right)(\mathrm{t}) = f(\mathrm{t} + [\zeta]) \tag{2.53}$$

and

$$\left(\overline{\mathbb{E}}_\zeta f\right)(\mathrm{t}) = f(\mathrm{t} - [\zeta]). \tag{2.54}$$

In these terms equation (2.50) can be presented as

$$\zeta\,\tau_{n-2}\left(\mathbb{E}_\zeta\tau_{n+1}\right) - \tau_{n-1}\left(\mathbb{E}_\zeta\tau_n\right) + \tau_n\left(\mathbb{E}_\zeta\tau_{n-1}\right) = 0 \tag{2.55}$$

or

$$\zeta\left(\overline{\mathbb{E}}_\zeta\tau_{n-2}\right)\tau_{n+1} - \left(\overline{\mathbb{E}}_\zeta\tau_{n-1}\right)\tau_n + \left(\overline{\mathbb{E}}_\zeta\tau_n\right)\tau_{n-1} = 0 \tag{2.56}$$

(which is equation (2.50) after the shift $t_k \to t_k - \zeta^k/k$).

3   Fay's identities.

After we have established the simplest relations describing the Miwa's shifts I am going to derive the superposition formulae, i.e. to calculate the result of combined action of several Miwa's shifts.

Writing down (2.55) with $\zeta = \xi$, applying $\mathbb{E}_\eta$, multiplying the result by $\eta\left(\mathbb{E}_\xi\tau_{n-2}\right)$ and then subtracting the similar expression with ξ and η being interchanged (i.e. performing the antisymmetrization with respect to ξ and η) one can get

$$X_n(\xi,\eta)\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right) - \widetilde{X}_n(\xi,\eta)\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_{n-1}\right) = 0 \tag{3.1}$$

where

$$X_n(\xi,\eta) = \xi\left(\mathbb{E}_\xi\tau_n\right)\left(\mathbb{E}_\eta\tau_{n-1}\right) - \eta\left(\mathbb{E}_\xi\tau_{n-1}\right)\left(\mathbb{E}_\eta\tau_n\right) \tag{3.2}$$

$$\widetilde{X}_n(\xi,\eta) = \xi\left(\mathbb{E}_\xi\tau_n\right)\left(\mathbb{E}_\eta\tau_{n-2}\right) - \eta\left(\mathbb{E}_\xi\tau_{n-2}\right)\left(\mathbb{E}_\eta\tau_n\right). \tag{3.3}$$

After the following calculations,

$$\begin{aligned}
\tau_{n-1}\,\widetilde{X}_n(\xi,\eta) &= \xi\left(\mathbb{E}_\xi\tau_n\right)\left[\tau_{n-2}\left(\mathbb{E}_\eta\tau_{n-1}\right) - \eta\tau_{n-3}\left(\mathbb{E}_\eta\tau_n\right)\right] \tag{3.4}\\
&+ \eta\left(\mathbb{E}_\eta\tau_n\right)\left[\xi\tau_{n-3}\left(\mathbb{E}_\xi\tau_n\right) - \tau_{n-2}\left(\mathbb{E}_\xi\tau_{n-1}\right)\right] \tag{3.5}\\
&= \tau_{n-2}\,X_n(\xi,\eta) \tag{3.6}
\end{aligned}$$

(where (2.55) with $n \to n-1$ has been used), equation (3.1) reads

$$\frac{X_n(\xi,\eta)}{\tau_{n-1}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right)} = \frac{X_{n-1}(\xi,\eta)}{\tau_{n-2}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_{n-1}\right)} \tag{3.7}$$

which means that $X_n(\xi,\eta)/\tau_{n-1}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right)$ is a constant with respect to n. Denoting this constant as $a(\xi,\eta)$ we come to the two-shift superposition formula

$$a(\xi,\eta)\,\tau_{n-1}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right) = \xi\left(\mathbb{E}_\xi\tau_n\right)\left(\mathbb{E}_\eta\tau_{n-1}\right) - \eta\left(\mathbb{E}_\xi\tau_{n-1}\right)\left(\mathbb{E}_\eta\tau_n\right). \tag{3.8}$$

By applying (2.55) one can derive from this relation the following ones:

$$a(\xi,\eta)\,\tau_{n-2}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right) = \xi\left(\mathbb{E}_\xi\tau_n\right)\left(\mathbb{E}_\eta\tau_{n-2}\right) - \eta\left(\mathbb{E}_\xi\tau_{n-2}\right)\left(\mathbb{E}_\eta\tau_n\right) \tag{3.9}$$

$$a(\xi,\eta)\,\tau_{n-3}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right) = \left(\mathbb{E}_\xi\tau_{n-1}\right)\left(\mathbb{E}_\eta\tau_{n-2}\right) - \left(\mathbb{E}_\xi\tau_{n-2}\right)\left(\mathbb{E}_\eta\tau_{n-1}\right) \tag{3.10}$$

where $a(\xi,\eta)$ is an antisymmetric function

$$a(\xi,\eta) = -a(\eta,\xi) \tag{3.11}$$

which should be determined from the boundary conditions.

Subtraction of equation (3.10) multiplied by η from (3.8) gives

$$a(\xi,\eta)\left[\tau_{n-1}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_n\right) - \eta\tau_{n-2}\left(\mathbb{E}_\xi\,\mathbb{E}_\eta\tau_{n+1}\right)\right] = (\xi-\eta)\left(\mathbb{E}_\xi\tau_n\right)\left(\mathbb{E}_\eta\tau_{n-1}\right) \tag{3.12}$$

which can be rewritten as a superposition formula for positive, $\mathbb{E}_\xi$, and negative, $\overline{\mathbb{E}}_\eta$, Miwa's shifts. This formula, together with two other ones which can also be derived from (3.8)–(3.10), are given by

$$c(\xi,\eta)\,\tau_n\left(\mathbb{E}_\xi\,\overline{\mathbb{E}}_\eta\tau_n\right) = \left(\overline{\mathbb{E}}_\eta\tau_n\right)\left(\mathbb{E}_\xi\tau_n\right) - \xi\eta\left(\overline{\mathbb{E}}_\eta\tau_{n-2}\right)\left(\mathbb{E}_\xi\tau_{n+2}\right) \tag{3.13}$$

$$c(\xi,\eta)\,\tau_{n+1}\left(\mathbb{E}_\xi\,\overline{\mathbb{E}}_\eta\tau_n\right) = \left(\overline{\mathbb{E}}_\eta\tau_n\right)\left(\mathbb{E}_\xi\tau_{n+1}\right) - \xi\left(\overline{\mathbb{E}}_\eta\tau_{n-1}\right)\left(\mathbb{E}_\xi\tau_{n+2}\right) \tag{3.14}$$

$$c(\xi,\eta)\,\tau_{n-1}\left(\mathbb{E}_\xi\,\overline{\mathbb{E}}_\eta\tau_n\right) = \left(\overline{\mathbb{E}}_\eta\tau_{n-1}\right)\left(\mathbb{E}_\xi\tau_n\right) - \eta\left(\overline{\mathbb{E}}_\eta\tau_{n-2}\right)\left(\mathbb{E}_\xi\tau_{n+1}\right) \tag{3.15}$$

where $c(\xi,\eta)$ is some symmetric function defined by

$$c(\xi,\eta) = c(\eta,\xi) = \frac{\xi-\eta}{a(\xi,\eta)}. \tag{3.16}$$

It is easy to note that the right-hand side of the antisymmetric Fay's identity (3.10) is a 2×2 determinant. It turns out that it can be extended to provide a superposition formula for N Miwa's shifts

$$\det\left|\;\mathbb{E}_{\xi_j}\,\tau_{n+1-k}\;\right|_{j,k=1,\ldots,N} = a(\xi_1,\ldots,\xi_N)\,\tau_{n-N}\ldots\tau_{n-2}\left(\mathbb{E}_{\xi_1}\ldots\mathbb{E}_{\xi_N}\tau_{n+N-1}\right) \tag{3.17}$$

where $a(\xi_1,\ldots,\xi_N)$ is given by

$$a(\xi_1,\ldots,\xi_N) = \prod_{1\le i<j\le N} a(\xi_i,\xi_j). \tag{3.18}$$

In a similar way, the multi-shift analog of (3.8) can be written as

$$\det\left|\;\xi_j^{N-k}\,\mathbb{E}_{\xi_j}\,\tau_{n+1-k}\;\right|_{j,k=1,\ldots,N} = a(\xi_1,\ldots,\xi_N)\,\tau_{n+1-N}\ldots\tau_{n-1}\left(\mathbb{E}_{\xi_1}\ldots\mathbb{E}_{\xi_N}\tau_n\right). \tag{3.19}$$

I give these formulae here without proof, which can be done, say, by induction (by reducing the $N \times N$ determinant to $(N-1) \times (N-1)$ one and using two-shift formulae (3.10) or (3.8)).

4 Baker-Akhiezer function.

The aim of this section is to derive (formal) solution of the auxiliary problem $\Psi_{n+1} = U_n\Psi_n$ (see (2.1) and (2.2)), which for a 2-vector $(\psi_n, \varphi_n)^T$ can be written as a system

$$\psi_{n+1} = \lambda\psi_n + u_n\varphi_n \tag{4.1}$$

$$\varphi_{n+1} = -\psi_n \tag{4.2}$$

or as second-order equations

$$\psi_{n+1} - \lambda\psi_n + u_n\psi_{n-1} = 0 \tag{4.3}$$

$$\varphi_{n+1} - \lambda\varphi_n + u_{n-1}\varphi_{n-1} = 0. \tag{4.4}$$

Our starting point are equations (2.55) and (2.56). Dividing the later by $\tau_{n-1}\tau_n$ it can be presented as

$$\frac{\overline{\mathbb{E}}_\zeta\tau_n}{\tau_n} - \frac{\overline{\mathbb{E}}_\zeta\tau_{n-1}}{\tau_{n-1}} + \zeta u_n\frac{\overline{\mathbb{E}}_\zeta\tau_{n-2}}{\tau_{n-2}} = 0 \tag{4.5}$$

where, recall, $u_n = \tau_{n-2}\tau_{n+1}/\tau_{n-1}\tau_n$ In a similar way, after the shift $n \to n+1$ (2.55) takes the form

$$\zeta\frac{\mathbb{E}_\zeta\tau_{n+2}}{\tau_n} - \frac{\mathbb{E}_\zeta\tau_{n+1}}{\tau_{n-1}} + u_n\frac{\mathbb{E}_\zeta\tau_n}{\tau_{n-2}} = 0. \tag{4.6}$$

Comparing these equations with (4.3) and (4.4) one can conclude that $\lambda^n\,\overline{\mathbb{E}}_\zeta\tau_{n-1}/\tau_{n-1}$ solves (4.3) while $\lambda^{-n}\,\mathbb{E}_\zeta\tau_n/\tau_{n-2}$ is a solution of (4.4) This means that, if τ_n is a satisfies (2.55) and (2.56), then the matrix Ψ,

$$\Psi_n = \begin{pmatrix} \dfrac{\overline{\mathbb{E}}_\zeta\tau_{n-1}}{\tau_{n-1}}\,\lambda^n & -\dfrac{1}{\lambda}\dfrac{\mathbb{E}_\zeta\tau_{n+1}}{\tau_{n-1}}\,\lambda^{-n} \\[3ex] -\dfrac{1}{\lambda}\dfrac{\overline{\mathbb{E}}_\zeta\tau_{n-2}}{\tau_{n-2}}\,\lambda^n & \dfrac{\mathbb{E}_\zeta\tau_n}{\tau_{n-2}}\,\lambda^{-n} \end{pmatrix} \qquad \zeta = \lambda^{-2} \tag{4.7}$$

solves the discrete auxiliary problem. The determinant of the matrix Ψ

$$\det\Psi_n = \frac{\left(\overline{\mathbb{E}}_\zeta\tau_{n-1}\right)\left(\mathbb{E}_\zeta\tau_n\right) - \zeta\left(\overline{\mathbb{E}}_\zeta\tau_{n-2}\right)\left(\mathbb{E}_\zeta\tau_{n+1}\right)}{\tau_{n-2}\tau_{n-1}} \tag{4.8}$$

is given by

$$\det\Psi_n = c(\zeta, \zeta)\frac{\tau_n}{\tau_{n-2}} \tag{4.9}$$

as follows from (3.15). So, $\det\Psi_n$ is non-zero, except for some special values of ζ, which means that the columns of Ψ_n are two linearly independent solutions of our scattering problem, i.e. its basis.

Thus we have come to the point which demonstrates one of the main advantages of functional representation of the integrable hierarchy: using the Miwa's shifts one can express solutions of the auxiliary problem (which are the central objects of the inverse scattering technique) in terms of the solutions of the original equation. In the following sections I use this fact to enhance results given by the IST, such as, e.g., the generating function for the conservation laws, presenting them in a compact form and directly in terms of the tau-functions, not invoking some intermediate quantities such as Jost functions or scattering data.

5   Conservation laws.

The Volterra model is an integrable system possessing an infinite number of constants of motion. A few first of them can be written as

$$I_1 = \sum u_n, \tag{5.1}$$

$$I_2 = \sum 2u_{n+1}u_n + u_n^2, \tag{5.2}$$

$$I_3 = \sum 3u_{n+1}u_nu_{n-1} + 3u_{n+1}u_n^2 + 3u_n^2u_{n-1} + u_n^3. \tag{5.3}$$

Let us introduce the conserved densities $U_n^{(j)}$ by

$$I_j = \sum_n U_n^{(j-1)}. \tag{5.4}$$

The first two ones can be presented as

$$U_n^{(0)} = u_n, \tag{5.5}$$

$$U_n^{(1)} = u_{n+1}u_n + u_nu_{n-1} + u_n^2. \tag{5.6}$$

In terms of the tau-functions $U_n^{(0)}$ is given by

$$U_n^{(0)} = \frac{\tau_{n-2}\tau_{n+1}}{\tau_{n-1}\tau_n} \tag{5.7}$$

while the second one, by means of the Volterra equation (2.18),

$$u_n = \partial_1 \ln \frac{\tau_n}{\tau_{n-1}}, \tag{5.8}$$

can be presented as

$$U_n^{(1)} = \frac{1}{\tau_{n-1}\tau_n} \left[\tau_{n-2}\,\partial_1\tau_{n+1} - \tau_{n+1}\,\partial_1\tau_{n-2} \right]. \tag{5.9}$$

One can note that $U_n^{(0)}$ and $U_n^{(1)}$ coincide with the cofactors of ζ^0 and ζ^1 in the Taylor expansion of the quantity

$$\left(\overline{\mathbb{E}}_\zeta \tau_{n-2} \right) \left(\mathbb{E}_\zeta \tau_{n+1} \right) = \tau_{n-2}\tau_{n+1} + \zeta \left(\tau_{n-2}\,\partial_1\tau_{n+1} - \tau_{n+1}\,\partial_1\tau_{n-2} \right) + O\left(\zeta^2 \right). \tag{5.10}$$

By straightforward (but rather cumbersome) calculations one can find that the same occurs for $U_n^{(2)}$, $U_n^{(3)}$ etc. It turns out that this is indeed the case: it can be shown that the series

$$U_n(\zeta) = \sum_{j=0}^\infty \zeta^j\, U_n^{(j)} \tag{5.11}$$

is nothing but $\left(\overline{\mathbb{E}}_\zeta \tau_{n-2} \right) \left(\mathbb{E}_\zeta \tau_{n+1} \right) / \tau_{n-1}\tau_n$:

$$U_n\left(\zeta, \mathrm{t} \right) = \frac{\tau_{n-2}\left(\mathrm{t} - [\zeta] \right) \tau_{n+1}\left(\mathrm{t} + [\zeta] \right)}{\tau_{n-1}\left(\mathrm{t} \right) \tau_n\left(\mathrm{t} \right)}. \tag{5.12}$$

The proof of this statement is based on the commutativity of differentiating and the Miwa's shifts and exploits the superposition formulae (3.13)–(3.15) in the particular case of $\xi = \eta$:

$$c(\zeta)\,\tau_n^2 \;=\; \left(\overline{\mathbb{E}}_\zeta \tau_n\right)\left(\mathbb{E}_\zeta \tau_n\right) - \zeta^2 \left(\overline{\mathbb{E}}_\zeta \tau_{n-2}\right)\left(\mathbb{E}_\zeta \tau_{n+2}\right) \tag{5.13}$$

$$c(\zeta)\,\tau_{n-1}\tau_n \;=\; \left(\overline{\mathbb{E}}_\zeta \tau_{n-1}\right)\left(\mathbb{E}_\zeta \tau_n\right) - \zeta \left(\overline{\mathbb{E}}_\zeta \tau_{n-2}\right)\left(\mathbb{E}_\zeta \tau_{n+1}\right) \tag{5.14}$$

where

$$c(\zeta) = c(\zeta, \zeta). \tag{5.15}$$

To make the following formulae more readable I will use in this section the $\pm$ designation for $\mathbb{E}_\zeta$ and $\overline{\mathbb{E}}_\zeta$:

$$\tau_n^+ = \mathbb{E}_\zeta \tau_n, \qquad \tau_n^- = \overline{\mathbb{E}}_\zeta \tau_n. \tag{5.16}$$

Expressing from (2.18) $\partial_1 \tau_{n-2}$ and $\partial_1 \tau_{n+1}$ in terms of $\partial_1 \tau_{n-1}$ and $\partial_1 \tau_n$

$$\partial_1 \tau_{n-2} \;=\; \frac{1}{\tau_{n-1}}\left(\tau_{n-2}\,\partial_1 \tau_{n-1} - \tau_{n-3}\tau_n\right) \tag{5.17}$$

$$\partial_1 \tau_{n+1} \;=\; \frac{1}{\tau_n}\left(\tau_{n+1}\,\partial_1 \tau_n - \tau_{n-1}\tau_{n+2}\right) \tag{5.18}$$

one can obtain for the derivative of $\tau_{n-2}^- \tau_{n+1}^+$ the following expression:

$$\partial_1 \tau_{n-2}^- \tau_{n+1}^+ = \frac{\tau_{n-2}^- \tau_{n+1}^+}{\tau_{n-1}^- \tau_n^+}\,\partial_1 \tau_{n-1}^- \tau_n^+ + \frac{\tau_{n-2}^- \tau_{n-1}^+ \tau_{n+2}^+}{\tau_n^+} - \frac{\tau_{n-3}^- \tau_n^- \tau_{n+1}^+}{\tau_{n-1}^-} \tag{5.19}$$

or,

$$\tau_{n-1}^- \tau_n^+ \,\partial_1 \tau_{n-2}^- \tau_{n+1}^+ - \tau_{n-2}^- \tau_{n+1}^+ \,\partial_1 \tau_{n-1}^- \tau_n^+ = \tau_{n-2}^- \tau_{n-1}^- \tau_{n-1}^+ \tau_{n+2}^+ - \tau_{n-3}^- \tau_n^- \tau_n^+ \tau_{n+1}^+. \tag{5.20}$$

Using (5.14) the left-hand side of (5.20) can be rewritten as

$$\begin{aligned}
\mathrm{lhs}(5.20) &= c(\zeta)\left(\tau_{n-1}\tau_n\,\partial_1 \tau_{n-2}^- \tau_{n+1}^+ - \tau_{n-2}^- \tau_{n+1}^+ \,\partial_1 \tau_{n-1}\tau_n\right) \\
&= c(\zeta)\,\tau_{n-1}^2 \tau_n^2 \,\partial_1 \frac{\tau_{n-2}^- \tau_{n+1}^+}{\tau_{n-1}\tau_n}.
\end{aligned} \tag{5.21}$$

At the same time, the right-hand side of (5.20), by virtue of (5.13), is

$$\begin{aligned}
\mathrm{rhs}(5.20) &= c(\zeta)\left(\tau_{n-1}^2 \tau_{n-2}^- \tau_{n+2}^+ - \tau_n^2 \tau_{n-3}^- \tau_{n+1}^+\right) \\
&= c(\zeta)\,\tau_{n-1}^2 \tau_n^2 \left(\frac{\tau_{n-2}^- \tau_{n+2}^+}{\tau_n^2} - \frac{\tau_{n-3}^- \tau_{n+1}^+}{\tau_{n-1}^2}\right).
\end{aligned} \tag{5.22}$$

Comparing (5.21) and (5.22) we come to the following result:

$$\partial_1 U_n\left(\zeta, \mathrm{t}\right) = W_n\left(\zeta, \mathrm{t}\right) - W_{n-1}\left(\zeta, \mathrm{t}\right) \tag{5.23}$$

where

$$W_n\left(\zeta, \mathrm{t}\right) = \frac{\tau_{n-2}\left(\mathrm{t} - [\zeta]\right)\,\tau_{n+2}\left(\mathrm{t} + [\zeta]\right)}{\tau_n^2\left(\mathrm{t}\right)}. \tag{5.24}$$

This means that $U_n\left(\zeta, \mathrm{t}\right)$ is indeed the conserved density of the first Volterra flow ∂_1 (i.e. of the Volterra equation (2.17)). A little bit more lengthy calculations (omitted here) show that the same is valid for all Volterra flows:

$$\sum_{j=1}^{\infty} \eta^{j-1} \partial_j\, U_n\left(\xi, \mathrm{t}\right) = W_n\left(\xi, \eta, \mathrm{t}\right) - W_{n-1}\left(\xi, \eta, \mathrm{t}\right) \tag{5.25}$$

where

$$W_n\left(\xi, \eta, \mathrm{t}\right) = \frac{1}{c(\eta)c^2(\xi, \eta)} \frac{\tau_{n-2}\left(\mathrm{t} - [\xi] - [\eta]\right)\, \tau_{n+2}\left(\mathrm{t} + [\xi] + [\eta]\right)}{\tau_n^2\left(\mathrm{t}\right)}. \tag{5.26}$$

Expanding (5.25) in power series in ξ and η one can get an infinite number of divergence-like conservation laws *for all* equations of the VH:

$$\partial_k\, U_n^{(j)} = W_n^{(j,k)} - W_{n-1}^{(j,k)}. \tag{5.27}$$

Note that the conserved densities $U_n^{(j)}$ are the same for all Volterra equations (as was expected) while the right-hand side of (5.27) depends on which Volterra flow we are dealing with.

After summing over all n, in the case of proper boundary conditions, one gets an infinite number of constants of motion $I^{(j)}$ given by (5.4) common for all equations of the hierarchy,

$$\partial_k\, I_j = 0 \tag{5.28}$$

and their generating function

$$I(\zeta) = \sum_{j=1}^{\infty} \zeta^j\, I_j \tag{5.29}$$

which is given by

$$I(\zeta) = \zeta \sum_n U_n(\zeta) \tag{5.30}$$

or, finally,

$$I(\zeta) = \zeta \sum_n \frac{\tau_{n-2}\left(\mathrm{t} - [\zeta]\right)\, \tau_{n+1}\left(\mathrm{t} + [\zeta]\right)}{\tau_{n-1}(\mathrm{t})\tau_n(\mathrm{t})}. \tag{5.31}$$

6 Backlund transformations.

The aim of this section is to discuss the Miura and Backlund transforms and to expose their inner structure which becomes transparent when we rewrite them in the terms of the Miwa's shifts.

Here I will follow the paper [7] by Kajinaga and Wadati. The discrete Miura transformation [8, 9] is defined by

$$u_n = \mu^2\left(1 + q_{n-1}\right)\left(1 - q_n\right) \tag{6.1}$$

where μ is an arbitrary constant parameter. It links solutions of the Volterra equation (2.17) u_n with solutions of the so-called modified Volterra lattice,

$$\dot{q}_n = \mu^2 \left(1 - q_n^2\right)(q_{n+1} - q_{n-1}) \tag{6.2}$$

The Miura transformation $u_n \to q_n$ can be used to construct the Backlund transformation $u_n \to u_n'$ of the Volterra equation (and in fact the whole VH). It is easy to verify that, if u_n solves (2.17) and q_n is defined by (6.1), then the function

$$u_n' = \mu^2 \left(1 - q_{n-1}\right)(1 + q_n) \tag{6.3}$$

is also a solution of the Volterra equation (and, again, of all equations of the VH).

Now let us look at the above construction from the viewpoint of the functional representation of the VH. By straightforward calculations one can show that in terms of the function

$$q_n' = 1 - 2 \frac{\tau_{n+1} \left(\mathbb{E}_\zeta \tau_n \right)}{\tau_n \left(\mathbb{E}_\zeta \tau_{n+1} \right)} \tag{6.4}$$

equation (2.55) becomes

$$4\zeta u_n = \left(1 + q_{n-1}'\right)\left(1 - q_n'\right) \tag{6.5}$$

which is exactly (6.1) with $\mu^2 = 1/4\zeta$, i.e the right-hand side of (6.4) is explicit realization of the Miura transform. Now the obvious question is to calculate u_n' given by (6.3). It is easy to see, by substituting (6.4) into the right-hand side of (6.3) and using once more relation (2.55) that

$$\left(1 - q_{n-1}'\right)\left(1 + q_n'\right) = 4\zeta \left(\mathbb{E}_\zeta u_{n+1} \right) \tag{6.6}$$

i.e.

$$u_n' = \mathbb{E}_\zeta u_{n+1}. \tag{6.7}$$

This means that the Backlund transform constructed by the recipe (6.1)–(6.3) with (6.4) is just the Miwa shift combined with the shift $n \to n + 1$.

But q_n' is not the only solution of (6.5) or (6.1). Indeed, starting from (2.56) one can show that the function q_n'' given by

$$q_n'' = 2 \frac{\left(\overline{\mathbb{E}}_\zeta \tau_n \right) \tau_{n-1}}{\left(\overline{\mathbb{E}}_\zeta \tau_{n-1} \right) \tau_n} - 1 \tag{6.8}$$

also satisfies

$$\left(1 + q_{n-1}''\right)\left(1 - q_n''\right) = 4\zeta u_n \tag{6.9}$$

while

$$\left(1 - q_{n-1}''\right)\left(1 + q_n''\right) = 4\zeta \left(\overline{\mathbb{E}}_\zeta u_{n-1} \right) \tag{6.10}$$

which gives the ζ-transform in the opposite direction

$$u_n \to u_n'' = \overline{\mathbb{E}}_\zeta u_{n-1} \tag{6.11}$$

However, these transformations are too simple to be interesting: they do not change the structure of solutions contrary to, say, Backlund-Darboux ones which add solitons. This is clearly seen if one acts by (6.7) or (6.11) on the simplest (constant) solution of the VH, $u_n = u_\infty = $ constant:

$$u_n' = u_n'' = u_n = u_\infty. \tag{6.12}$$

At the same time, one can get more rich transformations by *linear superposition* of (6.7) and (6.11) rewritten in terms of the tau-functions,

$$\tau_n \to \mathbb{E}_\zeta \tau_{n+1} \qquad \text{and} \qquad \tau_n \to \overline{\mathbb{E}}_\zeta \tau_{n-1}. \tag{6.13}$$

It is non-trivial moment: our equations are nonlinear and in general a linear combination of two solutions is not a solution. However in our case $\mathbb{E}_\zeta \tau_{n+1}$ and $\overline{\mathbb{E}}_\zeta \tau_{n-1}$ are not completely independent and it can be shown that the transformation $\tau_n \to \widetilde{\tau}_n$ defined by

$$\widetilde{\tau}_n(\mathrm{t}) = \tau_{n+1}\left(\mathrm{t} + [\eta]\right) + \eta^{-n} \exp\left[\chi(\eta, \mathrm{t})\right] \tau_{n-1}\left(\mathrm{t} - [\eta]\right) \tag{6.14}$$

which leads to the Miura transformation $u_n \to q_n$ that generalizes (6.4) and (6.8),

$$q_n = 1 - 2\frac{\tau_{n+1}\widetilde{\tau}_{n-1}}{\tau_n \widetilde{\tau}_n} = 2\eta \frac{\tau_{n-1}\widetilde{\tau}_{n+1}}{\tau_n \widetilde{\tau}_n} - 1, \tag{6.15}$$

is indeed a Backlund transformation: if τ_n solves (2.55), so does $\widetilde{\tau}_n$ provided

$$\chi(\eta, \mathrm{t} + [\zeta]) - \chi(\eta, \mathrm{t}) = \Gamma(\eta, \zeta) - \Gamma(\eta, 0) \tag{6.16}$$

with

$$\Gamma(\eta, \zeta) = \ln \frac{c(\eta, \zeta)}{a(\eta, \zeta)}. \tag{6.17}$$

It is easy to see that χ is a linear function of times

$$\chi(\eta, \mathrm{t}) = \sum_{j=1}^\infty \chi_j(\eta)\, t_j \tag{6.18}$$

with the coefficients $\chi_j(\eta)$ being determined by the Taylor series for $\Gamma(\eta, \zeta)$ as a function of ζ:

$$\chi_j(\eta) = \frac{1}{j}\, \Gamma_j(\eta), \qquad i = 1, 2, ... \tag{6.19}$$

where

$$\sum_{j=0}^\infty \Gamma_j(\eta)\, \zeta^j = \ln \frac{c(\eta, \zeta)}{a(\eta, \zeta)}. \tag{6.20}$$

Contrary to (6.7) and (6.11), transformation (6.14) gives non-trivial results. Say, applying it to the tau-function corresponding to the constant solution one gets nothing but the one-soliton one. The elementary Backlund transformations (6.14) are commutative and can be used to construct more complex ones (by superposition). I do not discuss this matter further because the main result of this section is (6.14): it has been shown that by means of the Miwa's shifts it is possible to derive the *explicit* form of the Backlund transformations which can hardly be done in the framework of other approaches.

7 Dark solitons.

Now I want to discuss the soliton solutions of the VH. After we have presented all differential equations of the hierarchy in the functional form (2.55) this problem becomes algebraic one and can be solved, as is shown below, by means of the elementary matrix calculus giving as a result common soliton solutions of all Volterra equations.

In what follows I will restrict myself to the case of the so-called finite-density boundary conditions

$$\lim_{n \to \infty} u_n = u_\infty. \tag{7.1}$$

In terms of the tau-functions (7.1) gives

$$\tau_n \sim u_\infty^{n^2/4} e^{n\varphi} \quad \text{as} \quad n \to \infty. \tag{7.2}$$

Note that a choice of φ does not affect the value of u_n since the latter is invariant under $\tau_n \to \tau_n q^n$ transforms. I will determine this function from the condition that the simplest (vacuum) solution is given by

$$\tau_n^{(vac)}(\mathrm{t}) = u_\infty^{n^2/4} e^{n\varphi(\mathrm{t})}. \tag{7.3}$$

After substituting (7.3) into, for example, (2.55) one comes to the equation

$$\exp\left\{\varphi\left(\mathrm{t} + [\zeta]\right) - \varphi\left(\mathrm{t}\right)\right\} = f\left(\zeta\right) \tag{7.4}$$

where $f(\zeta)$ is the solution of the equation

$$\zeta u_\infty f^2 - f + 1 = 0 \tag{7.5}$$

which satisfies condition $f(0) = 1$, i.e.

$$f(\zeta) = \frac{2}{1 + \sqrt{1 - 4u_\infty \zeta}}\,. \tag{7.6}$$

Equation (7.4) can be solved by the ansatz

$$\varphi\left(\mathrm{t}\right) = \sum_{j=1}^{\infty} \varphi_j\, t_j \tag{7.7}$$

where the coefficients φ_j should be determined from the condition $\sum_j \varphi_j \zeta^j / j = \ln f(\zeta)$ which, after applying $\zeta d/d\zeta$, can be rewritten as

$$\sum_{j=1}^{\infty} \varphi_j\, \zeta^j = \frac{1 - \sqrt{1 - 4u_\infty \zeta}}{2\sqrt{1 - 4u_\infty \zeta}}\,. \tag{7.8}$$

Now one can take into account the boundary conditions (7.1) by looking for the solution of (2.55) in the form

$$\tau_n = \tau_n^{(vac)}\,\omega_n, \qquad \lim_{n \to \infty} \omega_n = 1. \tag{7.9}$$

In terms of ω_n equation (2.55) can be written as

$$[f(\zeta) - 1] \omega_{n-2} \left(\overline{\mathbb{E}}_\zeta \omega_{n+1} \right) - f(\zeta) \omega_{n-1} \left(\overline{\mathbb{E}}_\zeta \omega_n \right) + \omega_n \left(\overline{\mathbb{E}}_\zeta \omega_{n-1} \right) = 0 \qquad (7.10)$$

and namely this equation will be solved in what follows.

I will not present a detailed textbook-like derivation of N-soliton solutions of (7.10) but show that they are given by

$$\omega_n = \omega(A_n), \qquad \omega(A) = \det |\mathbb{I} + A| \qquad (7.11)$$

where $\mathbb{I}$ is the $N \times N$ unit matrix and A_n is the matrix with the elements

$$A_n^{(jk)} = \frac{\ell_j \, a_{nk}(\mathrm{t})}{\xi_j - \eta_k} \qquad j, k = 1, ..., N \qquad (7.12)$$

(matrices of this type often arise in the soliton theory). Here ℓ_j, ξ_j and η_k are some constant parameters and $a_{nk}(\mathrm{t})$ are some functions of all times t_1, t_2, ..., which will be determined below. I will rely on the algebraic properties of the matrices (7.12) satisfying

$$LA - AR = |\ell\rangle\langle a| \qquad (7.13)$$

where $L = \mathrm{diag}(\xi_1, ..., \xi_N)$, $R = \mathrm{diag}(\eta_1, ..., \eta_N)$ and the bra-ket notation is used for the N-columns, $|\ell\rangle = (\ell_1, ..., \ell_N)^T$ and the N-rows, $\langle a| = (a_1, ..., a_N)$. In the appendix one can find some basic facts related to such matrices together with the derivation of the identity which we need for our purpose and which can be formulated as follows. If we have an one-parameter family of matrices M_ζ

$$M_\zeta = I_\zeta J_\zeta^{-1} \qquad (7.14)$$

with

$$I_\zeta = \mathbb{I} - \zeta L \quad \text{and} \quad J_\zeta = \mathbb{I} - \zeta R \qquad (7.15)$$

then the determinants of the matrices deformed by means of M_ζ,

$$\begin{aligned} \Omega_\alpha &= \det |\mathbb{I} + A M_\alpha|, & (7.16) \\ \Omega_{\alpha\beta} &= \det |\mathbb{I} + A M_\alpha M_\beta|, & (7.17) \end{aligned}$$

satisfy the identity

$$\alpha(\beta - \gamma) \Omega_\alpha \Omega_{\beta\gamma} + \beta(\gamma - \alpha) \Omega_\beta \Omega_{\alpha\gamma} + \gamma(\alpha - \beta) \Omega_\gamma \Omega_{\alpha\beta} = 0 \qquad (7.18)$$

which is an elementary version of the famous Fay's formula for the theta functions [10].

Now, to solve (7.10) one has to state that both the Miwa's shifts and the shifts of the index n can be obtained by means of the M-matrices:

$$\mathbb{E}_\zeta A_n = A_n M_\alpha \qquad (7.19)$$

where α is a function of ζ and

$$\begin{aligned} A_{n-1} &= A_n M_\beta & (7.20) \\ A_{n-2} &= A_n M_\gamma & (7.21) \end{aligned}$$

where, of course, M_β and M_γ should be related by $M_\gamma = M_\beta^2$. Then equation (7.10) can be rewritten as

$$[f(\zeta) - 1]\,\Omega_\alpha\Omega_{\beta\gamma} + \Omega_\beta\Omega_{\alpha\gamma} - f(\zeta)\,\Omega_\gamma\Omega_{\alpha\beta} = 0 \tag{7.22}$$

(here I used the definitions (7.16) and (7.17) with $A = A_{n+1}$) and to solve it one has only to determine α, β and γ by comparing it with (7.18). After some simple algebra one can conclude that the proper choice of the parameters is

$$\alpha(\zeta) = f(\zeta) - 1, \qquad \beta = -1, \qquad \gamma = \infty \tag{7.23}$$

while the restriction $M_\infty = M_{-1}^2$ leads to

$$R = L^{-1}. \tag{7.24}$$

This completes the solution of the problem. The final result can be rewritten as

$$\omega_n = \det\left| \delta_{jk} + \frac{a_k(\mathrm{t})\,\xi_k^{-n}}{1 - \xi_j\xi_k} \right| \tag{7.25}$$

(here the 'left' multipliers $\ell_j\xi_j$ has been incorporated into a_k by the gauge transform $A_n \to S^{-1}A_nS$ which does not change the determinants) with

$$a_k(\mathrm{t}) = a_k^{(0)}\exp\left[\phi_k(\mathrm{t})\right]. \tag{7.26}$$

The time dependence is given by

$$\phi_k(\mathrm{t}) = \phi\left(\xi_k, \mathrm{t}\right) \tag{7.27}$$

with

$$\phi\left(\xi, \mathrm{t}\right) = \sum_{j=1}^{\infty} \nu_j(\xi)\,t_j \tag{7.28}$$

where the coefficients ν_j should be determined from the series

$$\sum_{j=1}^{\infty} \nu_j(\xi)\,\frac{\zeta^j}{j} = \ln\frac{1 - \alpha(\zeta)\,\xi}{1 - \alpha(\zeta)\,\xi^{-1}} \tag{7.29}$$

which is a condensed form of *all dispersion laws* of the VH in the dark soliton case.

8 Volterra and other hierarchies

As the last application of the functional approach I am going to discuss an interesting question of interrelations between different integrable systems. The idea behind examples given below is that starting from the main equation of this work (2.50) one can derive some other equations which turn out to be closely related to hierarchies different from the VH.

Equation (2.50) is an functional-difference equation which relates *four* tau-functions with different index n. However by simple algebra one can derive, as its consequences, equations involving less number of τ_n's. One of them can be written as a *two*-point relation

$$D(\zeta)\ \tau_n \cdot \tau_{n-1} = \left(\overline{\mathbb{E}}_\zeta \tau_{n-1}\right)\left(\mathbb{E}_\zeta \tau_n\right) - \tau_{n-1}\tau_n \tag{8.1}$$

where $D(\zeta)$ is a "hierarchical" version of the Hirota's bilinear operators: $D(\zeta) = \sum_{j=1}^{\infty} \zeta^j D_j$ (all formulae of this section are written for the case of zero boundary conditions). Expanding this relation in the multidimensional bilinear Taylor series using the identity

$$\left(\mathbb{E}_\zeta a\right)\left(\overline{\mathbb{E}}_\zeta b\right) = F\left(\zeta, D\right)\ a \cdot b \tag{8.2}$$

where the bilinear operator F is given by

$$F\left(\zeta, D\right) = 1 + \zeta D_1 + \frac{\zeta^2}{2}\left(D_2 + D_{11}\right) + \frac{\zeta^3}{6}\left(2D_3 + 3D_{21} + D_{111}\right) + ... \tag{8.3}$$

(here $D_{j\,k...} = D_j D_k...$) one can get an infinite set of differential equations for the pair τ_n and τ_{n-1}. A few first of them are

$$\left(-D_2 + D_{11}\right)\ \tau_n \cdot \tau_{n-1} = 0 \tag{8.4}$$

$$\left(-4D_3 + 3D_{21} + D_{111}\right)\ \tau_n \cdot \tau_{n-1} = 0 \tag{8.5}$$

$$\left(-18D_4 + 8D_{31} + 3D_{22} + 6D_{211} + D_{1111}\right)\ \tau_n \cdot \tau_{n-1} = 0 \tag{8.6}$$

$$\cdots$$

Comparing these equations with the ones presented in [11] one can conclude that i) equation (8.1) can be viewed as the Miwa's representation for the 1st modified KP hierarchy (according to the classification of [11]) and that ii) this hierarchy can be 'embedded' into the VH in the sense that any tau-function of the VH is at the same time a solution of all modified KP equations.

Continuing the procedure of decreasing the number of tau-functions involved in our functional relations it is possible to derive the following one:

$$\frac{\zeta}{2}\ D(\zeta)D_1\ \tau_n \cdot \tau_n = \left(\overline{\mathbb{E}}_\zeta \tau_n\right)\left(\mathbb{E}_\zeta \tau_n\right) - \tau_n^2 \tag{8.7}$$

which contains only *one* tau-function and which can be rewritten as

$$\left[\frac{\zeta}{2}D(\zeta)D_1 + 1 - F(\zeta, D)\right]\ \tau_n \cdot \tau_n = 0 \tag{8.8}$$

where operator F was defined above. The simplest equation of this infinite set,

$$\left(-4D_{31} + 3D_{22} + D_{1111}\right)\ \tau_n \cdot \tau_n = 0, \tag{8.9}$$

is nothing but the KP equation. So we have come to an interesting fact that the KP equation can be 'embedded' into the VH and have obtained the functional representation (8.8) for the KP hierarchy.

9 Conclusion.

In this paper we have derived the so-called functional representation of the VH. To my knowledge this is a new result. Of course, it is closely related to the discrete time approach which has been developed for the Volterra model by different authors (see book of Suris [12] and references therein). However equations discussed here were considered not as discrete but as ones with analytic dependence on the deformation parameter and, secondly, can be viewed as a realization of the discrete maps in terms of the usual Volterra flows. In some sense, the functional representation is an intermediate method between the discretization of time and one of the Fay's identities. The later approach, which also uses the Miwa's shifts (usually a few simultaneously), is a rather common tool to deal with such equations as KP or 2D Toda lattice but, to my knowledge, has not been discussed in context of the VH. Moreover, what has been done in this work is not limited to the Volterra model: results of the section 8 devoted to the KP-like equations seem to be new and it is rather difficult to derive them from the standard Fay's identities (due to the appearance of the $D(\zeta)$ operator).

Another point that distinguishes approach of this work from the traditional IST-based studies of the integrable systems is the fact that here we have obtained some explicit realizations of the IST algorithms (see, e.g., section 5). In other words, one of the main results of this work is the 'eliminating' of the intermediate objects of the IST, such as Jost or Baker-Akhiezer functions. The fact that we have expressed the Backlund transforms and the generating function for the conservation laws in terms of the solutions of the Volterra equations themselves is interesting not only from the theoretical viewpoint. Say, these results can be useful in tackling the non-integrable problems related to the Volterra model by improving the averaging methods (exploiting the conservation laws) or the perturbation theories (which use the Jost functions to construct the inverse of the operator appearing after linearization of the our nonlinear equations).

Considering the practical aspects of these studies I would like to note that this approach can simplify the problem of finding solutions, as was demonstrated in section 7. Another example, which was not discussed here, is the derivation of the quasi-periodical ones. To do this one does not need now to develop the algebro-geometric scheme or to solve the inverse Jacobi problem. Instead one can use the straightforward method (applied, e.g., in [13] for the Ablowitz-Ladik and in [14] for the Hirota's bilinear discrete equations). By comparing equation (2.50) with the original Fay's identity for the theta functions [10] one can immediately establish the structure of the solution and determine the dispersion law by expanding in series some functions appearing in the latter in a way similar to one used in sections 6 and 7 (see (6.20) an (7.29)).

Finally, I would like to make the following remark. The functional representation is known now for many systems. In principle it can be obtained for any integrable system (this problem is analogous, and rather close, to elaboration of the so-called Sato's approach). However, there is no systematic procedure of its derivation. If one starts from the scattering problem, he could hardly predict beforehand how exactly the Miwa's shifts will appear. Say, referring to this work, it was a crucial moment to guess that namely the quantity (2.42) is what we need. So, to my opinion, the question of the relation between the IST-based formulation of integrable systems and one based on the functional equations, considered from a more general viewpoint, seems to be interesting and deserves

further studies.

Appendix.

The aim of this appendix is to derive the identity (7.18) which can be rewritten as

$$\sum_{c.p.} \alpha_i \left(\alpha_j - \alpha_k\right) \omega_{\alpha_i} \omega_{\alpha_j \alpha_k} = 0 \tag{A.1}$$

where $\sum_{c.p.}$ is the sum over the cyclic permutations of the indices i, j, k.

To do this I am going first to derive some formulae relating determinants $\omega(B)$ and $\omega(C)$ with $\omega(BM_\zeta^{-1})$ and $\omega(CM_\zeta)$ where M_ζ is given by (7.14) and B, C are matrices of the type (7.12):

$$LB - BR = |\ell\rangle\langle b|, \qquad LC - CR = |\ell\rangle\langle c|. \tag{A.2}$$

This can be done as follows. For any matrix (A.2) and any number ζ it is easy to derive the chain of identities

$$BJ_\zeta(\mathbb{I} + B) = BJ_\zeta - I_\zeta B + (I_\zeta + BJ_\zeta)B = BJ_\zeta - I_\zeta B + \left(\mathbb{I} + BM_\zeta^{-1}\right) I_\zeta B \tag{A.3}$$

which leads to

$$\left(\mathbb{I} + BM_\zeta^{-1}\right) I_\zeta BF(B) = BJ_\zeta + (I_\zeta B - BJ_\zeta)F(B) \tag{A.4}$$

with

$$F(B) = (\mathbb{I} + B)^{-1} \tag{A.5}$$

and

$$\left(\mathbb{I} + BM_\zeta^{-1}\right)(\mathbb{I} - I_\zeta BF(B)Y) = \mathbb{I} + BM_\zeta^{-1} - BJ_\zeta Y + (BJ_\zeta - I_\zeta B)F(B)Y \tag{A.6}$$

$$= \mathbb{I} + BJ_\zeta\left(I_\zeta^{-1} - Y\right) + (BJ_\zeta - I_\zeta B)F(B)Y \tag{A.7}$$

where Y is an arbitrary matrix. Choosing

$$Y = Y_\zeta = I_\zeta^{-1} - J_\zeta^{-1}D \tag{A.8}$$

where D is a diagonal matrix, $D = \text{diag}(c_j/b_j)$ (for simplicity I restrict myself to the case $b_j \neq 0$), one can continue (A.7) as

$$\left(\mathbb{I} + BM_\zeta^{-1}\right)(\mathbb{I} - I_\zeta BF(B)Y_\zeta) = \mathbb{I} + C + (BJ_\zeta - I_\zeta B)F(B)Y_\zeta \tag{A.9}$$

$$= [1 + (BJ_\zeta - I_\zeta B)F(B)Y_\zeta F(C)](\mathbb{I} + C). \tag{A.10}$$

Taking the determinant one can get

$$\omega\left(BM_\zeta^{-1}\right) \det|\mathbb{I} - I_\zeta BF(B)Y_\zeta| = \omega(C) \det|\mathbb{I} + (BJ_\zeta - I_\zeta B)F(B)Y_\zeta F(C)|. \tag{A.11}$$

From the other hand,

$$\det |\, \mathbb{I} - I_\zeta BF(B)Y_\zeta | \;\; = \;\; \frac{1}{\omega(B)} \det |\, \mathbb{I} + B - Y_\zeta I_\zeta B| \tag{A.12}$$

$$= \;\; \frac{1}{\omega(B)} \det |\, \mathbb{I} + B(1 - Y_\zeta I_\zeta)| \tag{A.13}$$

$$= \;\; \frac{1}{\omega(B)} \det |\, \mathbb{I} + CM_\zeta | \tag{A.14}$$

$$= \;\; \frac{\omega(CM_\zeta)}{\omega(B)} \tag{A.15}$$

As to the determinant appearing in the right-hand side of (A.11), it can be calculated using the identity

$$BJ_\zeta - I_\zeta B = \zeta |\, \ell \,\rangle\langle b\, | \tag{A.16}$$

which follows from (A.2) and the identity

$$\det |\, \mathbb{I} + |\, u \,\rangle\langle v|\, | = 1 + \langle v |\, u \,\rangle. \tag{A.17}$$

This leads to

$$\det |\, \mathbb{I} + (BJ_\zeta - I_\zeta B)\, F(B)Y_\zeta F(C)| = 1 + \zeta \langle b|F(B)Y_\zeta F(C)|\ell\rangle. \tag{A.18}$$

Gathering (A.15) and (A.18) we come to

$$\frac{\omega\left(BM_\zeta^{-1}\right)\omega(CM_\zeta)}{\omega(B)\omega(C)} = 1 + \zeta \langle b|\, F(B)Y_\zeta F(C)|\, \ell \,\rangle. \tag{A.19}$$

Taking this formula with

$$B = AM_\alpha M_\beta M_\gamma, \qquad C = A \qquad \text{and} \qquad \zeta = \alpha \tag{A.20}$$

one can present $\Omega_\alpha \Omega_{\beta\gamma}/\Omega\Omega_{\alpha\beta\gamma}$ as

$$\frac{\Omega_\alpha \Omega_{\beta\gamma}}{\Omega\Omega_{\alpha\beta\gamma}} \;\; = \;\; \frac{\omega\left(BM_\alpha^{-1}\right)\omega\left(CM_\alpha\right)}{\omega(B)\omega(C)} \tag{A.21}$$

$$= \;\; 1 + \alpha \langle b|\, F(B) \left[I_\alpha^{-1} - J_\alpha^{-1} M_\alpha^{-1} M_\beta^{-1} M_\gamma^{-1} \right] F(A)|\, \ell \,\rangle \tag{A.22}$$

$$= \;\; 1 + \alpha \langle b|\, F(B) I_\alpha^{-1} I_\beta^{-1} I_\gamma^{-1} \left[I_\beta I_\gamma - J_\beta J_\gamma \right] F(A)|\, \ell \,\rangle. \tag{A.23}$$

Here, of course, the row $\langle b|$ and the matrix B depend on α, β and γ, but depend *symmetrically* (so this dependence will not be indicated explicitly). Using the definition of I_ζ and J_ζ one can get

$$I_\beta I_\gamma - J_\beta J_\gamma = (\beta + \gamma)(R - L) + \beta\gamma \left(L^2 - R^2\right) \tag{A.24}$$

which gives

$$\Omega_\alpha \Omega_{\beta\gamma} = u + \alpha(\beta + \gamma)v \tag{A.25}$$

where, again, u and v depend on α, β and γ,

$$u \;=\; \Omega\,\Omega_{\alpha\beta\gamma}\left\{1+\alpha\beta\gamma\langle b|\,F(B)I_\alpha^{-1}I_\beta^{-1}I_\gamma^{-1}\left(L^2-R^2\right)F(A)|\,\ell\,\rangle\right\}, \qquad (A.26)$$

$$v \;=\; \Omega\,\Omega_{\alpha\beta\gamma}\,\langle b|\,F(B)I_\alpha^{-1}I_\beta^{-1}I_\gamma^{-1}\left(R-L\right)F(A)|\,\ell\,\rangle \qquad (A.27)$$

but depend *symmetrically*. Now it is easy to finish the derivation of (A.1):

$$\sum_{c.p.}\alpha_i\left(\alpha_j-\alpha_k\right)\Omega_{\alpha_i}\Omega_{\alpha_j\alpha_k} \;=\; u\sum_{c.p.}\alpha_i\left(\alpha_j-\alpha_k\right)+v\sum_{c.p.}\alpha_i^2\left(\alpha_j^2-\alpha_k^2\right) \qquad (A.28)$$

$$\qquad\qquad\qquad\qquad =\; 0 \qquad\qquad\qquad\qquad\qquad\qquad (A.29)$$

Of course, the calculations presented here hardly have anything specific for the theory of integrable systems: this is just an exercise in elementary matrix calculus. However the bilinear matrix identities of this kind are very useful in analysis of integrable equations, giving, for example, a possibility to derive the soliton solutions in a very easy and short way.

References

[1] Lotka A.J., Contribution to the theory of periodic reactions. *J. Phys. Chem.* **14** (1910), 271–274.

[2] Lotka A.J., Analytical note on certain rhythmic relations in organic systems. *Proc. Nat. Acad. Sci. USA* **6** (1920), 410–415.

[3] V. Volterra. *Leçons sur la théorie mathématique de la lutte pour la vie.* Gauthier-Villars, Paris, 1931.

[4] Manakov S.V., Complete integrability and stochastization of discrete dynamical systems. *Soviet Phys. JETP* **40** (1974), 269-274.

[5] Kac M. and van Moerbeke P., On an explicitly soluble system of non-linear differential equations related to certain Toda lattices. *Adv. in Math.* **16** (1975), 160–169.

[6] Faddeev L. D. and Takhtajan L. A., *Hamiltonian Methods in the Theory of Solitons.* Springer-Verlag, Berlin, 1987.

[7] Yasumasa Kajinaga and Miki Wadati, A new derivation of the Backlund transformation for the Volterra lattice. *J. Phys. Soc. of Japan* **67** (1998), 2237–2241.

[8] Wadati M., Transformation theories for nonlinear discrete systems. *Progr. Theor. Phys. Suppl.* **59** (1976), 36–63.

[9] Volkov A.Yu., Miura transformation on a lattice. *Theor. Math. Phys.* **74** (1988), 96–99. [Translated from *Teor. Mat. Fiz.* **74** (1988), 135].

[10] Fay J., Theta functions on Riemann surfaces. *Lect. Notes in Math.*, **352** (1973), 1–137.

[11] Jimbo M. and Miwa T., Solitons and infinite dimensional Lie algebras. *Publ. Res. Inst. Math. Sci.* **19** (1983), 943-1001.

[12] Suris Yu.B., *The Problem of Integrable Discretization: Hamiltonian Approach*, Birkhauser Verlag, Basel Boston Berlin 2003.

[13] Vekslerchik V.E., Finite genus solutions for the Ablowitz-Ladik hierarchy. *J. Phys. A* **32** (1999), 4983–4994.

[14] Vekslerchik V.E., Finite-genus solutions for the Hirota's bilinear difference equation. `nlin.SI/0002005`.