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Jacobi, Ellipsoidal Coordinates and Superintegrable Systems

E G KALNINS [†], *J M KRESS* [‡] *and W MILLER* [§]

[†] *Department of Mathematics and Statistics, University of Waikato, Hamilton New Zealand*
E-mail: e.kalnins@waikato.ac.nz

[‡] *School of Mathematics, The University of New South Wales, Sydney NSW 2052 Australia*
E-mail: j.kress@unsw.edu.au

[§] *School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA*
E-mail: miller@ima.umn.edu

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Abstract

We describe Jacobi's method for integrating the Hamilton-Jacobi equation and his discovery of elliptic coordinates, the generic separable coordinate systems for real and complex constant curvature spaces. This work was an essential precursor for the modern theory of second-order superintegrable systems to which we then turn. A Schrödinger operator with potential on a Riemannian space is **second-order superintegrable** if there are $2n-1$ (classically) functionally independent second-order symmetry operators. (The $2n-1$ is the maximum possible number of such symmetries.) These systems are of considerable interest in the theory of special functions because they are multiseparable, i.e., variables separate in several coordinate sets and are explicitly solvable in terms of special functions. The interrelationships between separable solutions provides much additional information about the systems. We give an example of a superintegrable system and then present very recent results exhibiting the general structure of superintegrable systems in all real or complex two-dimensional spaces and three-dimensional conformally flat spaces and a complete list of such spaces and potentials in two dimensions.

1 Introduction

During Carl Gustav Jacob Jacobi's brief life he contributed much to number theory, the theory of both ordinary and partial differential equations, the calculus of variations, the three-body problem and the development of classical mechanics and elliptic functions. His most celebrated researches relate to the study of elliptic functions which he and Abel established independently. Our work builds on Jacobi's researches in mechanics and overlaps with the notion of Jacobi elliptic functions. In 1842 Jacobi invented the method of generating functions for solving the Hamiltonian equations of classical mechanics [23],

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{1.1}$$

where $\{q_i, p_j\} = \delta_{ij}$ and $\{\cdot, \cdot\}$ is the Poisson Bracket. (Jacobi also allowed for explicitly time-dependent Hamiltonians. We do not discuss this extension here, although such systems can also be treated by the methods of separation of variables. Treatment of the appropriate type of separation, the so-called R-separation, would take us too far afield.) This method consists of finding a generating function, $S(\mathbf{q}, \alpha)$, such that $\mathbf{p} = \nabla_{\mathbf{q}} S(\mathbf{q}, \alpha), \beta = \nabla_\alpha S(\mathbf{q}, \alpha)$ and the Hamiltonian is transformed to α_1. The transformed equations have the form

$$\frac{d\beta}{dt} = \frac{\partial H}{\partial \alpha} = (1, 0, ..., 0), \quad \frac{d\alpha}{dt} = -\frac{\partial H}{\partial \beta} = 0,$$

where $H = H(\mathbf{q}(\alpha, \beta), \mathbf{p}(\alpha, \beta))$. The solutions have the particularly simple form

$$\beta(t) = (t + b_1, b_2, ..., b_n), \quad \alpha(t) = (a_1, a_2, ..., a_n).$$

The generating function that enables this transformation can be calculated using the relation $\mathbf{p} = \nabla_{\mathbf{q}} S(\mathbf{q}, \alpha)$ which results from $S(\mathbf{q}, \alpha)$ being a generating function. The other relation is $\beta = \nabla_\alpha S(\mathbf{q}, \alpha)$. The resulting equation for $S(\mathbf{q}, \alpha)$ is the (time-independent) **Hamilton-Jacobi equation**

$$H(\mathbf{q}, \nabla S(\mathbf{q}, \alpha)) = \alpha_1, \tag{1.2}$$

where it is usual to set $\alpha_1 = E$. If this equation can be solved for $S(\mathbf{q}, \alpha)$ in such a way that

$$\det \left(\frac{\partial^2 S(\mathbf{q}, \alpha)}{\partial q_i \partial \alpha_j} \right) \neq 0,$$

then a complete integral for the Hamiltonian system has been obtained, depending on n constants of the motion α. The key connection with separation of variables techniques comes from the ansatz of additive separation, $S(\mathbf{q}, \alpha) = \sum_{i=1}^n S_i(q_i, \alpha)$.

Hamiltonians that correspond to the usual H = Kinetic Energy + Potential Energy and are of the form $H = \frac{1}{2}\mathbf{p} \cdot \mathbf{p} + V(\mathbf{q})$ can be solved by this ansatz in many physically interesting cases. The most notable case is that of the motion of a single planet under the influence of the gravity of the Sun. Written in spherical coordinates the Hamilton-Jacobi equation has the form

$$\frac{1}{2}\left(p_r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2\right)\right) - \frac{G}{r} - \alpha_1 = 0$$

and can be solved via the substitution

$$S(r, \theta, \varphi, \alpha) = S_r(r, \alpha) + S_\theta(\theta, \alpha) + S_\varphi(\varphi, \alpha),$$

where $\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$. In order to solve other nontrivial problems in mechanics, Jacobi introduced his "remarkable change of variables", the generalized elliptical coordinates x_j in n dimensions, [23]. These can be defined by the relations

$$1 + \sum_{k=1}^{n} \frac{q_k^2}{z - e_k} = \frac{\Pi_{j=1}^{n}(z - x_j)}{\Pi_{k=1}^{n}(z - e_k)}, \tag{1.3}$$

where the q_k are Cartesian coordinates and the e_k are distinct constants. An equivalent definition is

$$q_k^2 = \frac{\Pi_{j=1}^{n}(e_k - x_j)}{\Pi_{j\neq k}(e_j - e_k)},$$

where $e_1 < x_1 < e_2 < \cdots < e_n < x_n$ and $k = 1, \cdots, n$. In the case that $n = 3, 4$ the elliptic coordinates admit expression in terms of Jacobi elliptic functions [57, 1]. For $n = 3$ we have

$$q_1 = k\mathrm{sn}\alpha\mathrm{sn}\beta\mathrm{sn}\gamma, \quad q_2 = i\frac{k}{k'}\mathrm{cn}\alpha\mathrm{cn}\beta\mathrm{cn}\gamma, \quad q_3 = \frac{1}{kk'}\mathrm{dn}\alpha\mathrm{dn}\beta\mathrm{dn}\gamma,$$

where we write $x_1 = \mathrm{sn}\alpha$, $x_2 = \mathrm{sn}\beta$ and $x_3 = \mathrm{sn}\gamma$ with normalized choice of e_i according to $e_1 = 0$, $e_2 = 1$ and $e_3 = k^{-2}$ with $k^2 < 1$, and the k dependence of the Jacobi elliptic functions has been suppressed, i.e., $\mathrm{sn}\delta = \mathrm{sn}(\delta, k)$. Typically the Jacobi elliptic function $\mathrm{sn}(\delta, k)$ is defined by

$$\delta = \int_0^{\mathrm{sn}(\delta,k)} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt.$$

These functions have properties analogous to trigonometric functions. The variables α, β and γ vary in the ranges $\alpha \in [-K, K], \beta \in [K - iK', K + iK']$ and $\gamma \in [iK' - K, iK' + K]$. In addition to elliptic coordinates in Euclidean space there are also elliptic coordinates on the n-dimensional sphere. These are defined by

$$\sum_{k=1}^{n+1} \frac{s_k^2}{z - e_k} = \frac{\Pi_{j=1}^{n+1}(z - x_j)}{\Pi_{k=1}^{n+1}(z - e_k)}, \tag{1.4}$$

where $s_1^2 + \cdots + s_{n+1}^2 = 1$. The inverse relations are

$$s_k^2 = \frac{\Pi_{j=1}^{n}(e_k - x_j)}{\Pi_{j\neq k}(e_j - e_k)},$$

where $k = 1, \cdots, n+1$ and the coordinates satisfy $e_1 < x_1 < e_2 < \cdots < e_n < x_n < e_{n+1}$. These coordinates enable the ansatz of separation of variables to be used for problems on the sphere analogously to those solved in Euclidean space. If $n = 2$, the coordinates can also be written in terms of Jacobi elliptic functions according to [1]

$$s_1 = k\mathrm{sn}\alpha\mathrm{sn}\beta, \quad s_2 = i\frac{k}{k'}\mathrm{cn}\alpha\mathrm{cn}\beta, \quad s_3 = \frac{1}{k'}\mathrm{dn}\alpha\mathrm{dn}\beta \tag{1.5}$$

with α and β varying in the same ranges as for Euclidean elliptical coordinates. The Jacobi elliptical coordinates enabled the problem of geodesic motion on an ellipsoid to be solved. It was on the basis of these investigations of Jacobi that subsequent investigations in the theory of separation of variables developed. Most notable among these were the mechanism of separation extended by Stäckel [55] to quite general systems of orthogonal coordinates. Subsequently Levi Civita [43] gave a set of nonlinear partial differential equations that must be satisfied if separation of variables is possible in a particular coordinate system. The next important results were obtained by Eisenhart [9] who gave an intrinsic characterization of orthogonal separable coordinate systems and also discussed the product separability of the Helmholtz or Schrödinger equation $\Delta\Psi + \lambda_1\Psi = 0$. Included in his analysis was the geometrical significance of the additional criterion for separation by products to occur, i.e., $\Psi = \Pi_{k=1}^{n}\Psi(q_k, \lambda)$ in some suitable coordinate system q. This condition was originally determined by Robertson [52] in a formal manner. In more recent times the study of separation of variables has advanced significantly both from the point of view of intrinsic characterization as well as classification of the various different kinds of separation that are possible on spaces of constant curvature. With regard to the latter problem it is in a sense true that "all" orthogonal separable systems on real or complex spaces of constant curvature are limiting cases of the original elliptic coordinates found by Jacobi.

Jacobi's discovery of elliptic coordinates, followed much later by the development of quantum mechanics, led to the interest in second-order superintegrable systems. In both classical mechanics and in its quantum extension there are some special mechanical systems on Riemannian manifolds, expressed as kinetic energy terms plus a potential, that can be solved via separation of variables in more than one coordinate system. Such multiseparable systems are not only integrable, they are multiply integrable and much additional information about the systems can be obtained by interrelating the separate separable solutions. These systems have a theory rich in structure .

Although our definition of second-order superintegrability does not mention multiseparability, we see that, for important classes of superintegrable systems, multiseparability is implied. We start by studying an important example of a superintegrable system in two-dimensional Euclidean space, with separation in elliptical coordinates, that illustrates the typical features of superintegrable systems. In the remainder of this paper we lay out the essentials of a structure and classification theory for all these systems in two-dimensional Riemannian spaces and important results for three-dimensional conformally flat spaces. These results are very recent and the extensive details of the proofs will appear elsewhere.

A classical superintegrable system

$$\mathcal{H} = \sum_{ij} g^{ij} p_i p_j + V(\mathbf{x}) \tag{1.6}$$

on an n-dimensional local Riemannian manifold is one that admits $2n-1$ functionally independent symmetries (constants of the motion) $\mathcal{S}_k$, $k = 1, \cdots, 2n-1$, where we choose $\mathcal{S}_1 = \mathcal{H}$ for convenience [59], that is, $\{\mathcal{H}, \mathcal{S}_k\} = 0$, where

$$\{f, g\} = \sum_{j=1}^{n} (\partial_{x_j} f \partial_{p_j} g - \partial_{p_j} f \partial_{x_j} g) \tag{1.7}$$

is the Poisson Bracket for functions $f(\mathbf{x}, \mathbf{p}), g(\mathbf{x}, \mathbf{p})$ on phase space [17, 13, 14, 15, 58, 44]. Comments:

1. We refer to these functions as symmetries because each leads to a conserved quantity for the associated physical system.

2. In general we assume that our local Riemannian manifolds are complex, e.g., the complex sphere. The various real Riemannian manfolds are restrictions of the complex case, e.g., the real sphere and real hyperbolic spaces.

Note that $2n-1$ is the maximum possible number of functionally independent symmetries and, locally, such symmetries always exist. The main interest is in symmetries that are polynomials in the p_k and are globally defined, except for lower-dimensional singularities such as poles and branch points. Many tools in the theory of Hamiltonian systems have been brought to bear on superintegrable systems such as R-matrix theory, Lax pairs, exact solvability and quasi-exact solvability, [53, 16, 56, 20, 42]. However, the most detailed and complete results are obtained from the methods of separation of variables in those cases for which they are applicable. Standard orthogonal separation of variables techniques are associated with second-order symmetries, e.g., [10, 11, 48, 31, 49, 24, 50, 2, 3, 8] and multiseparable Hamiltonian systems provide numerous examples of superintegrability. Here we concentrate on second-order superintegrable systems, that is those in which the symmetries take the form $\mathcal{S} = \sum a^{ij}(\mathbf{x})p_i p_j + W(\mathbf{x})$, i.e. they are quadratic in the momenta. Note: Many authors require that superintegrable systems be also integrable, i.e., they admit n linearly independent commuting symmetries. We do not require this because we prove that a second-order superintegrable system is necessarily integrable.

There is an analogous definition for second-order quantum superintegrable systems with the Schrödinger operator

$$H = \Delta + V(\mathbf{x}), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij})\partial_{x_j}, \tag{1.8}$$

where Δ is the Laplace-Beltrami operator on a complex Riemannian manifold, expressed in local coordinates x_j [11]. Here there are $2n-1$ second-order symmetry operators, $S_k = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} a^{ij}_{(k)})\partial_{x_j} + W_{(k)}$, $k = 1, \cdots, 2n-1$, with $S_1 = H$ and $[H, S_k] \equiv HS_k - S_k H = 0$. Again multiseparable systems yield many examples of superintegrability. There is also a quantization problem in extending the results for classical systems to operator systems. This problem turns out to be very easily solved in two dimensions and not difficult in higher dimensions for so-called nondegenerate potentials.

To illustrate the main features of superintegrable systems we give a simple example in Euclidean space. Consider the Schrödinger eigenvalue equation $H\Psi = E\Psi$ or [19, 33]

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi + \frac{1}{2}\left(\omega^2(x^2+y^2) + \frac{k_1^2 - \frac{1}{4}}{x^2} + \frac{k_2^2 - \frac{1}{4}}{y^2}\right)\Psi = E\Psi. \tag{1.9}$$

This equation admits multiplicatively separable solutions in three systems: **Cartesian** coordinates (x, y); **polar** coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and **elliptical** coordinates,

$$x^2 = c^2 \frac{(u_1 - e_1)(u_2 - e_1)}{(e_1 - e_2)}, \qquad y^2 = c^2 \frac{(u_1 - e_2)(u_2 - e_2)}{(e_2 - e_1)}.$$

The bound state energy levels are degenerate with energies $E_n = \omega(2n + 2 + k_1 + k_2)$ for integer n. The corresponding wave functions are

1. **Cartesian**:

$$\Psi_{n_1,n_2}(x,y) = 2\omega^{\frac{1}{2}(k_1+k_2+2)}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}}\,x^{(k_1+\frac{1}{2})}y^{(k_2+\frac{1}{2})} \quad (1.10)$$

$$\times e^{-\frac{\omega}{2}(x^2+y^2)}L_{n_1}^{k_1}(\omega x^2)L_{n_2}^{k_2}(\omega y^2), \qquad n = n_1+n_2,$$

and the $L_n^k(x)$ are Laguerre polynomials [12].

2. **Polar**:

$$\Psi(r,\theta) = \Phi_q^{(k_1,k_2)}(\theta)\omega^{\frac{1}{2}(2q+k_1+k_2+1)}\sqrt{\frac{2m!}{\Gamma(m+2q+k_1+k_2+1)}} \quad (1.11)$$

$$\times e^{(-\omega r^2/2)}r^{(2q+k_1+k_2+1)}L_m^{2q+k_1+k_2+1}(\omega r^2), \quad n = m+q,$$

$$\Phi_q^{(k_1,k_2)}(\theta) = \sqrt{2(2q+k_1+k_2+1)\frac{q!\Gamma(k_1+k_2+q+1)}{\Gamma(k_2+q+1)\Gamma(k_1+q+1)}}$$

$$\times(\cos\theta)^{k_1+(1/2)}(\sin\theta)^{k_2+(1/2)}P_q^{(k_1,k_2)}(\cos 2\theta)$$

and the $P_q^{(k_1,k_2)}(\cos 2\theta)$ are Jacobi polynomials [12] .

3. **Elliptical**:

$$\Psi = e^{-\omega(x^2+y^2)}x^{k_1+\frac{1}{2}}y^{k_2+\frac{1}{2}}\prod_{m=1}^{n}\left(\frac{x^2}{\theta_m-e_1}+\frac{y^2}{\theta_m-e_2}-c^2\right), \quad (1.12)$$

where

$$\frac{x^2}{\theta-e_1}+\frac{y^2}{\theta-e_2}-c^2 = -c^2\frac{(u_1-\theta)(u_2-\theta)}{(\theta-e_1)(\theta-e_2)}.$$

These are ellipsoidal wave functions [57, 1].

A basis for the second-order symmetry operators is

$$S_1 = \partial_x^2 + \frac{(\frac{1}{4}-k_1^2)}{x^2} - \omega^2x^2, \qquad S_2 = \partial_y^2 + \frac{(\frac{1}{4}-k_2^2)}{y^2} - \omega^2y^2 \quad (1.13)$$

$$S_3 = (x\partial_y - y\partial_x)^2 + \left(\tfrac{1}{4}-k_1^2\right)\frac{y^2}{x^2} + \left(\tfrac{1}{4}-k_2^2\right)\frac{x^2}{y^2} - \tfrac{1}{2}.$$

(Note that $-2H = S_1+S_2$.) The separable solutions are eigenfunctions of the symmetry operators S_1, S_3 and $S_3 + e_2S_1 + e_1S_2$ with corresponding eigenvalues

$$\lambda_c = -\omega(2n_1+k_1+1), \qquad \lambda_p = (2q+k_1+k_2+1)^2 + (1+k_1^2+k_2^2),$$

$$\lambda_e = 2(1-k_1)(1-k_2) - 2e_2\omega(k_1+1) - 2e_1\omega(k_2+1) - \omega^2e_1e_2$$

$$-4\sum_{m=1}^{q}\left[e_2\frac{k_1+1}{\theta_m-e_1}+e_1\frac{k_2+1}{\theta_m-e_2}\right],$$

respectively.

The algebra constructed by repeated commutators is

$$[S_1,S_3]=[S_3,S_2]\equiv R,\quad [S_i,R]=4<S_i,S_j>+16\omega^2S_3,\quad i\neq j,\ i,j=1,2,\quad (1.14)$$
$$[S_3,R]=4<S_1,S_3>-4<S_2,S_3>+8(1-k_2^2)S_1-8(1-k_1^2)S_2,$$
$$R^2=\frac{8}{3}<S_1,S_2,S_3>+\frac{64}{3}<S_1,S_2>-16\omega^2S_3^2-16(1-k_2^2)S_1^2$$
$$-16(1-k_1^2)S_2^2-\frac{128}{3}\omega^2S_3-64\omega^2(1-k_1^2)(1-k_2^2).$$

Note that all except the last relation is at most quadratic in the generators. Here $<A,B>$ and $<A,B,C>$ denote full symmetrization over all products of the argument operators.

The classical algebra has basis

$$\mathcal{S}_1=p_x^2+\frac{\frac14-k_1^2}{x^2}-\omega^2x^2,\quad \mathcal{S}_2=p_y^2+\frac{\frac14-k_2^2}{y^2}-\omega^2y^2,\qquad (1.15)$$
$$\mathcal{S}_3=(xp_y-yp_x)^2+\left(\tfrac14-k_1^2\right)\frac{y^2}{x^2}+\left(\tfrac14-k_2^2\right)\frac{x^2}{y^2},\quad -2\mathcal{H}=\mathcal{S}_1+\mathcal{S}_2.$$

The classical quadratic algebra relations (with $\{\cdot,\cdot\}$ the Poisson Bracket) are

$$\{\mathcal{S}_1,\mathcal{S}_3\}=\{\mathcal{S}_3,\mathcal{S}_2\}\equiv\mathcal{R},\qquad \{\mathcal{S}_i,\mathcal{R}\}=8\mathcal{S}_i\mathcal{S}_j+16\omega^2\mathcal{S}_3,\qquad i\neq j,\ i,j=1,2,$$
$$\{\mathcal{S}_3,\mathcal{R}\}=8\mathcal{S}_1\mathcal{S}_3-8\mathcal{S}_2\mathcal{S}_3+(4-16k_2^2)\mathcal{S}_1-(4-16k_1^2)\mathcal{S}_2,\qquad (1.16)$$
$$\mathcal{R}^2=16\mathcal{S}_1\mathcal{S}_2\mathcal{S}_3-16\omega^2\mathcal{S}_3^2+(4-16k_2^2)\mathcal{S}_1^2-(4-16k_1^2)\mathcal{S}_2^2+4\omega^2(1-4k_1^2)(1-4k_2^2).$$

Note the following features.

- The algebra generated by S_1,S_2,S_3,R is **closed under commutation** [18, 51]. This is remarkable but typical of superintegrable systems with so-called nondegenerate potentials. Closure is at level six in the momenta since we have to express the square of the third-order operator R in terms of the S_j basis of second-order operators.

- The eigenfunctions of one separable system can be expanded in terms of the eigenfunctions of another and this is a source of nontrivial expansion theorems for special function [45, 46, 47, 32].

- The quadratic algebra identities allow us to relate eigenbases and eigenvalues of one symmetry operator to those of another. Indeed the representation theory of the abstract quadratic algebra can be used to derive spectral properties of the generators L_j in a manner analogous to the use of Lie algebra representation theory to derive spectral properties of quantum systems that admit Lie symmetry algebras [7, 5, 54].

- A common feature of quantum superintegrable systems is that after splitting off a multiplicative functional factor,

$$x^{(k_1+\frac12)}y^{(k_2+\frac12)}e^{-\frac{\omega}{2}(x^2+y^2)}$$

  in the example, the Schrödinger and symmetry operators are acting on a space of polynomials, [38]. There is a Hilbert space structure and the separation of variables yields bases of multivariable orthogonal polynomials.

- There is a close relationship to the theory of exactly and quasi-exactly solvable systems, [42]. In the example the one-dimensional ordinary differential equations obtained by separation in the Cartesian and polar systems are exactly solvable in terms of hypergeometric functions and the energy eigenvalues are easily obtained. The elliptic system separated equations are quasi-exactly solvable and polynomial solutions are obtained for only particular values of E. However, these values are just the energy eigenvalues obtained in the Cartesian and polar systems!

In the example the potential is nondegenerate, i.e., it depends on three arbitrary parameters (or four if we include the trivial constant that we can always add to a potential). In $n \geq 2$ dimensions the nondegenerate potentials depend on $n + 2$ parameters. Systems with nondegenerate potentials have the most beautiful properties, but there are also superintegrable systems with degenerate potentials depending on fewer than $n + 2$ parameters. For $n = 2$ we show that all of these depending on at least one nonadditive parameter are in a certain sense specializations of the nondegenerate systems. For degenerate systems first-order symmetries may exist. Note that in the classical case the symmetries corresponding to a constant potential are just Killing tensors.

Many examples of such systems are known and lists of possible systems have been determined for real and complex spaces of constant curvature in two and three dimensions as well as a few other spaces, [19, 36, 30, 29, 51, 39]. Here rather than focus on particular spaces and systems we employ a theoretical method based on integrability conditions to derive structure common to all such systems. We firstly consider classical superintegrable systems on a general two-dimensional Riemannian manifold, real or complex, and uncover their common structure. We show that for superintegrable systems with nondegenerate potentials there exists a standard structure based on the algebra of 2×2 symmetric matrices, that such systems are necessarily multiseparable and that the quadratic algebra closes at level six. This is all done without making use of lists of such systems so that generalization to higher dimensions [39], where relatively few examples are known, is much easier.

Then we study the Stäckel transform, or coupling constant metamorphosis [6, 22], for two-dimensional classical superintegrable systems. This is a conformal transformation of a superintegrable system on one space to a superintegrable system on another space. We prove that all nondegenerate two-dimensional superintegrable systems are Stäckel transforms of constant curvature systems and give a complete classification of all two-dimensional superintegrable systems. We discuss briefly how to extend these results to three-dimensional systems and the quantum analogs of two-dimensional and three-dimensional classical systems.

2 Maximal dimensions of the spaces of polynomial constants in two dimensions

In the following, without loss of generality, we can assume that we have a basic set of coordinates $(x, y) \equiv (x_1, x_2)$ with respect to which the Hamiltonian takes the diagonal form $\mathcal{H} = (p_1^2 + p_2^2)/\lambda(x_1, x_2) + V(x_1, x_2)$. Thus the metric is $ds^2 = \lambda(dx_1^2 + dx_2^2)$. From the example in the preceding section we see that it is important to compute the dimensions of the spaces of symmetries of superintegrable systems that are of orders two, three, four and six. As illustrated by the example these symmetries are necessarily of a special type.

- The highest order terms in the momenta are independent of the parameters in the potential.
- The terms of order two less in the momenta are linear in these parameters, those of order four less are quadratic and those of order six less are cubic.

The system is **second-order order superintegrable** with **nondegenerate potential** if

1. it admits three functionally independent second-order symmetries and
2. the potential has three parameters (in addition to the usual additive parameter).

$$V(x,y) = \alpha_1 V^{(1)}(x,y) + \alpha_2 V^{(2)}(x,y) + \alpha_3 V^{(3)}(x,y), \tag{2.1}$$

that is, at each point where the potenial is defined and analytic we can prescribe the values of the derivatives V_x, V_y and V_{xx} arbitrarily. Nondegenerate potentials exhibit the most structure and one can show that superintegrable systems with potentials depending on one or two parameters are special cases or limits of three-parameter systems. The following result is proved using the integrabilty conditions for the requirement that a symmetry $\mathcal{S}$ of a nondegenerate superintegrable system must satisfy the condition, $\{\mathcal{H},\mathcal{S}\} = 0$, and the restrictions on the parameters listed above.

Theorem 1. *Let $\mathcal{H}$ be the Hamiltonian of a two-dimensional superintegrable system with nondegenerate potential.*

- *The space of second-order constants of the motion is exactly three-dimensional.*
- *The space of third-order constants of the motion is at most one-dimensional.*
- *The space of fourth-order constants of the motion is at most six-dimensional.*
- *The space of sixth-order constants is at most ten-dimensional.*

An ordered pair of complex numbers $\mathbf{x}_0 = (x_0, y_0)$ is a **regular point** for a superintegrable system if the potential is defined and analytic and the three basis symmetries are functionally independent in a neighborhood of $\mathbf{x}_0$.

Corollary 1. *The quadratic terms $a^{ij} = a^{ji}$ of a second-order symmetry*

$$\mathcal{S} = \sum a^{ij}(\mathbf{x}) p_i p_j + W(\mathbf{x})$$

are uniquely determined by their values $a^{ij}(\mathbf{x}_0)$ at a regular point $\mathbf{x}_0$.

By assumption every two-dimensional superintegrable system admits three functionally independent second-order symmetries. Our strategy is to choose a basis of three second-order symmetries and show that the second- and third-order polynomials in these basis elements form a basis for the fourth- and sixth-order symmetries, reaching the maximum dimensions given in the theorem. This implies closure of the quadratic algebra. Of course third-order symmetries cannot be expressed in terms of polynomials of second-order symmetries and we have to study this case separately. Again the result is obtained through a careful study of integrability conditions for the symmetry.

Theorem 2. *Let $\mathcal{K}$ be a third-order constant of the motion for a superintegrable system with nondegenerate potential V:*

$$\mathcal{K} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_k p_j p_i + \sum_{\ell=1}^{2} b^{\ell}(x,y)p_\ell. \tag{2.2}$$

Then

$$b^{\ell}(x,y) = \sum_{j=1}^{2} f^{\ell,j}(x,y)\frac{\partial V}{\partial x_j}(x,y)$$

with

$$f^{\ell,j} + f^{j,\ell} = 0, \quad 1 \le \ell, j \le 2,$$

and the a^{ijk} and b^{ℓ} are uniquely determined by the number

$$f^{1,2}(x_0,y_0)$$

at some regular point (x_0,y_0).

Let

$$\mathcal{S}_1 = \sum a^{kj}_{(1)} p_k p_j + W_{(1)}, \quad \mathcal{S}_2 = \sum a^{kj}_{(2)} p_k p_j + W_{(2)}$$

be second-order constants of the the motion for a superintegrable system with nondegenerate potential and let $\mathcal{A}_{(i)}(x,y) = \{a^{kj}_{(i)}(x,y)\}$, $i = 1,2$, be 2×2 matrix functions. Then the Poisson Bracket of these symmetries is given by

$$\{\mathcal{S}_1, \mathcal{S}_2\} = \sum_{k,j,i=1}^{2} a^{kji}(x,y)p_k p_j p_i + b^{\ell}(x,y)p_\ell, \tag{2.3}$$

where

$$f^{k,\ell} = 2\lambda \sum_j (a^{kj}_{(2)} a^{j\ell}_{(1)} - a^{kj}_{(1)} a^{j\ell}_{(2)}).$$

Thus $\{\mathcal{S}_1, \mathcal{S}_2\}$ is uniquely determined by the skew-symmetric matrix

$$[\mathcal{A}_{(2)}, \mathcal{A}_{(1)}] \equiv \mathcal{A}_{(2)}\mathcal{A}_{(1)} - \mathcal{A}_{(1)}\mathcal{A}_{(2)}, \tag{2.4}$$

hence by the constant matrix $[\mathcal{A}_{(2)}(x_0,y_0), \mathcal{A}_{(1)}(x_0,y_0)]$ evaluated at a regular point.

Corollary 2. *Let V be a superintegrable nondegenerate potential. Then the space of third-order constants of the motion is one-dimensional and is spanned by Poisson Brackets of the second-order constants of the motion.*

Corollary 3. *Let V be a superintegrable nondegenerate potential and $\mathcal{S}_1$ and $\mathcal{S}_2$ be second-order constants of the motion with matrices $\mathcal{A}_{(1)}$ and $\mathcal{A}_{(2)}$, respectively. Then*

$$\{\mathcal{S}_1, \mathcal{S}_2\} \equiv 0 \Longleftrightarrow [\mathcal{A}_{(1)}, \mathcal{A}_{(2)}] \equiv 0 \Longleftrightarrow [\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)] = 0 \tag{2.5}$$

at a regular point $\mathbf{x}_0$.

2.1 A standard form for two-dimensional superintegrable systems

For superintegrable nondegenerate potentials we see that there is a standard structure that allows the identification of the space of second-order constants of the motion with the space of 2×2 symmetric matrices and allows identification of the space of third-order constants of the motion with the space of 2×2 skew-symmetric matrices. Indeed,

- if $\mathbf{x}_0$ is a regular point, then there is a one-to-one linear correspondence between second-order operators S and their associated symmetric matrices $\mathcal{A}_(\mathbf{x}_0)$. Let $\{\mathcal{S}_1, \mathcal{S}_2\}' = \{\mathcal{S}_2, \mathcal{S}_1\}$ be the reversed Poisson Bracket. The map

  $$\{\mathcal{S}_1, \mathcal{S}_2\}' \Longleftrightarrow [\mathcal{A}_{(1)}(\mathbf{x}_0), \mathcal{A}_{(2)}(\mathbf{x}_0)] \tag{2.6}$$

  is an algebraic isomorphism.

- Let $\mathcal{E}^{ij}$ be the 2×2 matrix with a 1 in row i, column j and 0 for every other matrix element. Then the symmetric matrices

  $$\mathcal{A}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} + \mathcal{E}^{ji}) = \mathcal{A}^{(ji)}, \quad i, j = 1, 2, \tag{2.7}$$

  form a basis for the three-dimensional space of symmetric matrices.

- Moreover

  $$[\mathcal{A}^{(ij)}, \mathcal{A}^{(k\ell)}] = \frac{1}{2}\left(\delta_{jk}\mathcal{B}^{(i\ell)} + \delta_{j\ell}\mathcal{B}^{(ik)} + \delta_{ik}\mathcal{B}^{(j\ell)} + \delta_{i\ell}\mathcal{B}^{(jk)}\right), \tag{2.8}$$

  where

  $$\mathcal{B}^{(ij)} = \frac{1}{2}(\mathcal{E}^{ij} - \mathcal{E}^{ji}) = -\mathcal{B}^{(ji)}, \quad i, j = 1, 2.$$

  Here $\mathcal{B}^{(ii)} = 0$ and $\mathcal{B}^{(12)}$ forms a basis for the space of skew-symmetric matrices. Thus (2.8) gives the commutation relations for the second-order symmetries.

- We define a standard set of basis symmetries $\mathcal{S}_{(jk)} = \sum_{ih} a^{ih}_{(jk)}(\mathbf{x})p_i p_h + W_{(jk)}(\mathbf{x})$ corresponding to a regular point $\mathbf{x}_0$ by

  $$\begin{pmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \end{pmatrix}_{\mathbf{x}_0} = \lambda(\mathbf{x}_0)\begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}_{\mathbf{x}_0} = \lambda(\mathbf{x}_0)\mathcal{A}^{(jk)}, \quad W_{(jk)}(\mathbf{x}_0) = 0. \tag{2.9}$$

  Note that the symmetry $\mathcal{S}_{(jk)}$ restricts to $p_j p_k$ at the regular point $\mathbf{x}_0$, for fixed j and k. The condition on $W_{(jk)}$ is actually three conditions since $W_{(jk)}$ depends upon three parameters.

2.2 Multiseparability of two-dimensional systems

Necessary and sufficient conditions for variables to separate in the Hamilton-Jacobi equation for a classical system are well-known, e.g., [49, 50]. They require a second-order symmetry $\mathcal{S}$ as well as an algebraic condition on the matrix of $\mathcal{S}$. However, for superintegrable systems with nondegenerate potential the conditions simplify. Recall that a point $\mathbf{x}_0$ is a **regular point** for our superintegrable system if the potential $V(\mathbf{x})$ is defined and analytic in a neigborhood of this point and if the basis of symmetries is also functionally independent at the point.

Theorem 3. *Let V be a superintegrable nondegenerate potential and $\mathcal{S}$ be a second-order constant of the motion with matrix function, $\mathcal{A}(\mathbf{x})$. If at some regular point, $\mathbf{x}_0$, the matrix $\mathcal{A}(\mathbf{x}_0)$ has two distinct eigenvalues, then $\mathcal{H}$ and $\mathcal{S}$ characterize an orthogonal separable coordinate system.*

Note: Since a generic 2×2 symmetric matrix has distinct roots, it follows that any superintegrable nondegenerate potential is multiseparable.

2.3 The quadratic algebra

Theorem 4. *The six distinct monomials,*

$$(\mathcal{S}_{(11)})^2,\ (\mathcal{S}_{(22)})^2,\ (\mathcal{S}_{(12)})^2,\ \mathcal{S}_{(11)}\mathcal{S}_{(22)},\ \mathcal{S}_{(11)}\mathcal{S}_{(12)},\ \mathcal{S}_{(12)}\mathcal{S}_{(22)},$$

form a basis for the space of fourth-order symmetries. The ten distinct monomials,

$$(\mathcal{S}_{(ii)})^3,\ (\mathcal{S}_{(ij)})^3,\ (\mathcal{S}_{(ii)})^2\mathcal{S}_{(jj)},\ (\mathcal{S}_{(ii)})^2\mathcal{S}_{(ij)},\ (\mathcal{S}_{(ij)})^2\mathcal{S}_{(ii)},\ \mathcal{S}_{(11)}\mathcal{S}_{(12)}\mathcal{S}_{(22)},$$

$i, j = 1, 2,\ i \neq j$, *form a basis for the space of sixth-order symmetries.*

These theorems are proved by computing the values and first derivatives of the symmetries at a regular point to verify linear independence of the monomials. Since the number of monomials listed is the same as the maximum possible dimension of the space of symmetries, they must form a basis. Note that by use of the standard form for symmetries one can explicitly expand any fourth- or sixth-order symmetry in terms of the standard basis.

3 The Stäckel transform for two-dimensional systems

The Stäckel transform [6] or coupling constant metamorphosis [22] plays a fundamental role in relating superintegrable systems on different manifolds. Suppose we have a superintegrable system

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(x_1, x_2)} + V(x_1, x_2) \tag{3.1}$$

in local orthogonal coordinates with nondegenerate potential $V(x, y)$. This four-parameter family is uniquely characterized by a system of partial differential equations of the form

$$\begin{array}{rclcl} V_{22} & = & V_{11} & + & A^{22}V_1 + B^{22}V_2, \\ V_{12} & = & & & A^{12}V_1 + B^{12}V_2. \end{array} \tag{3.2}$$

Indeed these equations are straightforward consequences of the integrability conditions for a a basis of second-order symmetries and the requirement that the derivatives V_1, V_2 and V_{11} can be prescribed arbitrarily at any regular point. Now suppose that $U(x_1, x_2)$ is a particular solution of equations (3.2) which is nonzero in an open set. Then the transformed system

$$\tilde{\mathcal{H}} = \frac{p_1^2 + p_2^2}{\tilde{\lambda}(x_1, x_2)} + \tilde{V}(x_1, x_2) \tag{3.3}$$

with nondegenerate potential $\tilde{V}(x_1, x_2)$:

$$\begin{array}{rcl} \tilde{V}_{22} &=& \tilde{V}_{11} + \tilde{A}^{22}\tilde{V}_1 + \tilde{B}^{22}\tilde{V}_2, \\ \tilde{V}_{12} &=& \tilde{A}^{12}\tilde{V}_1 + \tilde{B}^{12}\tilde{V}_2 \end{array} \tag{3.4}$$

is also superintegrable, where

$$\tilde{\lambda} = \lambda U, \quad \tilde{V} = \frac{V}{U},$$

$$\tilde{A}^{12} = A^{12} - \frac{U_2}{U}, \ \tilde{A}^{22} = A^{22} + 2\frac{U_1}{U}, \ \tilde{B}^{12} = B^{12} - \frac{U_1}{U}, \ \tilde{B}^{22} = B^{22} - 2\frac{U_2}{U}.$$

Let $\mathcal{S} = \sum a^{ij} p_i p_j + W = S_0 + W$ be a (parameter-dependent) second-order symmetry of $\mathcal{H}$ and $\mathcal{S}_U = \sum a^{ij} p_i p_j + W_U = \mathcal{S}_0 + W_U$ be the special case of this (fixing of parameters) that is in involution with $\lambda^{-1}(p_1^2 + p_2^2) + U$. Then

$$\tilde{\mathcal{S}} = \mathcal{S}_0 - \frac{W_U}{U}\mathcal{H} + \frac{1}{U}\mathcal{H}$$

is the corresponding symmetry of $\tilde{\mathcal{H}}$. Since one can always add a constant to a nondegenerate potential, it follows that $1/U$ defines an inverse Stäckel transform of $\tilde{\mathcal{H}}$ to $\mathcal{H}$. See [6, 29] for many examples of this transform. We say that two superintegrable systems are **Stäckel equivalent** if one can be obtained from the other by a Stäckel transform.

If λ is the metric of a space that admits a nondegenerate superintegrable system, then it is always possible to choose coordinates x, y such that $\lambda_{12} = 0$ [41]. In [25] we prove the following basic result.

Theorem 5. *If $ds^2 = \lambda(dx^2 + dy^2)$ is the metric of a nondegenerate superintegrable system (expressed in coordinates (x, y) such that $\lambda_{12} = 0$), then $\lambda = \mu$ is a solution of the system*

$$\mu_{12} = 0, \qquad \mu_{22} - \mu_{11} = 3\mu_1 \left(\ln a^{12}\right)_1 - 3\mu_2 \left(\ln a^{12}\right)_2 + \left(\frac{a^{12}_{11} - a^{12}_{22}}{a^{12}}\right)\mu, \tag{3.5}$$

where either

$$I) \quad a^{12} = X(x)Y(y), \quad X'' = \alpha^2 X, \quad Y'' = -\alpha^2 Y \tag{3.6}$$

or

$$II) \quad a^{12} = \frac{2X'(x)Y'(y)}{C(X(x) + Y(y))^2}, \tag{3.7}$$

$$(X')^2 = F(X), \quad X'' = \frac{1}{2}F'(X), \quad (Y')^2 = G(Y), \quad Y'' = \frac{1}{2}G'(Y)$$

and

$$F(X) = \frac{\alpha}{24}X^4 + \frac{\gamma_1}{6}X^3 + \frac{\gamma_2}{2}X^2 + \gamma_3 X + \gamma_4, \tag{3.8}$$

$$G(Y) = -\frac{\alpha}{24}Y^4 + \frac{\gamma_1}{6}Y^3 - \frac{\gamma_2}{2}Y^2 + \gamma_3 Y - \gamma_4. \tag{3.9}$$

Conversely every solution λ of one of these systems defines a nondegenerate superintegrable system. If λ be a solution, then the remaining solutions, μ, are exactly the nondegenerate superintegrable systems that are Stäckel equivalent to λ.

Corollary 4. *Every nondegenerate superintegrable (real or complex) two-dimensional system is Stäckel equivalent to a nondegenerate superintegrable system on a space of constant curvature.*

The nondegenerate superintegrable potentials on complex two-dimensional spaces of constant curvature have already been classified [34, 35].

There is an extensive literature on what amount to superintegrable systems with constant (essentially zero) potential. In this case the second-order symmetries are called Killing tensors [40, 21]. In a tour de force Koenigs [41] classified all two-dimensional manifolds with only isolated singularites that admit exactly three second-order Killing tensors and listed them in two tables: Tableau VI and Tableau VII from his original paper. Note: Koenigs wrote his metrics in the Langrangian form $ds^2 = \lambda dz\, d\bar{z} = \lambda(dx^2 - dy^2)$ where $z = x + iy$, whereas we are using the Riemannian form for the metric. Since all of the variables are complex, it is trivial to transform from one form to the other.

TABLEAU VI

$$[1]\ ds^2 = \left[\frac{c_1 \cos x + c_2}{\sin^2 x} + \frac{c_3 \cos y + c_4}{\sin^2 y}\right](dx^2 - dy^2)$$

$$[2]\ ds^2 = \left[\frac{c_1 \cosh x + c_2}{\sinh^2 x} + \frac{c_3 e^y + c_4}{e^{2y}}\right](dx^2 - dy^2)$$

$$[3]\ ds^2 = \left[\frac{c_1 e^x + c_2}{e^{2x}} + \frac{c_3 e^y + c_4}{e^{2y}}\right](dx^2 - dy^2)$$

$$[4]\ ds^2 = \left[c_1(x^2 - y^2) + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4\right](dx^2 - dy^2)$$

$$[5]\ ds^2 = \left[c_1(x^2 - y^2) + \frac{c_2}{x^2} + c_3 y + c_4\right](dx^2 - dy^2)$$

$$[6]\ ds^2 = \left[c_1(x^2 - y^2) + c_2 x + c_3 y + c_4\right](dx^2 - dy^2)$$

TABLEAU VII

$$
\begin{aligned}
[1]\ ds^2 &= \left[c_1\left(\frac{1}{\mathrm{sn}^2(x,k)}-\frac{1}{\mathrm{sn}^2(y,k)}\right)+c_2\left(\frac{1}{\mathrm{cn}^2(x,k)}-\frac{1}{\mathrm{cn}^2(y,k)}\right)\right.\\
&+\left. c_3\left(\frac{1}{\mathrm{dn}^2(x,k)}-\frac{1}{\mathrm{dn}^2(y,k)}\right)+c_4\left(\mathrm{sn}^2(x,k)-\mathrm{sn}^2(y,k)\right)\right](dx^2-dy^2)\\
[2]\ ds^2 &= \left[c_1\left(\frac{1}{\sin^2 x}-\frac{1}{\sin^2 y}\right)+c_2\left(\frac{1}{\cos^2 x}-\frac{1}{\cos^2 y}\right)+c_3\left(\cos 2x-\cos 2y\right)\right.\\
&+\left. c_4\left(\cos 4x-\cos 4y\right)\right](dx^2-dy^2)\\
[3]\ ds^2 &= [c_1(\sin 4x-\sin 4y)+c_2(\cos 4x-\cos 4y)+c_3(\sin 2x-\sin 2y)\\
&+ c_4(\cos 2x-\cos 2y)]\,(dx^2-dy^2)\\
[4]\ ds^2 &= \left[c_1\left(\frac{1}{x^2}-\frac{1}{y^2}\right)+c_2(x^2-y^2)+c_3(x^4-y^4)+c_4(x^6-y^6)\right](dx^2-dy^2)\\
[5]\ ds^2 &= \left[c_1(x-y)+c_2(x^2-y^2)+c_3(x^3-y^3)+c_4(x^4-y^4)\right](dx^2-dy^2)
\end{aligned}
$$

Our theorem above shows easily that these are exactly the spaces that admit superintegrable systems with nondegenerate potentials. (We do not list the potentials here due to space requirements. One space may correspond to several distinct superintegrable systems.) Our derivation is very straightforward and simpler than that of Koenigs. From our point of view Koenigs' impressive contribution shows that every two-dimensional manifold that admits three second-order Killing tensors also admits at least one nondegenerate potential.

4 Nondegenerate quantum superintegrable systems in two dimensions

Now we consider the operator version of superintegrable systems. For a manifold with metric $ds^2 = \lambda(x, y)(dx^2 + dy^2)$ the Hamiltonian system

$$\mathcal{H} = \frac{p_1^2 + p_2^2}{\lambda(x, y)} + V(x, y)$$

is replaced by the Hamiltonian (Schrödinger) operator with potential

$$H = \frac{1}{\lambda(x, y)}(\partial_{11} + \partial_{22}) + V(x, y). \tag{4.1}$$

A second-order symmetry of the Hamiltonian system

$$\mathcal{S} = \sum_{k,j=1}^{2} a^{kj}(x, y)p_k p_j + W(x, y),$$

with $a^{kj} = a^{jk}$, corresponds to the operator

$$S = \frac{1}{\lambda(x,y)} \sum_{k,j=1}^{2} \partial_k (a^{kj}(x,y)\lambda(x,y)\partial_j) + W(x,y), \quad a^{kj} = a^{jk}. \tag{4.2}$$

Lemma 1.

$$\{\mathcal{H}, \mathcal{S}\} = 0 \iff [H, S] = 0.$$

(This lemma is not generally true for higher-dimensional manifolds, where the quantization problem requires a modification of the potential.) It follows from Lemma 1 that the classical results for the space of second-order symmetries corresponding to a nondegenerate potential can be adopted without change. Thus the maximal dimensions of the spaces of formally self-adjoint symmetry operators of orders two, three, four and six are the same as for the classical case. Also we can construct a basis of second-order symmetry operators $S_{(ij)}$ in the neighborhood of a regular point $\mathbf{x}_0$ in exact analogy with the classical symmetries $\mathcal{S}_{(ij)}$.

Recall that the fully symmetrized quadratic and cubic products of linear operators A, B and C are denoted $< A, B >$ and $< A, B, C >$, respectively.

Theorem 6. *The six distinct monomials,*

$$< S_{(11)}, S_{(11)} >, \; < S_{(22)}, S_{(22)} >, \; < S_{(12)}, S_{(12)} >,$$

$$< S_{(11)}, S_{(22)} >, \; < S_{(11)}, S_{(12)} >, \; < S_{(12)} S_{(22)} >,$$

form a basis for the space of fourth-order symmetry operators.

Theorem 7. *The ten distinct monomials,*

$$< S_{(ii)}, S_{(ii)}, S_{(ii)} >, \; < S_{(ij)}, S_{(ij)}, S_{(ij)} > < S_{(ii)}, S_{(ii)}, S_{(jj)} >, \; < S_{(ii)}, S_{(ii)}, S_{(ij)} >,$$

$$< S_{(ij)}, S_{(ij)}, S_{(ii)} >, \; < S_{(11)}, S_{(12)}, S_{(22)} >,$$

for $i, j = 1, 2, \; i \neq j$, form a basis for the space of sixth-order symmetries.

These theorems establish the closure of the quadratic algebra for two-dimensional quantum superintegrable potentials: All fourth-order and sixth-order symmetry operators can be expressed as symmetric polynomials in the second-order symmetry operators.

4.1 The Stäckel transform for two-dimensional quantum systems

The quantum analog of the Stäckel transform or coupling constant metamorphosis for Hamilton-Jacobi systems is straightforward in the two-dimensional case. Suppose that we have a superintegrable system

$$H = \frac{1}{\lambda(x,y)}(\partial_{11} + \partial_{22}) + V(x,y) = H_0 + V \tag{4.3}$$

in local orthogonal coordinates with nondegenerate potential $V(x,y)$ and suppose that $U(x,y)$ is a particular case of the three-parameter potential V, nonzero in an open set. Then the transformed system

$$\tilde{H} = \frac{1}{\tilde{\lambda}(x,y)}(\partial_{11} + \partial_{22}) + \tilde{V}(x,y) \tag{4.4}$$

is also superintegrable, where

$$\tilde{\lambda} = \lambda U, \quad \tilde{V} = \frac{V}{U}.$$

Theorem 8. *1.*

$$[\tilde{H}, \tilde{S}] = 0 \iff [H, S] = 0.$$

2.

$$\tilde{S} = \sum_{ij} \frac{1}{\lambda U}\partial_i \left[\left(a^{ij} + \delta^{ij}\frac{1 - W_U}{\lambda U})\lambda U \right) \right] \partial_j + \left(W - \frac{W_U V}{U} + \frac{V}{U} \right).$$

Corollary 5. *If $S_{(1)}$ and $S_{(2)}$ are second-order symmetry operators for H, then*

$$[\tilde{S}_{(1)}, \tilde{S}_{(2)}] = 0 \iff [S_{(1)}, S_{(2)}] = 0.$$

Theorem 9. *Every nondegenerate second-order quantum superintegrable system in two variables (real or complex) is Stäckel equivalent to a superintegrable system on a space of constant curvature.*

5 Conclusions and further results

In this paper we have described the classification of all two-dimensional superintegrable systems with nondegenerate potential. (In [26, 25] the details of the proofs are given and the results are extended to systems with degenerate potentials.) We have shown that all these systems are Stäckel equivalent to superintegrable systems on spaces of constant curvature, the potentials of which have already been classified in detail [36, 30, 29]. We have proved the closure of the quadratic algebra and have shown in principle how to compute the structure of the algebra in individual cases.

The integrability condition approach of §2 that works for superintegrable systems on two-dimensional complex Riemannian manifolds extends to three-dimensional complex conformally flat spaces (2n-1=5 functionally independent constants of the motion) with some complications. In two dimensions the quadratic form a^{ij} has three independent components and there are three functionally independent second-order symmetries. Thus the value of the quadratic form at any regular point can be prescribed and this uniquely defines a symmetry. For $n = 3$ there are five functionally independent second-order symmetries, but the quadratic form a^{ij} has six independent components. This is a major complication. In [27] we overcome this problem by proving a $5 \Longrightarrow 6$ Theorem, that is, five functionally independent second-order symmetries for a nondegenerate superintegrable three-dimensional system imply six linearly independent second-order symmetries. Then we demonstrate that for three-dimensional conformally flat superintegrable systems with nondegenerate potential the maximum possible dimensions of the spaces of second-, third-, fourth- and sixth-order symmetries are six, four, 26 and 56, respectively, and these dimensions are achieved. Again the three-dimensional quadratic algebra generated by the second-order symmetries always closes at level six and there is a standard structure for the algebra.

The passage from the three-dimensional conformally flat classical superintegrable systems to quantum superintegrable systems is still straightforward, but requires modifying the quantum potential by an additive term proportional to the scalar curvature [28]. Work is in progress to determine all three-dimensional superintegrable systems.

Jacobi's contribution remains central to this program. Indeed **all** orthogonal separable coordinates for n dimensional superintegrable systems on conformally flat manifolds are generalized Jacobi elliptic coordinates and their limiting cases [37].

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