

# The Structure of Gelfand-Levitan-Marchenko Type Equations for Delsarte Transmutation Operators of Linear Multi-Dimensional Differential Operators and Operator Pencils. Part 2

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## Abstract

The differential-geometric and topological structure of Delsarte transmutation operators their associated Gelfand-Levitan-Marchenko type equations are studied making use of the de Rham-Hodge-Skrypnik differential complex. The relationships with spectral theory and special Berezansky type congruence properties of Delsarte transmuted operators are stated. Some applications to multi-dimensional differential operators are done including the three-dimensional Laplace operator and the two-dimensional classical Dirac operator and its multi-dimensional affine extension, related with self-dual Yang-Mills equations. The soliton like solutions to the related set of nonlinear dynamical systems are discussed.

## 1 The generalized de Rham-Hodge theory aspects and related Delsarte-Darboux type binary transformations

A differential-geometric analysis of Delsarte-Darboux type transformations that was done in Part 1 for differential operator expressions acting in a functional space  $\mathcal{H} = L_2(\mathbb{T}; H)$ ,

where  $T = \mathbb{R}^2$  and  $H := L_2(\mathbb{R}^2; \mathbb{C}^2)$ , appears to have a deep relationship with a generalized de Rham-Hodge theory [3, 4, 5, 6, 27] devised in the midst of the past century for a set of commuting differential operators defined, in general, on a smooth compact  $m$ -dimensional metric space  $M$ . Concerning our problem of describing the differential-geometric and spectral structure of Delsarte-Darboux type transmutations acting in  $\mathcal{H}$ , we preliminarily consider some backgrounds of the generalized de Rham-Hodge theory devised formerly by I.V. Skrypnik [3, 4, 5, 6] for studying differential complexes. Consider a smooth metric space  $M$  being a suitably compactified form of the space  $\mathbb{R}^m$ ,  $m \in \mathbb{Z}_+$ . Then one can define on  $M_T := T \times M$  the standard Grassmann algebra  $\Lambda(M_T; \mathcal{H})$  of differential forms on  $T \times M$  and consider a generalized external anti-differentiation operator  $d_{\mathcal{L}} : \Lambda(M_T; \mathcal{H}) \rightarrow \Lambda(M_T; \mathcal{H})$  acting as follows: for any  $\beta^{(k)} \in \Lambda^k(M_T; \mathcal{H})$ ,  $k = \overline{0, m}$ ,

$$d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^2 dt_j \wedge L_j(t; x|\partial)\beta^{(k)} + \sum_{i=1}^m dx_i \wedge A_i(t; x; \partial)\beta^{(k)} \in \Lambda^{k+1}(M_T; \mathcal{H}), \tag{1.1}$$

where  $A_i \in C^2(T; \mathcal{L}(H))$ ,  $i = \overline{1, m}$ , are some differential operator mappings and

$$L_j(t; x|\partial) := \partial/\partial t_j - L_j(t; x|\partial) \tag{1.2}$$

$j = \overline{1, 2}$ , are suitably defined linear differential operators in  $\mathcal{H}$ , commuting with each other, that is

$$[L_1, L_2] = 0, [A_k, A_i] = 0 \text{ and } [L_j, A_i] = 0 \tag{1.3}$$

for all  $j = \overline{1, 2}$  and  $i, k = \overline{1, m}$ . We will put, in general, that differential expressions

$$L_j(t; x|\partial) := \sum_{|\alpha|=0}^{n_j(L)} a_{\alpha}^{(j)}(t; x) \frac{\partial^{|\alpha|}}{\partial x^{\alpha}}, \tag{1.4}$$

with coefficients  $a_{\alpha}^{(j)} \in C^1(T; C^{\infty}(M; End\mathbb{C}^N))$ ,  $|\alpha| = \overline{0, n_j(L)}$   $n_j^{\alpha} \in \mathbb{Z}_+$ ,  $j = \overline{0, 1}$ , are some closed normal densely defined operators in the Hilbert space  $H$  for any  $t \in T$ . It is easy to observe that the anti-differentiation  $d_{\mathcal{L}}$  defined by (1.1) is a generalization of the usual interior anti-differentiation

$$d = \sum_{j=1}^m dx_j \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^2 dt_s \wedge \frac{\partial}{\partial t_s} \tag{1.5}$$

for which, evidently, commutation conditions

$$\left[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right] = 0, \left[\frac{\partial}{\partial t_s}, \frac{\partial}{\partial t_l}\right] = 0, \left[\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t_s}\right] = 0 \tag{1.6}$$

hold for all  $j, k = \overline{1, m}$  and  $s, l = \overline{1, 2}$ . Substituting within (1.5)  $\partial/\partial x_j \rightarrow A_j$ ,  $\partial/\partial t_s \rightarrow L_s$ ,  $j = \overline{1, m}$ ,  $s = \overline{1, 2}$ , one gets the anti-differentiation

$$d_A := \sum_{j=1}^m dx_j \wedge A_j(t; x|\partial) + \sum_{j=1}^2 dt_s \wedge L_s(t; x|\partial), \tag{1.7}$$

where the differential expressions  $A_j, L_s : \mathcal{H} \rightarrow \mathcal{H}$  for all  $j, k = \overline{1, m}$  and  $s, l = \overline{1, 2}$ , satisfy the commutation conditions  $[A_j, A_k] = 0, [L_s, L_s] = 0, [A_j, L_s] = 0$ , then operation (1.7) defines on  $\Lambda(M_T; \mathcal{H})$  an anti-differential  $d_A$  with respect to which the co-chain complex.

$$\mathcal{H} \rightarrow \Lambda^0(M_T; \mathcal{H}) \xrightarrow{d_A} \Lambda^1(M_T; \mathcal{H}) \xrightarrow{d_A} \dots \xrightarrow{d_A} \Lambda^{m+2}(M_T; \mathcal{H}) \xrightarrow{d_A} 0 \tag{1.8}$$

is evidently closed, that is  $d_A d_A \equiv 0$ . As the anti-differential (1.1) is a particular case of (1.7), we obtain that the co-chain complex (1.8) corresponding to it is closed too.

Below we will follow ideas formerly developed in [3, 4, 5, 6, 30]. A differential form  $\beta \in \Lambda(M_T; \mathcal{H})$  will be called  $d_A$ -closed if  $d_A \beta = 0$  and a form  $\gamma \in \Lambda(M_T; \mathcal{H})$  will be called exact or  $d_A$ -cohomologous to zero if there exists on  $M_T$  such a form  $\omega \in \Lambda(M_T; \mathcal{H})$  that  $\gamma = d_A \omega$ .

Consider now the standard [29, 30, 8, 32] algebraic Hodge star-operation

$$* : \Lambda^k(M_T; \mathcal{H}) \rightarrow \Lambda^{m+2-k}(M_T; \mathcal{H}), \tag{1.9}$$

$k = \overline{0, m+2}$ , as follows: if  $\beta \in \Lambda^k(M_T; \mathcal{H})$ , then the form  $*\beta \in \Lambda^{m+2-k}(M_T; \mathcal{H})$  is such that:

- $(m - k + 2)$  - dimensional volume  $|*\beta|$  of the form  $*\beta$  equals  $k$ -dimensional volume  $|\beta|$  of the form  $\beta$ ;
- the  $(m + 2)$  -dimensional measure  $\bar{\beta}^\top \wedge *\beta > 0$  under the fixed orientation on  $M_T$ .

Define also on the space  $\Lambda(M_T; \mathcal{H})$  the following natural scalar product: for any  $\beta, \gamma \in \Lambda^k(M_T; \mathcal{H}), k = \overline{0, m}$ ,

$$(\beta, \gamma) := \int_{M_T} \bar{\beta}^\top * \gamma. \tag{1.10}$$

Subject to the scalar product (1.10) one can naturally construct the corresponding Hilbert space

$$\mathcal{H}_\Lambda(M_T) := \bigoplus_{k=0}^{m+2} \mathcal{H}_\Lambda^k(M_T) \tag{1.11}$$

well suitable for our further consideration. Notice also here, that the Hodge star  $*$ -operation satisfies the following easily checkable property: for any  $\beta, \gamma \in \mathcal{H}_\Lambda^k(M_T), k = \overline{0, m}$ ,

$$(\beta, \gamma) = (*\beta, *\gamma), \tag{1.12}$$

that is the Hodge operation  $* : \mathcal{H}_\Lambda(M_T) \rightarrow \mathcal{H}_\Lambda(M_T)$  is unitary and its standard adjoint with respect to the scalar product (1.10) operation satisfies the condition  $(*)' = (*)^{-1}$ .

Denote by  $d'_\mathcal{L}$  the formally adjoint expression to the weak differential operation (1.1). By means of the operations  $d'_\mathcal{L}$  and  $d_\mathcal{L}$  in the  $\mathcal{H}_\Lambda(M_T)$  one can naturally define [8, 29, 30, 3, 32] the generalized Laplace-Hodge operator  $\Delta_\mathcal{L} : \mathcal{H}_\Lambda(M_T) \rightarrow \mathcal{H}_\Lambda(M_T)$  as

$$\Delta_\mathcal{L} = d'_\mathcal{L} d_\mathcal{L} + d'_\mathcal{L} d_\mathcal{L} . \tag{1.13}$$

Take a form  $\beta \in \mathcal{H}_\Lambda(M_T)$  satisfying the equality

$$\Delta_{\mathcal{L}}\beta = 0. \tag{1.14}$$

Such a form is called [3, 30, 32, 8] harmonic. One can also verify that a harmonic form  $\beta \in \mathcal{H}_\Lambda(M_T)$  satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}}\beta = 0, \quad d_{\mathcal{L}}\beta = 0 \tag{1.15}$$

follows readily from (1.13) and (1.14).

It is easy to check that the following differential operators in  $\mathcal{H}_\Lambda(M_T)$

$$d_{\mathcal{L}}^* := *d'_{\mathcal{L}}(*)^{-1} \tag{1.16}$$

defines also a new external anti-differential operation in  $\mathcal{H}_\Lambda(M_T)$ .

**Lemma 1.1.** *The corresponding dual to (1.8) co-chain complex*

$$\mathcal{H} \longrightarrow \Lambda^0(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} \Lambda^1(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} \dots \xrightarrow{d_{\mathcal{L}}^*} \Lambda^{m+2}(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{L}}^*} 0 \tag{1.17}$$

is exact.

**Proof.** A proof follows owing to the property  $d_{\mathcal{L}}^*d_{\mathcal{L}}^* = 0$  holding due to the definition (1.16). ■

Denote further by  $\mathcal{H}_{\Lambda(\mathcal{L})}^k(M_T)$ ,  $k = \overline{0, m+2}$ , the cohomology groups of  $d_{\mathcal{L}}$ -closed and by  $\mathcal{H}_{\Lambda(\mathcal{L}^*)}^k(M_T)$ ,  $k = \overline{0, m+2}$ ,  $k = \overline{0, m+2}$ , the cohomology groups of  $d_{\mathcal{L}}^*$ -closed differential forms, respectively, and by  $\mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L})}^k(M_T)$ ,  $k = \overline{0, m+2}$ , the abelian groups of harmonic differential forms from the Hilbert sub-spaces  $\mathcal{H}_\Lambda^k(M_T)$ ,  $k = \overline{0, m+2}$ . Before formulating the next results, define the standard Hilbert-Schmidt rigged chain [12, 13] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}_{\Lambda,+}^k(M_T) \subset \mathcal{H}_\Lambda^k(M_T) \subset \mathcal{H}_{\Lambda,-}^k(M_T), \tag{1.18}$$

the corresponding hereditary rigged chains of harmonic forms:

$$\mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L}),+}^k(M_T) \subset \mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L})}^k(M_T) \subset \mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L}),-}^k(M_T) \tag{1.19}$$

and chains of cohomology groups:

$$\begin{aligned} \mathcal{H}_{\Lambda(\mathcal{L}),+}^k(M_T) &\subset \mathcal{H}_{\Lambda(\mathcal{L})}^k(M_T) \subset \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T), \\ \mathcal{H}_{\Lambda(\mathcal{L}^*),+}^k(M_T) &\subset \mathcal{H}_{\Lambda(\mathcal{L}^*)}^k(M_T) \subset \mathcal{H}_{\Lambda(\mathcal{L}^*),-}^k(M_T) \end{aligned} \tag{1.20}$$

for all  $k = \overline{0, m+2}$ . Assume also that the generalized Laplace-Hodge operator (1.13) is reduced upon the space  $\mathcal{H}_\Lambda^0(M)$ . Now by reasoning similar to that in [8, 30, 32] one can formulate a little generalized [4, 5, 6, 30] de Rham-Hodge theorem.

**Theorem 1.2.** *The groups of harmonic forms  $\mathcal{H}_{\Lambda,-}^k(M_T)$ ,  $k = \overline{0, m+2}$ , are, respectively, isomorphic to the cohomology groups  $(H^k(M_T; \mathbb{C}))^{|\Sigma|}$ ,  $k = \overline{0, m+2}$ , where  $H^k(M_T; \mathbb{C})$  is the  $k$ -th cohomology group of the manifold  $M_T$  with complex coefficients, a set  $\Sigma \subset \mathbb{C}^p$ ,  $p \in \mathbb{Z}_+$ , is the set of suitable "spectral" parameters marking the linear space of independent  $d_{\mathcal{L}}^*$ -closed 0-form from  $\mathcal{H}_{\Lambda(\mathcal{L}),-}^0(M_T)$  and, moreover, the following direct sum decompositions*

$$\begin{aligned} \mathcal{H}_{\Lambda,-}^k(M_T) &= \mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L}),-}^k(M_T) \oplus \Delta_{\mathcal{L}}\mathcal{H}_{\Lambda,-}^k(M_T) \\ &= \mathcal{H}_{\Lambda(\mathcal{L}^*\mathcal{L}),-}^k(M_T) \oplus d_{\mathcal{L}}\mathcal{H}_{\Lambda,-}^{k-1}(M_T) \oplus d'_{\mathcal{L}}\mathcal{H}_{\Lambda,-}^{k+1}(M_T) \end{aligned} \tag{1.21}$$

hold for any  $k = \overline{0, m+2}$ .

Another variant of the statement similar to that above was formerly formulated in [3, 4] and reads as the following generalized de Rham-Hodge theorem.

**Theorem 1.3.** *The generalized cohomology groups  $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T)$ ,  $k = \overline{0, m+2}$ , are isomorphic, respectively, to the cohomology groups  $(H^k(M_T; \mathbb{C}))^{|\Sigma|}$ ,  $k = \overline{0, m+2}$ .*

**Proof.** A proof of this theorem is based on some special sequence [3, 4, 5, 6, 7] of differential Lagrange type identities. ■

Define the following closed subspace

$$\mathcal{H}_0^* := \{\varphi^{(0)}(\eta) \in \mathcal{H}_{\Lambda(\mathcal{L}^*),-}^0(M_T) : d_{\mathcal{L}}^*\varphi^{(0)}(\eta) = 0, \varphi^{(0)}(\eta)|_{\Gamma}, \eta \in \Sigma\} \tag{1.22}$$

for some smooth  $(m+1)$ -dimensional hypersurface  $\Gamma \subset M_T$  and  $\Sigma \subset (\sigma(L) \cap \bar{\sigma}(L)) \times \Sigma_{\sigma} \subset \mathbb{C}^p$ , where  $\mathcal{H}_{\Lambda(\mathcal{L}^*),-}^0(M_T)$  is, as above, a suitable Hilbert-Schmidt rigged[12, 13] zero-order cohomology group Hilbert space from the co-chain given by (1.20),  $\sigma(L)$  and  $\sigma(L^*)$  are, respectively, mutual generalized spectra of the sets of differential operators  $\mathcal{L}$  and  $\mathcal{L}^*$  in  $H$  at  $t = 0 \in T$ . Thereby, the dimension  $\dim \mathcal{H}_0^* = \text{card } \Sigma := |\Sigma|$  is assumed to be known. The next lemma, being first stated by I.V. Skrypnik [3, 4], is essential for a proof of Theorem 1.3.

**Lemma 1.4.** *There exists a set of differential  $(k+1)$ -forms  $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , and a set of  $k$ -forms  $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , parametrized by the set  $\Sigma \ni \eta$ , being semilinear in  $(\varphi^{(0)}(\eta), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda,-}^k(M_T)$ , such that*

$$Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] = dZ^k[\varphi^{(0)}(\eta), \psi^{(k)}] \tag{1.23}$$

for all  $k = \overline{0, m+2}$  and  $\eta \in \Sigma$ .

**Proof.** A proof is based on the following Lagrange type identity generalizing that of Part 1 and holding for any pair  $(\varphi^{(0)}(\eta), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda,-}^k(M_T)$ :

$$\begin{aligned} 0 &= \langle d_{\mathcal{L}}^*\varphi^{(0)}(\eta), *(\psi^{(k)} \wedge \bar{\gamma}) \rangle = \\ &= \langle *d'_{\mathcal{L}}(*)^{-1}\varphi^{(0)}(\eta), *(\psi^{(k)} \wedge \bar{\gamma}) \rangle = \\ &= \langle *d'_{\mathcal{L}}(*)^{-1}\varphi^{(0)}(\eta), \psi^{(k)} \wedge \bar{\gamma} \rangle = \\ &= \langle (*)^{-1}\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} \rangle + Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \wedge \bar{\gamma} = \\ &= \langle (*)^{-1}\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)} \wedge \bar{\gamma} \rangle + dZ^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \wedge \bar{\gamma}, \end{aligned} \tag{1.24}$$

where  $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , and  $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , are some semilinear differential forms on  $M_T$  parametrized by a parameter  $\lambda \in \Sigma$ , and  $\bar{\gamma} \in \Lambda^{m+1-k}(M_T; \mathbb{C})$  is arbitrary constant  $(m+1-k)$ -form. Thereby, the semilinear differential  $(k+1)$ -forms  $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_T; \mathbb{C})$  and  $k$ -forms  $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ ,  $\lambda \in \Sigma$ , constructed above exactly constitute those searched for in the Lemma. ■

Based now on Lemma 1.4 one can construct the cohomology group isomorphism claimed in Theorem 1.3 formulated above. Namely, following [3, 4], let us take some singular simplicial [29, 30, 31, 32] complex  $\mathcal{K}(M_T)$  of the compact metric space  $M_T$  and introduce a set of linear mappings  $B_{\lambda}^{(k)} : \mathcal{H}_{\Lambda,+}^k(M_T) \longrightarrow C_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ ,  $\lambda \in \Sigma$ , where  $C_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , are free abelian groups over the field  $\mathbb{C}$  generated, respectively, by all  $k$ -chains of singular simplexes  $S^{(k)} \in \mathcal{K}(M_T)$ ,  $k = \overline{0, m+2}$ , from the simplicial complex  $\mathcal{K}(M_T)$ , as follows:

$$B_{\lambda}^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M_T; \mathbb{C})} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{1.25}$$

with  $\psi^{(k)} \in \mathcal{H}_{\Lambda,-}^k(M_T)$ ,  $k = \overline{0, m+2}$ . The following theorem [3, 4] based on mappings (1.25) holds.

**Theorem 1.5.** *The set of operators (1.25) parametrized by  $\lambda \in \Sigma$  realizes the cohomology group isomorphism formulated in Theorem 1.3*

**Proof.** A proof of this theorem is obtained by passing over in (1.25) to the corresponding cohomology  $\mathcal{H}_{\Lambda(\mathcal{L}),+}^k(M_T)$  and homology  $H_k(M_T; \mathbb{C})$  groups of  $M_T$  for every  $k = \overline{0, m+2}$ . If one takes an element  $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T)$ ,  $k = \overline{0, m+2}$ , solving the equation  $d_{\mathcal{L}}\psi^{(k)}(\mu) = 0$  with  $\mu \in \Sigma_k$  being some set of the related "spectral" parameters marking elements of the subspace  $\mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T)$ , then one finds easily from (1.25) and identity (1.23) that  $dZ^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = 0$  for all  $(\lambda, \mu) \in \Sigma \times \Sigma_k$ ,  $k = \overline{0, m+2}$ . This, in particular, means due to the Poincaré lemma [28, 29, 30] that there exist differential  $(k-1)$ -forms  $\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \in \Lambda^{k-1}(M; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , such that

$$Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] = d\Omega^{(k-1)}[\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)] \tag{1.26}$$

for all pairs  $(\varphi^{(0)}(\lambda), \psi^{(k)}(\mu)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T)$  parametrized by  $(\lambda, \mu) \in \Sigma \times \Sigma_k$ ,  $k = \overline{0, m+2}$ . As a result of passing on the right hand-side of (1.25) to the homology groups  $H_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , one gets due to the standard Stokes theorem [28, 30, 29] that the mappings

$$\hat{B}_{\lambda}^{(k)} : \mathcal{H}_{\Lambda(\mathcal{L}),-}^k(M_T) \longrightarrow H_k(M_T; \mathbb{C}) \tag{1.27}$$

are isomorphisms for every  $k = \overline{0, m+2}$  and  $\lambda \in \Sigma$ . Making further use of the Poincaré duality [8, 29, 30] between the homology groups  $H_k(M_T; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , and the cohomology groups  $H^k(M; \mathbb{C})$ ,  $k = \overline{0, m+2}$ , respectively, one obtains finally the statement claimed in Theorem 1.4. ■

## 2 The spectral structure of Delsarte-Darboux type transmutation operators in multidimension

Take now into account that our differential operators  $L_j : \mathcal{H} \rightarrow \mathcal{H}$ ,  $j = \overline{1, 2}$ , are of the special form (1.2). Assume also that differential expressions (1.4) are normal closed operators defined on dense subspace  $D(\mathcal{L}) \subset L_2(M; \mathbb{C}^N)$ .

Then due to Theorem 1.3 one can find such a pair  $(\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda(\mathcal{L}), -}^m(M_T)$  parametrized by elements  $(\lambda, \mu) \in \Sigma \times \Sigma$ , for which the equality

$$B_\lambda^{(m)}(\psi^{(0)}(\mu)dx) = S_{(t;x)}^{(m)} \int_{\partial S_{(t;x)}^{(m)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \tag{2.1}$$

holds, where  $S_{(t;x)}^{(m)} \in H_m(M_T; \mathbb{C})$  is some arbitrary but fixed element parametrized by a running point  $(t; x) \in M_T \cap \partial S_{(t;x)}^{(m)}$ . Consider the next integral expressions

$$\begin{aligned} \Omega_{(t;x)}(\lambda, \mu) &:= \int_{\sigma_{(t;x)}^{(m-1)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \\ \Omega_{(t_0;x_0)}(\lambda, \mu) &:= \int_{\sigma_{(t_0;x_0)}^{(m-1)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \end{aligned} \tag{2.2}$$

with a point  $(t_0; x_0) \in M_T \cap \partial S_{(t_0;x_0)}^{(m)}$  being taken fixed on the boundaries  $\sigma_{(t;x)}^{(m-1)} := \partial S_{(t;x)}^{(m)}$ ,  $\sigma_{(t_0;x_0)}^{(m-1)} := \partial S_{(t_0;x_0)}^{(m)}$  assumed to be homological to each other as  $(t; x_0) \rightarrow (t; x) \in M_T$ ,  $(\lambda, \mu) \in \Sigma \times \Sigma$ , and interpret them as the kernels [12, 13, 33] of the corresponding invertible integral operators of Hilbert-Schmidt type  $\Omega_{(t;x)}, \Omega_{(t_0;x_0)} : L_2^{(\rho)}(\Sigma; \mathbb{C}) \rightarrow L_2^{(\rho)}(\Sigma; \mathbb{C})$ , where  $\rho$  is some finite Borel measure on the parameters set  $\Sigma$ . Define now the invertible operators expressions

$$\Omega_\pm : \psi^{(0)}(\mu) \rightarrow \tilde{\psi}^{(0)}(\mu) \tag{2.3}$$

for  $\psi^{(0)}(\mu)dx \in \mathcal{H}_{\Lambda(\mathcal{L}), -}^m(M_T)$  and some  $\tilde{\psi}^{(0)}(\mu)dx \in \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}), -}^m(M_T)$ ,  $\mu \in \Sigma$ , where, by definition, for any  $\eta \in \Sigma$

$$\begin{aligned} \tilde{\psi}^{(0)}(\eta) &:= \psi^{(0)}(\eta) \cdot \Omega_{(t;x)}^{-1} \cdot \Omega_{(t_0;x_0)} \\ &= \int_\Sigma d\rho(\mu) \int_\Sigma d\rho(\xi) \psi^{(0)}(\mu) \Omega_{(t;x)}^{-1}(\mu, \xi) \Omega_{(t_0;x_0)}(\xi, \eta), \end{aligned} \tag{2.4}$$

being motivated by the expression (2.1). Namely, consider the following diagram

$$\begin{array}{ccc} \mathcal{H}_{\Lambda(\mathcal{L}), -}^m(M_T) & \xrightarrow{\Omega_\pm} & \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}), -}^m(M_T), \\ \hat{B}_\lambda^{(m)} \downarrow & \swarrow \tilde{B}_\lambda^{(m)} & \\ H_m(M_T; \mathbb{C}) & & \end{array} \tag{2.5}$$

which is assumed to be commutative for another co-chain complex

$$\mathcal{H} \rightarrow \Lambda^0(M_T; \mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^1(M_T; \mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} \dots \xrightarrow{d_{\tilde{\mathcal{L}}}} \Lambda^{m+2}(M_T; \mathcal{H}) \xrightarrow{d_{\tilde{\mathcal{L}}}} 0. \tag{2.6}$$

Here, by definition, the generalized anti-differentiation is

$$d_{\tilde{\mathcal{L}}} := \sum_{j=1}^2 dt_j \wedge \tilde{L}_j(t; x|\partial) + \sum_{i=1}^m dx_i \wedge \tilde{c}_i \mathbb{I} \tag{2.7}$$

with  $\tilde{c}_i \in \mathbb{C}$ ,  $i = \overline{1, m}$ , and

$$\begin{aligned} \tilde{L}_j &= \partial/\partial t_j - \tilde{L}_j(t; x|\partial), \\ \tilde{L}_j(t; x|\partial) &:= \sum_{|\alpha|=0}^{n_j(\tilde{L})} \tilde{a}_\alpha^{(j)}(t; x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \end{aligned} \tag{2.8}$$

where coefficients  $\tilde{a}_\alpha^{(j)} \in C^1(\mathbb{T}; C^\infty(M; \text{End } \mathbb{C}^N)$ ,  $|\alpha| = \overline{0, n_j(\tilde{L})}$ ,  $n_j(\tilde{L}) := n_j(L) \in \mathbb{Z}_+$ ,  $j = \overline{1, 2}$ . Assume that the corresponding mappings  $\tilde{B}_\lambda^{(m)} : \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}), -}^m(M_{\mathbb{T}}) \rightarrow C_m(M_{\mathbb{T}}; \mathbb{C})$ ,  $\lambda \in \Sigma$ , act on some  $\tilde{\psi}^{(0)}(\mu)dx \in \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}), -}^m(M_{\mathbb{T}})$  as follows:

$$\tilde{B}_\lambda^{(m)}(\tilde{\psi}^{(0)}(\mu)dx) = S_{(t;x)}^{(m)} \int_{\partial S_{(t;x)}^{(m)}} \tilde{\Omega}^{(m-1)}[\tilde{\varphi}^{(0)}(\lambda), \tilde{\psi}^{(0)}(\mu)dx], \tag{2.9}$$

where  $\tilde{\varphi}^{(0)}(\lambda) \in \tilde{\mathcal{H}}_0^* \subset \mathcal{H}_{\Lambda(\mathcal{L}^*), -}^0(M_{\mathbb{T}})$ ,  $\lambda \in (\sigma(\tilde{L}) \cap \bar{\sigma}(\tilde{L}^*)) \times \Sigma_\sigma$ , and

$$\tilde{\mathcal{H}}_0^* := \{\tilde{\varphi}^{(0)}(\lambda) \in \mathcal{H}_{\Lambda(\mathcal{L}^*), -}^m(M_{\mathbb{T}}) : d_{\tilde{\mathcal{L}}}^* \tilde{\varphi}^{(0)}(x) = 0, \tilde{\varphi}^{(0)}(\lambda)|_{\tilde{\Gamma}} = 0, \lambda \in \Sigma\} \tag{2.10}$$

for some hypersurface  $\tilde{\Gamma} \subset M_{\mathbb{T}}$ . Respectively, one defines the following closed subspace

$$\tilde{\mathcal{H}}_0 := \{\tilde{\psi}^{(0)}(\mu) \in \mathcal{H}_{\Lambda(\mathcal{L}^*), -}^0(M_{\mathbb{T}}) : d_{\tilde{\mathcal{L}}}^* \tilde{\psi}^{(0)}(\lambda) = 0, \tilde{\psi}^{(0)}(\mu)|_{\tilde{\Gamma}} = 0, \mu \in \Sigma\} \tag{2.11}$$

for the hyperspace  $\tilde{\Gamma} \subset M_{\mathbb{T}}$ , introduced above.

Suppose now that the elements (2.4) belong to the closed subspace (2.11), that is

$$d_{\tilde{\mathcal{L}}} \tilde{\psi}^{(0)}(\mu) = 0 \tag{2.12}$$

Define similarly to (2.11) a closed subspace  $\tilde{\mathcal{H}}_0^* \subset \mathcal{H}_{\Lambda(\tilde{\mathcal{L}}^*), -}^m(M_{\mathbb{T}})$  as follows:

$$\mathcal{H}_0 := \{\psi^{(0)}(\lambda) \in \mathcal{H}_{\Lambda(\mathcal{L}^*), -}^0(M_{\mathbb{T}}) : d_{\mathcal{L}} \psi^{(0)}(\lambda) = 0, \psi^{(0)}(\lambda)|_{\Gamma} = 0, \lambda \in \Sigma\} \tag{2.13}$$

for all  $\mu \in \Sigma$ . Then due to the commutativity of the diagram (2.5) there exist the corresponding two invertible mappings

$$\Omega_\pm : \mathcal{H}_0 \rightarrow \tilde{\mathcal{H}}_0, \tag{2.14}$$

depending on ways in which they are extended the whole Hilbert space  $\mathcal{H}_{\Lambda, -}^m(M_{\mathbb{T}})$ . Extend now operators (2.14) upon the whole Hilbert space  $\mathcal{H}_{\Lambda, -}^m(M_{\mathbb{T}})$  by means of the standard method [21, 23] of variation of constants, taking into account that for kernels



$\Omega_{(t;x)}(\lambda, \mu), \Omega_{(t_0;x_0)}(\lambda, \mu) \in L_2^{(p)}(\Sigma; \mathbb{C}) \otimes L_2^{(p)}(\Sigma; \mathbb{C})$ ,  $\lambda, \mu \in \Sigma$ , one can write down the following relationships:

$$\begin{aligned} & \Omega_{(t;x)}(\lambda, \mu) - \Omega_{(t_0;x_0)}(\lambda, \mu) = \tag{2.15} \\ &= \int_{\partial S_{(t;x)}^{(m)}} \Omega^{(m-1)}[\varphi^{(0)}(x), \psi^{(0)}(\mu)dx] - \int_{\partial S_{(t_0;x_0)}^{(m)}} \Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &= \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} d\Omega^{(m-1)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx] \\ &= \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} Z^{(m)}[\varphi^{(0)}(\lambda), \psi^{(0)}(\mu)dx], \end{aligned}$$

where, by definition,  $m$ -dimensional open surfaces  $S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) \subset M_T$  are spanned smoothly without self-intersection between two homological cycles  $\sigma_{(t;x)}^{(m-1)} = \partial S_{(t;x)}^{(m)}$  and  $\sigma_{(t_0;x_0)}^{(m-1)} = \partial S_{(t_0;x_0)}^{(m)} \in C_{m-1}(M_T; \mathbb{C})$  in such a way that the boundary  $\partial(S_{+}^{(m)}(\sigma_{(t_0;x_0)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) \cup S_{-}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})) = \emptyset$ . Making use of the relationship (2.15), one can thereby find easily the following integral operator expressions in  $\mathcal{H}_{-}$ :

$$\begin{aligned} \Omega_{\pm} &= \mathbf{1} - \int_{\Sigma} d\rho(\eta) \tilde{\psi}^{(0)}(\xi) \Omega_{(t_0;x_0)}^{-1}(\xi, \eta) \tag{2.16} \\ &\quad \times \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} Z^{(m)}[\varphi^{(0)}(\eta), (\cdot)dx] \end{aligned}$$

defined for fixed pairs  $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$  and  $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\mu)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$ ,  $\lambda, \mu \in \Sigma$ , being bounded invertible operators of Volterra type [18, 19, 14, 33] on the whole Hilbert space  $\mathcal{H}$ . Moreover, for the differential operators  $\tilde{L}_j : \mathcal{H} \rightarrow \mathcal{H}$ ,  $j = \overline{1, 2}$ , one can get easily the following expressions

$$\tilde{L}_j = \Omega_{\pm} L_j \Omega_{\pm}^{-1}, \tag{2.17}$$

where the left hand-side of (2.17) does not depend on signs "±" of the right-hand sides. Thereby, the Volterra integral operators (2.16) are the Delsarte-Darboux transmutation operators, mapping a given set  $\mathcal{L}$  of differential operators into a new set  $\tilde{\mathcal{L}}$  of differential operators transformed via the Delsarte expressions (2.17).

Suppose now that all of differential operators  $L_j(t; x|\partial)$ ,  $j = \overline{1, 2}$ , considered above do not depend on the variable  $t \in T$ . Then, evidently, one can take

$$\begin{aligned} \mathcal{H}_0 &:= \{ \psi_{\mu}^{(0)}(\xi) \in L_{2,-}(M; \mathbb{C}^N) : L_j \psi_{\mu}^{(0)}(\xi) = \mu_j \psi_{\mu}^{(0)}(\xi), \\ &\quad j = \overline{1, 2}, \psi_{\mu}^{(0)}(\xi)|_{\tilde{\Gamma}} = 0, \mu = (\mu_1, \mu_2) \in \sigma(\tilde{L}) \cap \overline{\sigma(L^*)}, \xi \in \Sigma_{\sigma} \} \\ \tilde{\mathcal{H}}_0 &:= \{ \tilde{\psi}_{\mu}^{(0)}(\xi) \in L_{2,-}(M; \mathbb{C}^N) : \tilde{L}_j \tilde{\psi}_{\mu}^{(0)}(\xi) = \mu_j \tilde{\psi}_{\mu}^{(0)}(\xi), \\ &\quad j = \overline{1, 2}, \tilde{\psi}_{\mu}^{(0)}(\xi)|_{\tilde{\Gamma}} = 0, \mu = (\mu_1, \mu_2) \in \sigma(\tilde{L}) \cap \overline{\sigma(L^*)}, \xi \in \Sigma_{\sigma} \} \\ \mathcal{H}_0^* &:= \{ \varphi_{\lambda}^{(0)}(\eta) \in L_{2,-}(M; \mathbb{C}^N) : L_j^* \varphi_{\lambda}^{(0)}(\eta) = \bar{\lambda}_j \varphi_{\lambda}^{(0)}(\eta), j = \overline{1, 2}, \\ &\quad \varphi_{\lambda}^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \lambda = (\lambda_1, \lambda_2) \in \sigma(\tilde{L}) \cap \overline{\sigma(L^*)}, \eta \in \Sigma_{\sigma} \} \tag{2.18} \\ \tilde{\mathcal{H}}_0^* &:= \{ \tilde{\varphi}_{\lambda}^{(0)}(\eta) \in L_{2,-}(M; \mathbb{C}^N) : \tilde{L}_j^* \tilde{\varphi}_{\lambda}^{(0)}(\eta) = \bar{\lambda}_j \varphi_{\lambda}^{(0)}(\eta), \\ &\quad j = \overline{1, 2}, \tilde{\varphi}_{\lambda}^{(0)}(\eta)|_{\tilde{\Gamma}} = 0, \lambda = (\lambda_1, \lambda_2) \in \sigma(\tilde{L}) \cap \overline{\sigma(L^*)}, \eta \in \Sigma_{\sigma} \} \end{aligned}$$

and construct the corresponding Delsarte-Darboux transmutation operators

$$\begin{aligned} \Omega_{\pm} &= 1 - \int_{\sigma(\tilde{L}) \cap \overline{\sigma}(L^*)} d\rho_{\sigma}(\lambda) \int_{\Sigma_{\sigma} \times \Sigma_{\sigma}} d\rho_{\Sigma_{\sigma}}(\xi) d\rho_{\Sigma_{\sigma}}(\eta) \\ &\times \int_{S_{\pm}^{(m)} \sigma_{(t_0; x_0)}^{(m-1)}; \sigma_{(t_0; x_0)}^{(m-1)}} dx \tilde{\psi}_{\lambda}^{(0)}(\xi) \Omega_{x_0}^{-1}(\lambda; \xi; \eta) \bar{\varphi}_{\lambda}^{(0), \top}(\eta)(\cdot) \end{aligned} \tag{2.19}$$

acting already in the suitably rigged Hilbert space  $L_{2,-}(M; \mathbb{C}^N)$ , where for any  $(\lambda; \xi, \eta) \in (\sigma(\tilde{L}) \cap \overline{\sigma}(L^*) \times \Sigma_{\sigma}^2)$  kernels

$$\Omega_{(x_0)}(\lambda; \xi, \eta) := \int_{\sigma_{x_0}^{(m-1)}} \Omega^{(m-1)}[\varphi_{\lambda}^{(0)}(\xi), \psi_{\lambda}^{(0)}(\eta) dx] \tag{2.20}$$

for  $(\xi, \eta) \in \Sigma_{\sigma}^2$  and every  $\lambda \in \sigma(\tilde{L}) \cap \overline{\sigma}(L^*)$  belong to  $L_2^{(\rho)}(\Sigma_{\sigma}; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_{\sigma}; \mathbb{C})$ . Moreover, as  $\partial \Omega_{\pm} / \partial t_j = 0, j = \overline{1, 2}$ , one gets easily the set of differential expressions

$$\tilde{L}_j(x|\partial) := \Omega_{\pm} L_j(x|\partial) \Omega_{\pm}^{-1} \tag{2.21}$$

$j = \overline{1, 2}$ , also commuting, evidently, with each other.

The Volterra operators (2.19) possess some additional properties. Namely, define the following Fredholm type integral operator in  $H$  :

$$\Omega := \Omega_{+}^{-1} \Omega_{-}, \tag{2.22}$$

which can be written in the form

$$\Omega = \mathbf{1} + \Phi(\Omega), \tag{2.23}$$

where the operator  $\Phi(\Omega) \in \mathcal{B}_{\infty}(H)$  is compact. Moreover, due to the relationships (2.21) one gets easily that the following commutator conditions

$$[\Omega, L_j] = 0 \tag{2.24}$$

hold for  $j = \overline{1, 2}$ .

Denote now by  $\hat{\Phi}(\Omega) \in H_{-} \otimes H_{-}$  and  $\hat{K}_{+}(\Omega), \hat{K}_{-}(\Omega) \in H_{-} \otimes H_{-}$  the kernels corresponding [12, 13] to operators  $\Phi(\Omega) \in \mathcal{B}_{\infty}(H)$  and  $\Omega_{\pm} - \mathbf{1} \in \mathcal{B}_{\infty}(H)$ . Then due to the fact that  $supp K_{+}(\Omega) \cap supp K_{-}(\Omega) = \sigma_x^{(m-1)} \cup \sigma_{x_0}^{(m-1)}$ , one gets from (2.22) and (2.23) the well known Gelfand-Levitan-Marchenko linear integral equation

$$\hat{K}_{+}(\Omega) + \hat{\Phi}(\Omega) + \hat{K}_{+}(\Omega) \cdot \hat{\Phi}(\Omega) = \hat{K}_{-}(\Omega), \tag{2.25}$$

which enables one to factor the Fredholm operator (2.22) kernel  $\hat{K}_{+}(\Omega)(x; y) \in H_{-} \otimes H_{-}$  for all  $y \in supp K_{+}(\Omega)$ . The conditions (2.24) can be rewritten suitably as follows:

$$(L_{j,ext} \otimes \mathbf{1}) \hat{\Phi}(\Omega) = (1 \otimes L_{j,ext}^*) \hat{\Phi}(\Omega) \tag{2.26}$$

for  $j = \overline{1, 2}$ , where  $L_{j,ext} \in \mathcal{L}(H_{-})$ ,  $j = \overline{1, 2}$ , and their adjoints  $L_{j,ext}^* \in \mathcal{L}(H_{-})$ ,  $j = \overline{1, 2}$ , are the corresponding extensions [12, 24, 13] of the differential operators  $L_j$  and  $L_j^* \in \mathcal{L}(H)$ ,  $j = \overline{1, 2}$ .

Concerning the relationships (2.21) one can write down [12, 24] kernel conditions similar to (2.26):

$$(\tilde{L}_{j,ext} \otimes \mathbf{1})\hat{K}_{\pm}(\Omega) = (\mathbf{1} \otimes L_{j,ext}^*)\hat{K}_{\pm}(\Omega), \tag{2.27}$$

where as above,  $\tilde{L}_{j,ext} \in \mathcal{L}(H_-)$ ,  $j = \overline{1, 2}$ , are the corresponding rigging extensions of the differential operators  $\tilde{L}_j \in \mathcal{L}(H)$ ,  $j = \overline{1, 2}$ .

Proceed now to analyzing the question about the general differential and spectral structure of the transformed operator expression (2.17). It is evident that the found above conditions (2.25) and (2.26) on the kernels  $\hat{K}_{\pm}(\Omega) \in \mathcal{H}_- \otimes \mathcal{H}_-$  of Delsarte- Darboux transmutation operators are necessary for the operator expressions (2.17) to exist and be differential. It is natural to ask whether these conditions are also sufficient?

For studying this question let us consider Volterra operators (2.16) and (2.19) with kernels satisfying the conditions (2.25) and (2.26), assuming that suitable oriented piecewise smooth surfaces  $S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) \in C_m(M_T; \mathbb{C})$  can be given as follows:

$$\begin{aligned} S_+^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) &= \{(t'; x') \in M_T : \\ & \quad t' = P(t; x|x'), t \in T\}, \\ S_-^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) &= \{(t'; x') \in M_T : \\ & \quad t' = P(t; x|x') \in T \setminus [t_0, t]\}, \end{aligned} \tag{2.28}$$

where a mapping  $P \in C^\infty(M_T \times M; T)$  is smooth and such that the boundaries  $\partial S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)}) = \pm(\sigma_{(t;x)}^{(m-1)} - \sigma_{(t_0;x_0)}^{(m-1)})$  with cycles  $\sigma_{(t;x)}^{(m-1)}$  and  $\sigma_{(t_0;x_0)}^{(m-1)} \in \mathcal{K}(M_T)$  are homological to each other for any choice of points  $(t_0; x_0)$  and  $(t; x) \in M_T$ . Then one can see by means of some simple but cumbersome calculations, based on considerations from [35] and [9], that the resulting expressions on the right hand-sides of

$$\tilde{L} = L + [K_{\pm}(\Omega), L] \cdot \Omega_{\pm}^{-1} \tag{2.29}$$

are exactly equal to each other and differential if there is such an expression for an operator  $L \in \mathcal{L}(\mathcal{H})$ .

Concerning the inverse operators  $\Omega_{\pm}^{-1} \in \mathcal{B}(\mathcal{H})$  present in (2.29) one can notice here that due to the functional symmetry between closed subspaces  $\mathcal{H}_0$  and  $\tilde{\mathcal{H}}_0 \subset \tilde{\mathcal{H}}_-$ , the defining relationships (2.14) and (2.4) are reversible, that is there exist the inverse operator mappings  $\Omega_{\pm}^{-1} : \tilde{\mathcal{H}}_0 \rightarrow \mathcal{H}_0$ , such that

$$\Omega_{\pm}^{-1} : \tilde{\psi}^{(0)}(\lambda) \longrightarrow \psi^{(0)}(\lambda) := \tilde{\psi}^{(0)}(\lambda) \cdot \tilde{\Omega}_{(t;x)}^{-1} \tilde{\Omega}_{(t;x)} \tag{2.30}$$

for some suitable kernels  $\tilde{\Omega}_{(t;x)}(\lambda, \mu)$  and  $\tilde{\Omega}_{(t_0;x_0)}(\lambda, \mu) \in L_2^{(\rho)}(\Sigma; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma; \mathbb{C})$ , related naturally with the transformed differential expression  $\tilde{L} \in \mathcal{L}(\mathcal{H})$ . Thereby, due to the expressions (2.30) one can write down similar to (2.19) the following inverse integral operators:

$$\begin{aligned} \Omega_{\pm}^{-1} &= \mathbf{1} - \int_{\Sigma} d\rho(\xi) \int_{\Sigma} d\rho(\eta) \psi^{(0)}(\xi) \tilde{\Omega}_{t_0;x_0}^{-1}(\xi, \eta) \\ & \quad \times \int_{S_{\pm}^{(m)}(\sigma_{(t;x)}^{(m-1)}, \sigma_{(t_0;x_0)}^{(m-1)})} \tilde{Z}^{(m)}[\tilde{\varphi}^{(0)}(\eta), (\cdot) dx] \end{aligned} \tag{2.31}$$

defined for fixed pairs  $(\tilde{\varphi}^{(0)}(\xi), \tilde{\psi}^{(0)}(\eta)) \in \tilde{\mathcal{H}}_0^* \times \tilde{\mathcal{H}}_0$  and  $(\varphi^{(0)}(\xi), \psi^{(0)}(\eta)) \in \mathcal{H}_0^* \times \mathcal{H}_0$ ,  $\xi, \eta \in \Sigma$ , and being bounded invertible operators of Volterra type in the whole Hilbert space  $\mathcal{H}$ . In particular, the compatibility conditions  $\Omega_{\pm} \Omega_{\pm}^{-1} = \mathbf{1} = \Omega_{\pm}^{-1} \Omega_{\pm}$  must be fulfilled identically in  $\mathcal{H}$ , involving some restrictions identifying measures  $\rho$  and  $\Sigma$  and possible asymptotic conditions of coefficient functions of the differential expression  $L \in \mathcal{L}$ . Such kinds of restrictions were already mentioned before in [37, 38, 39], where in particular the relationships with the local and nonlocal Riemann problems were discussed.

Within the framework of the general construction presented above one can give a natural interpretation of so called Bäcklund transformations for coefficient functions of a given differential operator expression  $L \in \mathcal{L}(\mathcal{H})$ . Namely, following the symbolic considerations in [41], we reinterpret the approach devised there for constructing the Bäcklund transformations making use of the techniques based on the theory of Delsarte transmutation operators. Let us define two different Delsarte-Darboux transformed differential operator expressions

$$L_1 = \Omega_{1,\pm} L \Omega_{1,\pm}^{-1}, \quad L_2 = \Omega_{2,\pm} L \Omega_{2,\pm}^{-1}, \tag{2.32}$$

where  $\Omega_{1,+}, \Omega_{2,-} \in \mathcal{B}(\mathcal{H})$  are some Delsarte transmutation Volterra operators in  $\mathcal{H}$  with Borel spectral measures  $\rho_1$  and  $\rho_2$  on  $\Sigma$ , such that the following conditions

$$\Omega_{1,+}^{-1} \Omega_{1,-} = \Omega = \Omega_{2,+}^{-1} \Omega_{2,-} \tag{2.33}$$

hold. Making use now of the conditions (2.32) and relationships (2.33) one finds easily that the operator  $B := \Omega_{2,-} \Omega_{1,+}^{-1} \in \mathcal{B}(\mathcal{H})$  satisfies the following operator equations:

$$L_2 B = B L_1, \quad \Omega_{2,\pm} B = B \Omega_{1,\pm}, \tag{2.34}$$

which motivate the next definition.

**Definition 1.** An invertible symbolic mapping  $B : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$  will be called a Darboux-Bäcklund transformation of an operator  $L_1 \in \mathcal{L}(\mathcal{H})$  into the operator  $L_2 \in \mathcal{L}(\mathcal{H})$  if there holds the condition

$$[QB, L_1] = 0 \tag{2.35}$$

for some linear differential expression  $Q \in \mathcal{L}(\mathcal{H})$ .

The condition (2.35) can be realized as follows. Take any differential expression  $q \in \mathcal{L}()$  satisfying the symbolic equation

$$[qB, L] = 0. \tag{2.36}$$

Then, making use of the transformations like (2.32), from (2.33) one finds that

$$[QB, L_1] = 0, \tag{2.37}$$

where owing to (2.34)

$$QB := \Omega_{1,+} q B \Omega_{1,+}^{-1} = \Omega_{1,+} q \Omega_{2,+}^{-1} B. \tag{2.38}$$

Therefore, the expression  $Q = \Omega_{1,+}q\Omega_{2,+}^{-1}$  appears also to be differential owing to the conditions (2.34).

The consideration above related with the symbolic mapping  $B : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  gives rise to an effective tool of constructing self-Bäcklund transformations for coefficients of differential operator expressions  $L_1, L_2 \in \mathcal{L}(\mathcal{H})$  having many applications [15, 11, 26, 33, 23] in spectral and soliton theories.

Return now back to studying the structure Delsarte-Darboux transformations for a polynomial differential operators pencil

$$L(\lambda; x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\lambda^j, \tag{2.39}$$

where  $n(L) \in \mathbb{Z}_+$  and  $\lambda \in \mathbb{C}$  is a complex-valued parameter. It is asked to find the corresponding to (2.39) Delsarte-Darboux transformations  $\Omega_{\lambda,\pm} \in \mathcal{B}(\mathcal{H})$ ,  $\lambda \in \mathbb{C}$ , such that for some polynomial differential operators pencil  $\tilde{L}(\lambda; x|\partial) \in \mathcal{L}(\mathcal{H})$  the following Delsarte-Lions [2] transmutation condition

$$\tilde{L}\Omega_{\lambda,\pm} = \Omega_{\lambda,\pm}L \tag{2.40}$$

holds for almost all  $\lambda \in \mathbb{C}$ . For such transformations  $\Omega_{\lambda,\pm} \in \mathcal{B}(\mathcal{H})$  to be found, let us consider a parameter  $\tau \in \mathbb{R}$  dependent differential operator  $L_\tau(x|\partial) \in \mathcal{L}(\mathcal{H}_\tau)$ , where

$$L_\tau(x|\partial) := \sum_{j=0}^{n(L)} L_j(x|\partial)\partial^j / \partial\tau^j, \tag{2.41}$$

acting in the functional space  $\mathcal{H}_\tau = C^{q(L)}(\mathbb{R}_\tau; \mathcal{H})$  for some  $q(L) \in \mathbb{Z}_+$ . Then one can easily construct the corresponding Delsarte-Darboux transformations  $\Omega_{\tau,\pm} \in \mathcal{B}(\mathcal{H}_\tau)$  of Volterra type for some differential operator expression

$$\tilde{L}_\tau(x|\partial) := \sum_{j=0}^{n(L)} \tilde{L}_j(x|\partial)\partial^j / \partial\tau^j, \tag{2.42}$$

if the following Delsarte-Lions [2] transmutation conditions

$$\tilde{L}_\tau\Omega_{\tau,\pm} = \Omega_{\tau,\pm}L_\tau \tag{2.43}$$

hold in  $\mathcal{H}_\tau$ . Thus, making use of the results obtained above, one can write down that

$$\begin{aligned} \Omega_{\tau,\pm} &= \mathbf{1} - \int_{\Sigma} d\rho_{\Sigma}(\xi) \int_{\Sigma} d\rho_{\Sigma}(\eta) \tilde{\psi}_{\tau}^{(0)}(\lambda; \xi) \Omega_{(\tau_0; x_0)}^{-1}(\lambda; \xi, \eta) \\ &\quad \times \int_{S_{\pm}^{(m)}(\sigma_{(\tau; x)}^{(m-1)}, \sigma_{(\tau_0; x_0)}^{(m-1)})} Z^{(m)}[\varphi_{\tau}^{(0)}(\lambda; \eta), (\cdot)dx] \end{aligned} \tag{2.44}$$

defined by means of the following closed subspaces  $\mathcal{H}_{\tau,0} \subset \mathcal{H}_{\tau,-}$  and  $\mathcal{H}_{\tau,0}^* \subset \mathcal{H}_{\tau,-}^*$  :

$$\begin{aligned} \mathcal{H}_{\tau,0} &:= \{ \psi_{\tau}^{(0)}(\lambda; \xi) \in \mathcal{H}_{\tau,-} : L_{\tau}\psi_{\tau}^{(0)}(\lambda; \xi) = 0, \\ \psi_{\tau}^{(0)}(\lambda; \xi)|_{\tau=0} &= \psi^{(0)}(\lambda; \xi) \in \mathcal{H}, L\psi^{(0)}(\lambda; \xi) = 0, \\ \psi^{(0)}(\lambda; \xi)|_{\Gamma} &= 0, \lambda \in \mathbb{C}, \xi \in \Sigma \}, \end{aligned}$$

$$\begin{aligned}
 \mathcal{H}_{\tau,0}^* &:= \{ \varphi_{\tau}^{(0)}(\lambda; \eta) \in \mathcal{H}_{\tau,-}^* : L_{\tau} \varphi_{\tau}^{(0)}(\lambda; \eta) = 0, \\
 \varphi_{\tau}^{(0)}(\lambda; \eta)|_{\tau=0} &= \varphi^{(0)}(\lambda; \eta) \in \mathcal{H}^*, L \varphi^{(0)}(\lambda; \eta) = 0, \\
 \varphi^{(0)}(\lambda; \eta)|_{\Gamma} &= 0, \lambda \in \mathbb{C}, \eta \in \Sigma \}.
 \end{aligned}
 \tag{2.45}$$

Recalling now that our operators  $L_j \in \mathcal{L}(\mathcal{H})$ ,  $j = \overline{0, r(L)}$ , do not depend on the parameter  $\tau \in \mathbb{R}$ , one can derive easily from (2.44)

$$\begin{aligned}
 \mathbf{\Omega}_{\pm} &= \mathbf{1} - \int_{\Sigma} d\rho_{\Sigma}(\xi) \int_{\Sigma} d\rho_{\Sigma}(\eta) \tilde{\psi}^{(0)}(\lambda; \xi) \Omega_{(x_0)}^{-1}(\lambda; \xi, \eta) \\
 &\quad \times \int_{S_{\pm}^{(m)}(\sigma_{(x)}^{(m-1)}, \sigma_{(x_0)}^{(m-1)})} Z_0^{(m)}[\varphi^{(0)}(\lambda; \eta), (\cdot) dx],
 \end{aligned}
 \tag{2.46}$$

where we put  $\sigma_x^{(m-1)} := \sigma_{(\tau_0; x)}^{(m-1)}$ ,  $\sigma_{x_0}^{(m-1)} := \sigma_{(\tau_0; x_0)}^{(m-1)} \in C_{m-1}(\mathbb{R}^m; \mathbb{C})$  and

$$Z_0^{(m)}[\varphi^{(0)}(\lambda; \eta), \psi^{(0)} dx] := Z^{(m)}[\varphi_{\tau}^{(0)}(\lambda; \eta), \psi_{\tau}^{(0)} dx]|_{d\tau=0}.
 \tag{2.47}$$

The corresponding to (2.46) closed subspaces  $\mathcal{H}_0 \in \mathcal{H}_-$  and  $\mathcal{H}_0^* \in \mathcal{H}_-^*$  are given as follows:

$$\begin{aligned}
 \mathcal{H}_0 &:= \{ \psi^{(0)}(\lambda; \xi) \in \mathcal{H}_- : L\psi^{(0)}(\lambda; \xi) = 0, \\
 &\quad \psi^{(0)}(\lambda; \xi)|_{\Gamma} = 0, \lambda \in \mathbb{C}, \xi \in \Sigma \}, \\
 \mathcal{H}_{\tau,0}^* &:= \{ \varphi^{(0)}(\lambda; \eta) \in \mathcal{H}_-^* : L\varphi^{(0)}(\lambda; \eta) = 0, \\
 &\quad \varphi^{(0)}(\lambda; \eta)|_{\Gamma} = 0, \lambda \in \mathbb{C}, \eta \in \Sigma \}.
 \end{aligned}
 \tag{2.48}$$

Thereby, making use of the expressions (2.46) one can construct the Delsarte-Darboux transformed linear differential pencil  $\tilde{L} \in \mathcal{L}(\mathcal{H})$ , whose coefficients are related with those of the pencil  $L \in \mathcal{L}(\mathcal{H})$  via some Bäcklund type relationships useful for applications (see [23, 20, 42, 43, 38]) in the soliton theory.

### 3 Delsarte-Darboux type transmutation operators for special multi-dimensional expressions and their applications

#### 3.1 A perturbed self-adjoint Laplace operator in $\mathbb{R}^n$

Consider the Laplace operator  $-\Delta_m$  in  $H := L(\mathbb{R}^m; \mathbb{C})$  perturbed by the multiplication operator on a function  $q \in W_2^2(\mathbb{R}^m; \mathbb{C})$ , that is the operator

$$L(x|\partial) := -\Delta_m + q(x),
 \tag{3.1}$$

where  $x \in \mathbb{R}^m$ . The operator (3.1) is self-adjoint in  $H$ . Applying the results from Section 1 to the differential expression (3.1) in the Hilbert space  $H$ , one can write down the following invertible Delsarte-Darboux transmutation operators:

$$\begin{aligned}
 \mathbf{\Omega}_{\pm} &= \mathbf{1} - \int_{\sigma(L)} d\rho_{\sigma}(\xi) \int_{\sigma(L)} d\rho_{\sigma}(\xi) \int_{\Sigma_{\sigma}} d\rho_{\Sigma_{\sigma}}(\xi) \int_{\Sigma_{\sigma}} d\rho_{\Sigma_{\sigma}}(\eta) \\
 &\quad \times \tilde{\psi}^{(0)}(\lambda; \xi) \Omega_{(x_0)}^{-1}(\lambda; \xi, \eta) \int_{S_{\pm}^{(m)}(\sigma_{(x)}^{(m-1)}, \sigma_{(x_0)}^{(m-1)})} dy \bar{\varphi}^{(0)\top}(\lambda; \eta), (\cdot),
 \end{aligned}
 \tag{3.2}$$

where  $\sigma_x^{(m-1)} \in \mathcal{K}(\mathbb{R}^m)$  is some closed maybe non-compact simplicial hyper-surface in  $\mathbb{R}^m$  parametrized by a running point  $x \in \sigma_x^{(m-1)}$ , and  $\sigma_{x_0}^{(m-1)} \in \mathcal{K}(\mathbb{R}^m)$  is a suitable homological to  $\sigma_x^{(m-1)}$  simplicial hypersurface in  $\mathbb{R}^m$  parametrized by a point  $x_0 \in \sigma_{x_0}^{(m-1)}$ . There exist exactly two  $m$ -dimensional subspaces spanning them, say  $S_{\pm}^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) \in \mathcal{K}(\mathbb{R}^m)$ , such that  $S_+^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) \cup S_-^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)}) = \mathbb{R}^m$ . Taking into account these subspaces, one can concisely rewrite the Delsarte-Darboux transmutation operators (3.2) for (3.1):

$$\Omega_{\pm} = \mathbf{1} + \int_{S_{\pm}^{(m)}(\sigma_x^{(m-1)}, \sigma_{x_0}^{(m-1)})} dy \hat{K}_{\pm}(\Omega)(x; y)(\cdot), \tag{3.3}$$

where, as before,  $x \in \sigma_x^{(m-1)}$  and kernels  $\hat{K}_{\pm}(\Omega) \in H_- \otimes H_-$  satisfy the equations (2.27), or equivalently,

$$\begin{aligned} & -\Delta_m(x; \partial) \hat{K}_{\pm}(\Omega)(x; y) + \Delta_m(y; \partial) \hat{K}_{\pm}(\Omega)(x; y) \\ & = (q(y) - \tilde{q}(x)) \hat{K}_{\pm}(\Omega)(x; y) \end{aligned} \tag{3.4}$$

for all  $x, y \in \text{supp} \hat{K}_{\pm}(\Omega)$ . Take for simplicity, a non-compact closed simplicial hypersurface  $\sigma_x^{(m-1)} = \sigma_{x, \gamma}^{(m-1)} := \{y \in \mathbb{R}^m : \langle x - y, \gamma \rangle = 0\}$  and the degenerate simplicial cycle  $\sigma_{x_0}^{(m-1)} := \{x_0 = \infty\} \subset \mathbb{R}^m$ , where  $\gamma \in \mathbb{S}^{m-1}$  is an arbitrary versor,  $\|\gamma\| = 1$ . Then, evidently,

$$S_{\pm}^{(m)}(\sigma_{x, \gamma}^{(m-1)}, \sigma_{\infty}^{(m-1)}) := S_{\pm \gamma, x}^{(m)} = \{y \in \mathbb{R}^m : \langle x - y, \pm \gamma \rangle \geq 0\} \tag{3.5}$$

and our transmutation operators (3.3) take the form

$$\Omega_{\pm \gamma} = \mathbf{1} + \int_{S_{\pm \gamma, x}^{(m)}} dy \hat{K}_{\pm \gamma}(\Omega)(x; y)(\cdot), \tag{3.6}$$

where  $\text{supp} \hat{K}_{\pm \gamma}(\Omega) = S_{\pm \gamma, x}^{(m)}$ ,  $S_{+ \gamma, x}^{(m)} \cap S_{- \gamma, x}^{(m)} = \sigma_{x, \gamma}^{(m-1)} \cup \sigma_{\infty}^{(m-1)}$  and  $S_{+ \gamma, x}^{(m)} \cup S_{- \gamma, x}^{(m)} = \mathbb{R}^m$  for any direction  $\gamma \in \mathbb{S}^{m-1}$ .

The invertible transmutation Volterra operators like (3.6) were constructed before by L.D. Faddeev [9] for the self-adjoint perturbed Laplace operator (3.1) in  $\mathbb{R}^3$ . He called them [9] transformation operators with a Volterra direction  $\gamma \in \mathbb{S}^{m-1}$ . It is easy to see that Faddeev's expressions (3.6) are very special cases of the general expressions (3.3) obtained above.

Define now making use of (3.3) the following Fredholm operator in the Hilbert space  $H$ :

$$\Omega := (\mathbf{1} + K_+(\Omega))^{-1} (\mathbf{1} + K_-(\Omega)) = \mathbf{1} + \Phi(\Omega) \tag{3.7}$$

with the compact part  $\Phi(\Omega) \in \mathcal{B}_{\infty}(H)$ . Then the commutation equality

$$[L, \Phi(\Omega)] = \mathbf{0} \tag{3.8}$$

together with the Gelfand-Levitan-Marchenko equation

$$K_+(\Omega) + \hat{\Phi}(\Omega) + \hat{K}_+(\Omega) \cdot \hat{\Phi}(\Omega) = \hat{K}_-(\Omega) \tag{3.9}$$

for the corresponding kernels  $\hat{K}_\pm(\Omega)$  and  $\hat{\Phi}(\Omega) \in H_- \otimes H_-$  hold.

In [9] there was thoroughly analyzed the spectral structure of kernels  $\hat{K}_\pm(\Omega) \in H_- \otimes H_-$  in (3.6) making use of the analytical properties of the corresponding Green’s functions of the operator (3.1). As one can see from (3.2), these properties depend strongly both on the structure of the spectral measures  $\rho_\sigma$  on  $\sigma(L)$  and  $\rho_{\Sigma_\sigma}$  on  $\Sigma_\sigma$  and on analytical behavior of the kernel  $\Omega_\infty(\lambda; \xi, \eta) \in L_2^{(\rho)}(\Sigma_\sigma; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_\sigma; \mathbb{C})$ ,  $\xi, \eta \in \Sigma_\sigma$ , for all  $\lambda \in \sigma(L)$ . In [9] there was stated for any direction  $\gamma \in \mathbb{S}^{m-1}$  the dependence of kernels  $\hat{K}_\pm(\Omega) \in H_- \otimes H_-$  on the regularized determinant of the resolvent  $R_\mu(L) \in \mathcal{B}(H)$ ,  $\mu \in \mathbb{C}/\sigma(L)$  is a regular point for the operator (3.1). This dependence can be also clarified if one makes use of the approach from Section 2.

### 3.2 A two-dimensional Dirac type operator

Let us define in  $H := L_2(\mathbb{R}^2; \mathbb{C}^2)$  a two-dimensional Dirac type operator

$$\tilde{L}_1(x; \partial) := \begin{pmatrix} \partial/\partial x_1 & \tilde{u}_1(x) \\ \tilde{u}_2(x) & \partial/\partial x_2 \end{pmatrix}, \tag{3.10}$$

where  $x := (x_1, x_2) \in \mathbb{R}^2$ , and coefficients  $\tilde{u}_j \in W_2^1(\mathbb{R}^2; \mathbb{C})$ ,  $j = \overline{1, 2}$ . The transformation properties of the operator (3.10) were studied [16] thoroughly by L.P. Nizhnik. In particular, he constructed some special class of the Delsarte-Darboux transmutation operators in the form

$$\Omega_\pm = \mathbf{1} + \int_{S_\pm^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})} dy \hat{K}_\pm(\Omega)(x; y)(\cdot), \tag{3.11}$$

where for two orthonormal versors  $\gamma_1$  and  $\gamma_2 \in \mathbb{S}^1$ ,  $\|\gamma_1\| = 1 = \|\gamma_2\|$ ,

$$\begin{aligned} S_+^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)}) &:= \{y \in \mathbb{R}^2 : \langle x - y, \gamma_1 \rangle \geq 0\} \\ &\quad \cap \{y \in \mathbb{R}^2 : \langle x - y, \gamma_2 \rangle \geq 0\}, \\ S_-^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)}) &:= \{y \in \mathbb{R}^2 : \langle x - y, \gamma_1 \rangle \leq 0\} \\ &\quad \cup \{y \in \mathbb{R}^2 : \langle x - y, \gamma_2 \rangle \leq 0\}. \end{aligned} \tag{3.12}$$

In the case when  $\langle x, \gamma_j \rangle = x_j \in \mathbb{R}$ ,  $j = \overline{1, 2}$ , the corresponding kernel

$$\hat{K}_+(\Omega) = \begin{pmatrix} K_{+,11}^{(1)} \delta_{\langle y-x, \gamma_1 \rangle} + K_{+,11}^{(0)}(x; y) & K_{+,12}^{(1)} \delta_{\langle y-x, \gamma_2 \rangle} + K_{+,12}^{(0)} \\ K_{+,21}^{(1)} \delta_{\langle y-x, \gamma_1 \rangle} + K_{+,21}^{(0)}(x; y) & K_{+,22}^{(1)} \delta_{\langle y-x, \gamma_2 \rangle} + K_{+,22}^{(0)} \end{pmatrix} \tag{3.13}$$

is Dirac delta-function singular, being, in part, localized on half-lines  $\langle y - x, \gamma_2 \rangle = 0$  and  $\langle y - x, \gamma_1 \rangle = 0$ , with regular coefficients  $K_{+,ij}^{(l)} \in C^1(\mathbb{R}^2 \times \mathbb{R}^2; \mathbb{C})$  for all  $i, j = \overline{1, 2}$  and  $l = \overline{0, 1}$ . Such a property of the transmutation kernels for the perturbed Laplace operator (3.1) was also observed in [9], where it was motivated by the necessary condition for the transformed operator  $\tilde{L}(x; \partial) \in \mathcal{L}(H)$  to be differential. As one can check, the same reason for the existence of singularities holds in (3.13).

Let us now consider the general expression like (3.3) for the corresponding hypersurfaces  $S_\pm^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})$  spanning between a closed non-compact smooth cycle  $\sigma_x^{(1)} \in \mathcal{K}(\mathbb{R}^2)$  and the infinite point  $\sigma_\infty^{(1)} := \infty \in \mathcal{K}(\mathbb{R}^2)$ . A running point  $x \in \sigma_x^{(1)}$  is taken to be arbitrary



but, as usual, fixed. The kernels  $\hat{K}_\pm(\Omega) \in H_- \times H_-$  in (3.11) satisfy the standard conditions (2.26) and (2.27), that is

$$\begin{aligned} (\tilde{L}_{1,ext} \otimes \mathbf{1})\hat{K}_\pm(\Omega) &= (\mathbf{1} \otimes L_{1,ext}^*)\hat{K}_\pm(\Omega), \\ [L_1, \Phi(\Omega)] &= 0 \end{aligned} \tag{3.14}$$

for some matrix differential Dirac type operator  $L_1 \in \mathcal{L}(H)$  of the form (3.1). Together with this Dirac operator the following matrix second order differential operator

$$\tilde{L}_2(x; \partial) := \mathbf{1} \frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \tilde{v}_2 & -2\frac{\partial \tilde{u}_1}{\partial x_2} \\ -2\frac{\partial \tilde{u}_2}{\partial x_1} & \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \tilde{v}_1 \end{pmatrix} \tag{3.15}$$

in the parametric space  $\mathcal{H} := C^1(\mathbb{R}; H)$  was studied in [16, 17] for which scattering theory was developed and its an application was given for constructing soliton-like exact solutions to the so called Davey-Stewartson nonlinear dynamical system in partial derivatives. The latter was based on the fact that two operators  $\tilde{L}_1$  and  $\tilde{L}_2 \in \mathcal{L}(H)$  commute with each other.

Namely, consider the Volterra operators  $\Omega_\pm \in \mathcal{B}(\mathcal{H})$  realizing the following Delsarte-Darboux transmutations:

$$\tilde{L}_1 \Omega_\pm = \Omega_\pm L_1, \quad \tilde{L}_2 \Omega_\pm = \Omega_\pm L_2. \tag{3.16}$$

Here we put

$$\begin{aligned} L_1(x; \partial) &: = \begin{pmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \end{pmatrix}, \\ L_2(x; \partial) &: = \mathbf{1} \frac{\partial}{\partial t} + \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \alpha_2(x_2) & 0 \\ 0 & \frac{\partial^2}{\partial x_1^2} \pm \frac{\partial^2}{\partial x_2^2} - \alpha_1(x_1) \end{pmatrix}, \end{aligned} \tag{3.17}$$

where  $\alpha_j \in W_2^1(\mathbb{R}; \mathbb{C})$ ,  $j = \overline{1, 2}$ , are some given functions. It is evident that the operators (3.17) commute. Then, if the operators  $\Omega_\pm \in \mathcal{B}(H)$  exist and satisfy (3.16), the following commutation condition

$$[\tilde{L}_1, \tilde{L}_2] = 0 \tag{3.18}$$

holds, exactly as claimed above and effectively exploited before in [16, 17].

Recall now that for the operators  $\Omega_\pm \in \mathcal{B}(H)$  to exist they must satisfy additionally the kernel conditions (3.14) and

$$\begin{aligned} (\tilde{L}_{2,ext} \otimes \mathbf{1})\hat{K}_\pm(\Omega) &= (\mathbf{1} \otimes L_{2,ext}^*)\hat{K}_\pm(\Omega), \\ [L_2, \Phi(\Omega)] &= 0, \end{aligned} \tag{3.19}$$

where, as before, the operator  $\Phi(\Omega) \in \mathcal{B}_\infty(H)$  is defined by (3.7) as

$$\Omega := \mathbf{1} + \Phi(\Omega). \tag{3.20}$$

Owing to the evident commutation condition (3.18) the set of equations (3.14) and (3.19) is compatible giving rise to the expression like (3.11), where the kernel  $\hat{K}_+(\mathbf{\Omega}) \in H_- \otimes H_-$  satisfies the set of differential equations generalizing those from [16, 17]:

$$\begin{aligned}
 & \frac{\partial K_{+,11}}{\partial x_1} + \frac{\partial K_{+,11}}{\partial y_1} + \tilde{u}_1 K_{+,21} = 0, \\
 & \frac{\partial K_{+,12}}{\partial x_1} + \frac{\partial K_{+,12}}{\partial y_1} + \tilde{u}_1 K_{+,22} = 0, \\
 & \frac{\partial K_{+,21}}{\partial x_2} + \frac{\partial K_{+,21}}{\partial x_1} + \tilde{u}_2 K_{+,11} = 0, \\
 & \frac{\partial K_{+,22}}{\partial x_2} + \frac{\partial K_{+,22}}{\partial y_2} + \tilde{u}_2 K_{+,12} = 0, \\
 & \pm \frac{\partial \tilde{u}_1}{\partial x_2} K_{+,21} = \frac{\partial K_{+,11}}{\partial t} + \left[ \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \pm \left( \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \right] K_{+,11} \\
 & + (\alpha_2(x_2) - \tilde{v}_2(x)) K_{+,11} \\
 & \pm \frac{\partial \tilde{u}_1}{\partial x_2} K_{+,21} = \frac{\partial K_{+,22}}{\partial t} + \left[ \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \pm \left( \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \right] K_{+,22} \\
 & + (\alpha_1(x_1) - \tilde{v}_1(x)) K_{+,22}, \\
 & \mp 2 \frac{\partial \tilde{u}_1}{\partial x_2} K_{+,22} = \frac{\partial K_{+,12}}{\partial t} + \left[ \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \pm \left( \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \right] K_{+,12} \\
 & + (\alpha_1(x_1) - \tilde{v}_2(x)) K_{+,22}, \\
 & 2 \frac{\partial \tilde{u}_2}{\partial x_1} K_{+,22} = \frac{\partial K_{+,21}}{\partial t} + \left[ \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \pm \left( \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \right] K_{+,21} \\
 & + (\alpha_2(x_2) - \tilde{v}_1(x)) K_{+,11}.
 \end{aligned} \tag{3.21}$$

Moreover, the following conditions

$$\begin{aligned}
 \tilde{u}_1(x) &= -K_{+,12}^{(0)}|_{y=x}, & \tilde{u}_2(x) &= -K_{+,21}^{(0)}|_{y=x}, \\
 \tilde{v}_2(x)|_{x_1=-\infty} &= \alpha_2(x_2), & \tilde{v}_1(x)|_{x_2=-\infty} &= \alpha_1(x_1)
 \end{aligned} \tag{3.22}$$

hold for all  $x \in \mathbb{R}^2$  and  $y \in \text{supp} \hat{K}_+(\mathbf{\Omega})$ , where we took into account the singular series expansion

$$\hat{K}_+(\mathbf{\Omega}) = \sum_{s=0}^{p(K_+)} K_+^{(s)} \delta_{\sigma_x^{(1)}}^{(s-1)} \tag{3.23}$$

for some finite integer  $p(K_+) \in \mathbb{Z}_+$  with respect to the Dirac function  $\delta_{\sigma_x^{(1)}} : W_2^q(\mathbb{R}^2; \mathbb{C}) \rightarrow \mathbb{R}$ ,  $q \in \mathbb{Z}_+$ , and its derivatives, having the support (see [35], Chapter 3) coinciding with the closed cycle  $\sigma_x^{(1)} \in \mathcal{K}(\mathbb{R}^2)$ .

**Remark 3.1.** Concerning the special case (3.13) discussed before in [16, 17], one gets easily that  $p(K_+) = 1$  and  $\sigma_x^{(1)} = \partial(\cap_{j=1,2} \{y \in \mathbb{R}^2 : \langle y - x, \gamma_j \rangle = 0\}) \subset \text{supp} \hat{K}_+(\mathbf{\Omega})$ . It was shown also before that equations like (3.21) and (3.22) possess solutions if the Gelfand-Levitan-Marchenko equation (2.25) does.

Making use also of the exact forms of operators  $L_1$  and  $L_2 \in \mathcal{L}(\mathcal{H})$ , one obtains easily from (3.14) and (3.19) the corresponding set of differential equations for components of the kernel  $\hat{\Phi}(\Omega) \in H_- \otimes H_- :$

$$\begin{aligned} \frac{\partial \Phi_{11}}{\partial x_1} + \frac{\partial \Phi_{11}}{\partial y_1} &= 0, & \frac{\partial \Phi_{12}}{\partial x_1} + \frac{\partial \Phi_{12}}{\partial y_1} &= 0, \\ \frac{\partial \Phi_{21}}{\partial x_2} + \frac{\partial \Phi_{21}}{\partial y_2} &= 0, & \frac{\partial \Phi_{22}}{\partial x_2} + \frac{\partial \Phi_{22}}{\partial y_2} &= 0, \\ \frac{\partial \Phi_{11}}{\partial t} \pm \left( \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \Phi_{11} + (\alpha_2(y_2) - \alpha_2(x_2)) \Phi_{11} &= 0, \\ \frac{\partial \Phi_{12}}{\partial t} \pm \left( \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \Phi_{12} + (\alpha_1(y_1) - \alpha_2(x_2)) \Phi_{12} &= 0, \\ \frac{\partial \Phi_{21}}{\partial t} + \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \Phi_{21} + (\alpha_2(y_2) - \alpha_1(x_1)) \Phi_{21} &= 0, \\ \frac{\partial \Phi_{22}}{\partial t} + \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} \right) \Phi_{22} + (\alpha_1(y_1) - \alpha_1(x_1)) \Phi_{22} &= 0 \end{aligned} \tag{3.24}$$

for all  $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ . The obtained above equations (3.24) generalize those before found in [16, 17] and used for exactly integrating the well known Devey-Stewartson differential equation [37, 11, 10] and finding so called soliton like solutions. Concerning our generalized case the kernel (3.23) is a solution to the following Gelfand-Levitan-Marchenko type equations:

$$\begin{aligned} K_+^{(0)}(x; y) + \Phi^{(0)}(x; y) + \int_{S_+^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})} K_+^{(0)}(x; \xi) \Phi^{(0)}(\xi; y) d\xi \\ + \int_{\sigma_x^{(1)}} K_+^{(1)}(x; \xi) \Phi^{(0)}(\xi; y) d\sigma_x^{(1)} &= 0, \\ K_+^{(1)}(x; y) + \Phi^{(1)}(x; y) + \int_{S_+^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})} K_+^{(0)}(x; \xi) \Phi^{(1)}(\xi; y) d\xi \\ + \int_{\sigma_x^{(1)}} K_+^{(1)}(x; \xi) \Phi^{(1)}(\xi; y) d\sigma_x^{(1)} &= 0, \end{aligned} \tag{3.25}$$

where  $y \in S_+^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})$  for all  $x \in \mathbb{R}^2$  and, by definition,

$$\hat{\Phi}(\Omega) := \Phi^{(0)} + \Phi^{(1)} \delta_{\sigma_x^{(1)}} \tag{3.26}$$

is the corresponding to (3.23) kernel expansion. Since the kernel (3.26) is singular, the differential equations (3.24) naturally must be treated in the distributional sense [35].

Taking into account the exact forms of "dressed" differential operators  $L_j \in \mathcal{L}(\mathcal{H})$ ,  $j = \overline{1, 2}$ , given by (3.10) and (3.15) one gets easily that the commutativity condition (3.18) which gives rise to that of  $\tilde{L}_j \in \mathcal{L}(\mathcal{H})$ ,  $j = \overline{1, 2}$ , being equivalent to the mentioned before Devey-Stewartson dynamical system

$$\begin{aligned} d\tilde{u}_1/dt &= -(\tilde{u}_{1,xx} + \tilde{u}_{1,yy}) + 2(\tilde{v}_1 - \tilde{v}_2), \\ d\tilde{u}_2/dt &= \tilde{u}_{2,xx} + \tilde{u}_{2,yy} + 2(\tilde{v}_2 - \tilde{v}_1), \\ \tilde{v}_{1,x} &= (\tilde{u}_1 \tilde{u}_2)_y, \quad \tilde{v}_{2,x} = (\tilde{u}_1 \tilde{u}_2)_x \end{aligned} \tag{3.27}$$

on a functional infinite-dymensioanl manifold  $M_u \subset \mathcal{S}(\mathbb{R}^2; \mathbb{C}^2)$ . The exact soliton-like solutions to (3.27) are given by expressions (3.22), where the kernel  $K_+^{(1)}(\Omega)$  solves the second linear integral equation of (3.25). On the other hand-side, there exists the exact expression (2.4) which solves the set of "dressed" equations

$$\tilde{L}_1 \tilde{\psi}^{(0)}(\eta) = 0, \quad \tilde{L}_2 \tilde{\psi}^{(0)}(\eta) = 0. \tag{3.28}$$

Since the kernels  $\Omega(\lambda, \mu) \in L_2^{(\rho)}(\Sigma; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma; \mathbb{C})$ , for  $\lambda, \mu \in \Sigma, (t; x) \in M_T \cap S_+^{(2)}(\sigma_x^{(1)}, \sigma_\infty^{(1)})$  are given by means of exact expressions (2.2), one can find via simple calculations the corresponding analytical expression for the functions  $(\tilde{u}_1, \tilde{u}_2) \in M_u$ , solving the dynamical system (3.27). This procedure is often called the Darboux type transformation and was recently extensively used as a particular case of the construction above in [23] for finding soliton-like solutions to the Devey-Stewartson (3.27) and related with it two-dimensional modified Korteweg-de Wries flows on  $M_u$ . Moreover, as it can be observed from the technique used for constructing the Delsarte-Darboux transmutation operators  $\Omega_\pm \in \mathcal{B}(H)$ , the set of solutions to (3.27) obtained by means of Delsarte-Darboux transmutations coincides completely with the corresponding set of solutions obtaien by means of solving the realted set of Gelfand-Levitan-Marchenko integral equations (3.24) and (3.25).

### 4 A generalized affine de Rham-Hodge type differential complex and related self-dual Yang-Mills flows.

Consider the following set of affine differential expressions in  $\mathcal{H} := C^1(\mathbb{R}^{m+1}; H), H := L_2(\mathbb{R}^m; \mathbb{C}^N)$  :

$$L_i(\lambda) := \mathbf{1} \frac{\partial}{\partial p_i} - \lambda \frac{\partial}{\partial x_i} + A_i(x; p|t), \tag{4.1}$$

where  $x \in \mathbb{R}^m, (t, p) \in \mathbb{R}^{m+1}$ , matrices  $A_i \in C^1(\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N))$ ,  $i = \overline{1, m}$ , and a parameter  $\lambda \in \mathbb{C}$ . One can easily now construct an exact generalized affine de Rham-Hodge differential complex on  $M_T := \mathbb{R}^{m+1} \times \mathbb{R}^m$  as

$$\mathcal{H} \rightarrow \Lambda(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{L}(\lambda)}} \Lambda^1(M_T; \mathcal{H}) \rightarrow \xrightarrow{d_{\mathcal{L}(\lambda)}} \Lambda^{2m+1}(M_T; \mathcal{H}) \xrightarrow{d_{\mathcal{L}(\lambda)}} 0, \tag{4.2}$$

where, by definition, the differentiation

$$d_{\mathcal{L}(\lambda)} := dt \wedge B(\lambda) + \sum_{i=1}^m dp_i \wedge L_i(\lambda) \tag{4.3}$$

and the affine matrix

$$B(\lambda) := \partial/\partial t - \sum_{s=0}^{n(B)+q} B_s(x; p|t) \lambda^{n(B)-s} \tag{4.4}$$

with matrices  $B_s \in C^1(\mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N))$ ,  $s = \overline{0, n(B) + q}$ ,  $n(B), q \in \mathbb{Z}_+$ . The affine complex (4.2) will be exact for all  $\lambda \in \mathbb{C}$  iff the following generalized self-dual Yang-Mills equations [43]

$$\partial A_i / \partial p_j - \partial A_j / \partial p_i - [A_i, A_j] = 0, \quad \partial A_i / \partial x_j - \partial A_j / \partial x_i = 0,$$

$$\begin{aligned} \partial B_0/\partial x_i &= 0, \quad \partial B_{n(B)+q}/\partial p_i = 0, \quad \partial B_s/\partial x_i = \partial B_{s-1}/\partial p_i + [A_i, B_{s-1}] = 0, \\ \partial A_i/\partial t + \partial B_{n(B)}/\partial p_i - \partial B_{n(B)+1}/\partial x_i + [A_i, B_{n(B)}] &= 0 \end{aligned} \tag{4.5}$$

hold for all  $i, j = \overline{1, m}$  and  $s = \overline{0, n(B) \vee n(B) + q, n(B) + 2}$ . Assume now that the conditions (4.5) are satisfied on  $M_T$ . Then, making the change  $\mathbb{C} \ni \lambda \rightarrow \partial/\partial \tau : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\tau \in \mathbb{R}$ , one finds the following set of pure differential expressions

$$\begin{aligned} L_{i(\tau)} &: = \mathbf{1} \frac{\partial}{\partial p_i} - \frac{\partial^2}{\partial \tau \partial x_i} + A_i(x; p|t), \\ B_{(\tau)} &: = \partial/\partial t - \sum_{s=0}^{n(B)+q} B_s(x; p|t) \left(\frac{\partial}{\partial \tau}\right)^{n(B)-s}, \end{aligned} \tag{4.6}$$

where matrices  $A_i, i = \overline{1, m}$ , and  $B_s, s = \overline{0, n(B) + q}$ , do not depend on the variable  $\tau \in \mathbb{R}$ . By means of operator expressions (4.6) one can now naturally construct a new differential complex related with that of (4.2):

$$\mathcal{H}_{(\tau)} \rightarrow \Lambda(M_{T,\tau}; \mathcal{H}_{(\tau)}) \xrightarrow{d_{\mathcal{L}}} \Lambda^1(M_{T,\tau}; \mathcal{H}_{(\tau)}) \rightarrow \dots \rightarrow \Lambda^{2m+2}(M_{T,\tau}; \mathcal{H}_{(\tau)}) \xrightarrow{d_{\mathcal{L}}} 0, \tag{4.7}$$

where, by definition,  $\mathcal{H}_{(\tau)} := C^1(\mathbb{R}^{m+1}; H_{(\tau)})$ ,  $H_{(\tau)} := L_2(\mathbb{R}^m \times \mathbb{R}_\tau; \mathbb{C}^N)$  and

$$d_{\mathcal{L}} := dt \wedge B_{(\tau)} + \sum_{i=1}^m dp_i \wedge L_{i(\tau)}. \tag{4.8}$$

Owing to the condition (4.5) the following lemma holds.

**Lemma 4.1.** *The differential complex (4.7) is exact.*

Therefore, one can construct the standard generalized de Rham-Hodge type Hilbert space decomposition

$$\mathcal{H}_{\Lambda}(M_{T,\tau}) := \bigoplus_{k=0}^{k=2m+2} \mathcal{H}_{\Lambda}^k(M_{T,\tau}) \tag{4.9}$$

as well as the corresponding Hilbert-Schmidt rigging

$$\mathcal{H}_{\Lambda,+}(M_{T,\tau}) \subset \mathcal{H}_{\Lambda}(M_{T,\tau}) \subset \mathcal{H}_{\Lambda,-}(M_{T,\tau}). \tag{4.10}$$

Making use now of the results obtained in Subsection 1.5, one can define the Delsarte closed subspaces  $\mathcal{H}_{0(\tau)}$  and  $\tilde{\mathcal{H}}_{0(\tau)} \subset \mathcal{H}_{(\tau)-}$ , related with the exact complex (4.7):

$$\begin{aligned} \mathcal{H}_{0(\tau)} &: = \{\psi_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{T,\tau}) : L_{j(\tau)}\psi_{(\tau)}^{(0)}(\xi) = 0, \\ B_{(\tau)}\psi_{(\tau)}^{(0)}(\xi) &= 0, \quad \psi_{(\tau)}^{(0)}(\xi)|_{\Gamma} = 0, \quad \psi_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{\lambda\tau}\psi_{\lambda}^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\ L_j(\lambda)\psi_{\lambda}^{(0)}(\eta) &= 0, \quad \xi = (\lambda; \eta) \in \Sigma := \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\}, \\ \tilde{\mathcal{H}}_{0(\tau)} &: = \{\tilde{\psi}_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{T,\tau}) : \tilde{L}_{j(\tau)}^{(0)}\tilde{\psi}_{(\tau)}^{(0)}(\xi) = 0, \\ \tilde{B}_{(\tau)}\tilde{\psi}_{(\tau)}^{(0)}(\xi) &= 0, \quad \tilde{\psi}_{(\tau)}^{(0)}(\xi)|_{\Gamma} = 0, \quad \tilde{\psi}_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{\lambda\tau}\tilde{\psi}_{\lambda}^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\ \tilde{L}_j(\lambda)\tilde{\psi}_{\lambda}^{(0)}(\eta) &= 0, \quad \xi = (\lambda; \eta) \in \Sigma := \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}\}, \end{aligned} \tag{4.11}$$

where  $\Gamma$  and  $\tilde{\Gamma} \subset M_{T,\tau}$  are some smooth hyper-surfaces. The similar expressions correspond to the adjoint closed subspaces  $\mathcal{H}_{0(\tau)}^*$  and  $\tilde{\mathcal{H}}_{0(\tau)}^* \subset \mathcal{H}_{\tau,-}^*$  :

$$\begin{aligned}
 \tilde{\mathcal{H}}_{0(\tau)} & : = \{\varphi_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{T,\tau}) : L_{j(\tau)}^* \varphi_{(\tau)}^{(0)}(\xi) = 0, & (4.12) \\
 B_{(\tau)} \varphi_{(\tau)}^{(0)}(\xi) & = 0, \varphi_{(\tau)}^{(0)}(\xi)|_{\Gamma} = 0, \varphi_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{-\bar{\lambda}\tau} \varphi_{\lambda}^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\
 L_j^*(\lambda) \varphi_{\lambda}^{(0)}(\eta) & = 0, \xi = (\lambda; \eta) \in \Sigma := \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}, \\
 \tilde{\mathcal{H}}_{0(\tau)} & : = \{\tilde{\varphi}_{(\tau)}^{(0)}(\xi) \in \mathcal{H}_{\Lambda,-}^0(M_{T,\tau}) : \tilde{L}_{j(\tau)}^* \tilde{\varphi}_{(\tau)}^{(0)}(\xi) = 0, \\
 \tilde{B}_{(\tau)}^* \tilde{\varphi}_{(\tau)}^{(0)}(\xi) & = 0, \tilde{\varphi}_{(\tau)}^{(0)}(\xi)|_{\tilde{\Gamma}} = 0, \tilde{\varphi}_{(\tau)}^{(0)}(\xi)|_{t=0} = e^{-\bar{\lambda}\tau} \tilde{\varphi}_{\lambda}^{(0)}(\eta) \in \mathcal{H}_{\Lambda,-}^0(M_{\mathbb{R}^m,\tau}), \\
 \tilde{L}_j^*(\lambda) \tilde{\varphi}_{\lambda}^{(0)}(\eta) & = 0, \xi = (\lambda; \eta) \in \Sigma := \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}.
 \end{aligned}$$

Based on the closed subspaces (4.12) and (4.11), one can suitably construct the Darboux type kernel  $\tilde{\Omega}_{(t,x;\tau)}(\eta, \xi) \in L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)}; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)}; \mathbb{C})$ ,  $\eta, \xi \in \Sigma_{\mathbb{C}}^{(m)}$ , and further, the corresponding Delsarte transmutation mappings  $\hat{\Omega}_{\pm} \in \mathcal{B}(H_{(\tau)})$ . Namely, assume that the following conditions

$$\psi_{(\tau)}^{(0)}(\xi) := \tilde{\psi}_{(\tau)}^{(0)}(\xi) \cdot \tilde{\Omega}_{(t,p;x;\tau)}^{-1} \tilde{\Omega}_{(t_0,p_0,x_0;\tau)} \tag{4.13}$$

for any  $\xi \in \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}$  hold, where

$$\begin{aligned}
 \tilde{\Omega}_{(t,x;\tau)}(\mu, \xi) & := \int_{\sigma_{(t,x;\tau)}} \tilde{\Omega}_{(\tau)}^{(2m+1)} [e^{-\bar{\lambda}\tau} \tilde{\varphi}^{(0)}(\mu), e^{\lambda\tau} \tilde{\psi}^{(0)}(\eta) dx \wedge dp \wedge dt], \\
 \tilde{Z}_{(\tau)}^{(2m+1)} & [e^{-\bar{\lambda}\tau} \tilde{\varphi}^{(0)}(\mu), \sum_{i=1}^m e^{\lambda\tau} \tilde{\psi}^{(0)}(\xi_{(i)}) \wedge d\tau \wedge dx \wedge_{j \neq i}^m dp_j] \\
 : & = d\tilde{\Omega}_{(\tau)}^{(2m)} [e^{-\bar{\lambda}\tau} \tilde{\varphi}^{(0)}(\mu), \sum_{i=1}^m e^{\lambda\tau} \tilde{\psi}^{(0)}(\xi_{(i)}) \wedge d\tau \wedge dx \wedge_{j \neq i}^m dp_j],
 \end{aligned} \tag{4.14}$$

and, similarly to (1.25), there holds the relationship

$$\begin{aligned}
 & \langle d_{\tilde{\mathcal{L}}}^* \tilde{\varphi}^{(0)}(\mu) e^{-\bar{\lambda}\tau}, * \sum_{i=1}^m e^{\lambda\tau} \tilde{\psi}^{(0)}(\xi_{(i)}) dt \wedge d\tau \wedge dx \wedge_{j \neq i}^m dp_j \rangle \\
 & = \langle (* )^{-1} \tilde{\varphi}^{(0)}(\mu) e^{-\bar{\lambda}\tau}, d_{\tilde{\mathcal{L}}} (\sum_{i=1}^m e^{\lambda\tau} \tilde{\psi}^{(0)}(\xi_{(i)}) dt \wedge d\tau \wedge dx \wedge_{j \neq i}^m dp_j) \rangle \\
 & \quad + d\tilde{Z}_{(\tau)}^{(2m+1)} [\tilde{\varphi}^{(0)}(\mu) e^{-\bar{\lambda}\tau}, \sum_{i=1}^m e^{\lambda\tau} \tilde{\psi}^{(0)}(\xi_{(i)}) dt \wedge d\tau \wedge dx \wedge_{j \neq i}^m dp_j],
 \end{aligned} \tag{4.15}$$

defining the exact  $(2m + 1)$ -form  $\tilde{Z}_{(\tau)}^{(2m+1)} \in \Lambda^{2m+1}(M_{T,\tau}; \mathbb{C})$ . Compute now the Delsarte transformed differential expressions

$$L_{j(\tau)} := \hat{\Omega}_{(\tau)\pm}^{-1} \tilde{L}_{j(\tau)} \hat{\Omega}_{(\tau)\pm}, \quad B_{(\tau)} := \hat{\Omega}_{(\tau)\pm}^{-1} \tilde{B}_{(\tau)} \hat{\Omega}_{(\tau)\pm} \tag{4.16}$$

for any  $j = \overline{1, m}$ , where, by definition,

$$\begin{aligned} \tilde{L}_{j(\tau)} & : = \mathbf{1} \frac{\partial}{\partial p_j} - \frac{\partial^2}{\partial \tau \partial x_j} + \bar{A}_j, \\ B_{(\tau)} & : = \partial / \partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s \left( \frac{\partial}{\partial \tau} \right)^{n(B)-s} \end{aligned} \tag{4.17}$$

with all matrices  $\bar{A}_j \in \text{End} \mathbb{C}^m$ ,  $j = \overline{1, m}$ , and  $\bar{B}_s \in \text{End} \mathbb{C}^m$ ,  $s = \overline{0, n(B) + q}$ , being constant. This means, in particular, the commuting relationships

$$[\tilde{L}_{j(\tau)}, \tilde{L}_{i(\tau)}] = 0, [\tilde{L}_{j(\tau)}, \tilde{B}_{(\tau)}] = 0 \tag{4.18}$$

hold for all  $i, j = \overline{1, m}$ . Owing to the expressions (4.16) the induced commuting relationships

$$[L_{j(\tau)}, L_{i(\tau)}] = 0, [L_{j(\tau)}, B_{(\tau)}] = 0 \tag{4.19}$$

evidently hold, coinciding exactly with relationships (4.5). Moreover, reducing our differential expressions (4.16) upon functional subspaces  $\mathcal{H}_{(\lambda)} := e^{\lambda \tau} \mathcal{H}$ ,  $\lambda \in \mathbb{C}$ , one gets easily the set of affine differential expressions (4.1) and (4.4). Write down now the respectively reduced Delsarte transmutation operators

$$\begin{aligned} \hat{\Omega}_{\pm} & = \mathbf{1} - \int_{\Sigma_{\mathbb{C}}^{(m)}} d\rho_{\Sigma_{\mathbb{C}}^{(m)}}(\nu) \int_{\Sigma_{\mathbb{C}}^{(m)}} d\rho_{\Sigma_{\mathbb{C}}^{(m)}}(\eta) \psi^{(0)}(\lambda; \nu) \tilde{\Omega}_{(t_0, p_0; x_0)}^{-1}(\lambda; \nu, \eta) \\ & \quad \times \int_{S_{\pm}^{(2m+1)}(\sigma_{(t, p; x)}^{(2m)}, \sigma_{(tt_0, p_0; x_0)}^{(2m)})} \tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda; \nu), (\cdot) \sum_{i=1}^m dt \wedge dx \bigwedge_{j \neq i}^m dp_j], \end{aligned} \tag{4.20}$$

where  $\sigma_{(t, p; x)}^{(2m)}$  and  $\sigma_{(tt_0, p_0; x_0)}^{(2m)} \in \mathcal{K}(M_{\Gamma})$  are some  $2m$ -dimensional closed singular simplexes, and by definition,

$$\begin{aligned} & \tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda; \nu), \sum_{i=1}^m \tilde{\psi}^{(0)}(\lambda; \eta_{(i)}) dt \wedge dx \bigwedge_{j \neq i}^m dp_j] \\ : & = \tilde{Z}_{(\tau)}^{(2m+1)}[e^{-\bar{\lambda} \tau} \tilde{\varphi}^{(0)}(\lambda; \nu), \sum_{i=1}^m e^{\lambda \tau} \tilde{\psi}^{(0)}(\lambda; \eta_{(i)}) d\tau \wedge dt \wedge dx \bigwedge_{j \neq i}^m dp_j] |_{d\tau=0}, \\ d\tilde{\Omega}_{(t, p; x)}(\lambda; \nu, \eta) & := \tilde{Z}^{(2m+1)}[\tilde{\varphi}^{(0)}(\lambda; \nu), \sum_{i=1}^m \tilde{\psi}^{(0)}(\lambda; \eta_{(i)}) dt \wedge dx \bigwedge_{j \neq i}^m dp_j], \end{aligned} \tag{4.21}$$

since the  $(2m + 1)$ -form (4.21) is owing to (4.15) also exact for any  $(\lambda; \nu, \eta) \in \mathbb{C} \times (\Sigma_{\mathbb{C}}^{(m)} \times \Sigma_{\mathbb{C}}^{(m)})$ . Thus, the operator expression (4.20) if applied to the operators (4.17) reduced upon the functional subspace  $\mathcal{H}_{(\lambda)} \simeq \mathcal{H}$ ,  $\lambda \in \mathbb{C}$ , gives rise to the differential expressions

$$L_j(\lambda) := \hat{\Omega}_{\pm}^{-1} \tilde{L}_j(\lambda) \hat{\Omega}_{\pm} \quad B(\lambda) := \hat{\Omega}_{\pm}^{-1} \tilde{B}(\lambda) \hat{\Omega}_{\pm}, \tag{4.22}$$

where  $L_j(\lambda) \mathcal{H}_{(\lambda)} = L_{j(\tau)} \mathcal{H}_{(\lambda)}$ ,  $B(\lambda) \mathcal{H}_{(\lambda)} = B_{(\tau)}(\lambda) \mathcal{H}_{(\lambda)}$ ,  $j = \overline{1, m}$ , coinciding with affine differential expressions (4.1) and (4.4). Concerning application of these results to finding

exact soliton like solutions to self-dual Yang-Mills equations (4.5), it is enough to mention that the relationship (4.13) reduced upon the subspace  $\mathcal{H}_{(\lambda)} \simeq \mathcal{H}$ ,  $\lambda \in \mathbb{C}$ , gives rise to the following mapping:

$$\psi^{(0)}(\lambda; \eta) := \tilde{\psi}^{(0)}(\lambda; \eta) \cdot \tilde{\Omega}_{(t,p;x)}^{-1} \tilde{\Omega}_{(t_0,p_0;x_0)}, \tag{4.23}$$

where kernels  $\tilde{\Omega}_{(t,p;x;\tau)}(\lambda; \eta, \xi) \in L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)}; \mathbb{C}) \otimes L_2^{(\rho)}(\Sigma_{\mathbb{C}}^{(m)}; \mathbb{C})$ ,  $\eta, \xi \in \Sigma_{\mathbb{C}}^{(m)}$ , for all  $(t, p; x) \in M_T$  and  $\lambda \in \mathbb{C}$ . Since the element  $\psi^{(0)}(\lambda; \eta) \in \mathcal{H}_-$  for any  $(\lambda; \xi) \in \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)}$  satisfies the set of differential equations

$$L_i(\lambda)\psi^{(0)}(\lambda; \eta) = 0, \quad B(\lambda)\psi^{(0)}(\lambda; \eta) = 0, \tag{4.24}$$

for all  $i = \overline{1, m}$ , from (4.23) and (4.24) one finds easily exact expressions for the corresponding matrices  $A_j$  and  $B_s \in C^1(\mathbb{R} \times \mathbb{R}^{m+1}; S(\mathbb{R}^m; End\mathbb{C}^N))$ ,  $j = \overline{1, m}$ ,  $s = \overline{0, n(B) + q}$ , satisfying the self-dual Yang-Mills equations (4.5). This leads to the following result.

**Theorem 4.2.** *The integral expressions (4.20) in  $\mathcal{H}$  are the Delsarte transmutation operators corresponding to the affine differential expressions (4.1), (4.5) and constant operators*

$$\tilde{L}_i(\lambda) := \mathbf{1} \frac{\partial}{\partial p_i} - \lambda \frac{\partial}{\partial x_i} + \bar{A}, \quad \tilde{B}(\lambda) := \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s \lambda^{n(B)-s} \tag{4.25}$$

for any  $\lambda \in \mathbb{C}$ . The mapping (4.23) realizes the isomorphisms between the closed subspaces

$$\begin{aligned} \mathcal{H}_0 & : = \{ \psi^{(0)}(\lambda; \eta) \in \mathcal{H}_- : d_{\mathcal{L}(\lambda)}\psi^{(0)}(\lambda; \eta) = 0, \psi^{(0)}(\lambda; \eta)|_{t=0} \\ & = \psi_{\lambda}^{(0)}(\eta) \in H_-, \psi^{(0)}(\lambda; \eta)|_{\Gamma} = 0, (\lambda; \eta) \in \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)} \} \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} \tilde{\mathcal{H}}_0 & : = \{ \tilde{\psi}^{(0)}(\lambda; \eta) \in \mathcal{H}_- : d_{\tilde{\mathcal{L}}(\lambda)}^{(0)}\tilde{\psi}(\lambda; \eta) = 0, \tilde{\psi}^{(0)}(\lambda; \eta)|_{t=0} \\ & = \tilde{\psi}_{\lambda}^{(0)}(\eta) \in H_-, \tilde{\psi}^{(0)}(\lambda; \eta)|_{\tilde{\Gamma}} = 0, (\lambda; \eta) \in \mathbb{C} \times \Sigma_{\mathbb{C}}^{(m)} \} \end{aligned} \tag{4.27}$$

for any parameter  $\lambda \in \mathbb{C}$ . Moreover, the expressions (4.23) generate the standard Darboux type transformations for the set of operators (4.25) and (4.1), (4.4) via the corresponding set of linear equations (4.24), thereby producing exact soliton-like solutions to the self-dual Yang-Mills equations (4.5).

As a simple partial consequence from Theorem 3.2 one retrieves all of the results obtained in [43], where the Delsarte-Darboux mapping (4.23) was chosen completely a priori without any proof and motivation in the form of some affine gauge transformation.

Results similar to the above can be with a minor change applied also to the affine generalized differential de Rham-Hodge complex (4.2) with the external differentiation (4.3), where

$$\begin{aligned} L_i(\lambda) & : = \mathbf{1} \frac{\partial}{\partial p_i} - \left( \sum_{k=0}^{n_i(L)} a_{ik} \lambda^{k+1} \right) \frac{\partial}{\partial x_i} + \sum_{k=0}^{n_i(L)} A_{ik} \lambda^k, \\ \tilde{B}(\lambda) & : = \partial/\partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s \lambda^{n(B)-s}, \end{aligned} \tag{4.28}$$



or

$$\begin{aligned} L_i(\lambda) &: = \mathbf{1} \frac{\partial}{\partial p_i} - \left( \sum_{k=0}^{n_i(L)} a_{ik}^{(j)} \lambda^{k+1} \right) \frac{\partial}{\partial x_j} + \sum_{k=0}^{n_i(L)} A_{ik} \lambda^k, \\ \tilde{B}(\lambda) &: = \partial / \partial t - \sum_{s=0}^{n(B)+q} \bar{B}_s \lambda^{n(B)-s}, \end{aligned} \quad (4.29)$$

for  $i = \overline{1, m}$ ,  $\lambda \in \mathbb{C}$ . The case (4.28) was analyzed recently also in [42] by means of similar gauge type transformations which was used before in [43]. Regrettably, the results obtained there are too complicated and unwieldy, so one needs to use more mathematically motivated, clear and less cumbersome techniques for finding Delsarte-Darboux transformations and related soliton-like exact solutions.

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