

# A Lie Symmetry Connection between Jacobi's Modular Differential Equation and Schwarzian Differential Equation

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## Abstract

In [18] Jacobi introduced a third-order nonlinear ordinary differential equation which links two different moduli of an elliptic integral. In the present paper Lie group analysis is applied to that equation named Jacobi's modular differential equation. A six-dimensional Lie symmetry algebra is obtained and its symmetry generators are found to be given in terms of elliptic integrals. As a consequence the transformation between Jacobi's modular differential equation and the well-known Schwarzian differential equation is derived.

## 1 Introduction

In 1829 Carl Gustav Jacob Jacobi (1804-1851) published *Fundamenta Nova Theoriae Functionum Ellipticarum* through the Borntrager Brothers in Königsberg. In this book [18] – which we have translated into Italian with the addition of explanatory notes and comments [27] – he presents part of his studies on the elliptic functions that he began in 1826.

The theory of elliptic functions originated with Adrien-Marie Legendre (1752-1838) as was acknowledged by Jacobi in § 1 of *Fundamenta Nova*. Legendre demonstrated that elliptic integrals can be expressed by elementary functions and also reduced to three different kinds. Actually Legendre called them elliptic functions and in a letter dated August 19, 1829, to Legendre [17] Jacobi proposed naming them elliptic integrals. The complete elliptic integral of first kind is defined as

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad (1.1)$$

while the complementary complete elliptic integral of first kind is

$$K'(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(1-k^2)x^2)}}. \quad (1.2)$$

The quantity  $k$  is called the modulus and  $k' = \sqrt{1 - k^2}$  is the complementary modulus. A different modulus, say  $\lambda$ , generates completely different elliptic integrals of the first kind, i.e.:

$$\Lambda(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda^2 x^2)}} \quad \text{and} \quad (1.3)$$

$$\Lambda'(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(1-\lambda^2)x^2)}}, \quad (1.4)$$

respectively. There exist many systems of different moduli which reduce to each other. These systems are as many as the prime numbers, namely infinite [21].

Count Giulio Carlo (di) Fagnano (1682-1766) made the discovery that the integral that yields the length of the arc of a lemniscate has properties similar to those of the integral that yields the arc of a circle [10]. Leonhard Euler (1707-1783) generalized Fagnano's result and found that the sum of two complete elliptic integrals of first kind is an integral and its upper limit depends algebraically on the upper limits of the addends. This is the famous addition theorem of Euler [9].

Niels Henrik Abel (1802-1829) was studying elliptic functions during the same time as Jacobi [1], although Abel could dedicate to it just few years before his untimely death while Jacobi was able to continue his own studies for another twenty years. They both followed two main ideas: to extend the consideration of the new analytic entities to complex values and to replace the study of the elliptic integrals with the study of the elliptic functions which derive from the inverse of the elliptic integrals of first kind. Thus Abel and Jacobi are recognized jointly for developing the theory of the elliptic functions in their current form [21]. A "noble competition" between these two young men may have developed [21], but subsequent polemics about the unquestionable priority of Abel and the alleged speculation that Jacobi may have been unfair and not cognizant of Abel's original contribution ([28], [29]) may be countered if one recalls that on March 14, 1829, nearly a month before Abel's death, Jacobi wrote to Legendre [16]

"Quelle découverte de M. Abel que cette généralisation de l'intégrale d'Euler!  
A-t-on jamais vu pareille chose! Mais comment s'est-il fait que cette découverte,  
peut-être la plus importante de ce qu'a fait dans les mathématiques le siècle  
dans lequel nous vivons, étant communiquée à votre académie il y a deux  
ans, elle a pu échapper à l'attention de vous et de vos confrères?"<sup>1</sup>

Actually the first mathematician who studied elliptic functions as inverses of elliptic integrals was Johann Carl Friedrich Gauss (1777-1855) in 1809 [12]. This was established by Schering who posthumously edited Gauss' papers.

It is well-known that Sophus Lie (1842-1899) was influenced by Abel's and especially Galois' work on algebraic equations as well as Jacobi's work on partial differential equations [13], [14]. Less well-known<sup>2</sup> may be his involvement with the republication of Abel's work

<sup>1</sup>What a discovery that generalization of Euler's integral – namely, Euler's addition theorem – by Mr. Abel! Nothing similar has ever been seen before! How could it happen that this discovery, perhaps the most important that has been made in mathematics during the century in which we live, and communicated to your academy more than two years ago, has escaped your own and your fellow-academicians' attention?

<sup>2</sup>Even lesser known is that Lie's wife was a relative of Abel, her maternal grandfather being Abel's uncle [28].

[2] and unfruitful search for Abel's original manuscript [29] that Jacobi prized so much and went unnoticed and was subsequently lost at the Paris Academy<sup>3</sup>.

In this paper we apply Lie group analysis to Jacobi's modular differential equation, which he derived for linking two different moduli of an elliptic integral [18], and obtain a six-dimensional Lie symmetry algebra isomorphic to  $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$  which transforms it into the well-known Schwarzian differential equation [15]. To our knowledge this result has not been previously reported. Also it is noteworthy that the symmetry generators are in terms of elliptic integrals. This is the first instance of the appearance of special functions in the Lie symmetry algebra admitted by a nonlinear equation.

In order to honor Jacobi's bicentennial, in Section 2 we present the English translation of the two relevant paragraphs of *Fundamenta Nova*, namely §32 and §33, in which Jacobi's modular differential equation is obtained. Note that our comments to *Fundamenta Nova* are in boldface within the text while the sentences in italics are Jacobi's. We have also corrected any obvious misprints not all of which were listed in the Corrigenda of *Fundamenta Nova*. In [18] few formulas are numbered. Therefore we have successively numbered those addressed in our comments. Our numbering scheme follows the style of the current journal, consecutively numbering subordinate to the section number and appearing at the right of each formula, e.g. (2.1). We have also kept Jacobi's original consecutive numbering which ignores the section number, e.g., 1), and appears to the left. The two footnotes in Section 2 are Jacobi's and these are in their original notation.

In Section 3 we apply Lie group analysis to Jacobi's modular differential equation and obtain the transformation to the Schwarzian differential equation. In Section 4 some final remarks are made.

## 2 Two paragraphs from *Fundamenta Nova*

### 32.

However, among the properties of modular equations, it seems to me that it is important to point out and recall that *they all satisfy the same third-order differential equation*.

This research has to be carried a bit further.

It is quite known that \*) , assuming  $aK + bK' = Q$ , then it will be:

$$k(1 - k^2) \frac{d^2 Q}{dk^2} + (1 - 3k^2) \frac{dQ}{dk} = kQ,$$

where  $a$ ,  $b$  are two arbitrary constants.

**Using Maple V is easy to show that the substitution  $Q = aK + bK'$  identically satisfies this second-order ordinary differential equation.**

Thus assuming moreover  $a'K + b'K' = Q'$ , where  $a'$ ,  $b'$  are arbitrary constants, then it will be:

$$k(1 - k^2) \frac{d^2 Q'}{dk^2} + (1 - 3k^2) \frac{dQ'}{dk} = kQ'.$$

<sup>3</sup>It was found in Guglielmo Libri's library, although some pages are still missing [8].

\*) Cf. Legendre *Traité des F. E.* Tom. I. Cap. XIII.

By combining these equations one obtains:

$$k(1-k^2) \left\{ Q \frac{d^2 Q'}{dk^2} - Q' \frac{d^2 Q}{dk^2} \right\} + (1-3k^2) \left\{ Q \frac{dQ'}{dk} - Q' \frac{dQ}{dk} \right\} = 0$$

and after integrating:

$$k(1-k^2) \left\{ Q \frac{dQ'}{dk} - Q' \frac{dQ}{dk} \right\} = (ab' - a'b)k(1-k^2) \left\{ K \frac{dK'}{dk} - K' \frac{dK}{dk} \right\} = (ab' - a'b)C. \quad (2.1)$$

In a particular case the constant  $C$  has been found by Legendre to be  $= -\frac{1}{2}\pi$ , hence

$$\left\{ Q \frac{dQ'}{dk} - Q' \frac{dQ}{dk} \right\} = \frac{-\frac{1}{2}\pi(ab' - a'b)}{k(1-k^2)}, \quad \text{or :}$$

$$d \frac{Q'}{Q} = \frac{-\frac{1}{2}\pi(ab' - a'b)dk}{k(1-k^2)QQ}.$$

**If we put  $a = 1, b = 0, a' = 0, b' = 1, k = 1/\sqrt{2}$  into (2.1), then it is easy to obtain  $C = -\pi/2$ , as we have verified with Maple V.**

In the same way, being  $\lambda$  any other modulus, and assuming  $\alpha\Lambda + \beta\Lambda' = L$  and  $\alpha'\Lambda + \beta'\Lambda' = L'$ , one will have:

$$d \frac{L'}{L} = \frac{-\frac{1}{2}\pi(\alpha\beta' - \alpha'\beta)d\lambda}{\lambda(1-\lambda^2)LL}.$$

Let  $\lambda$  be the modulus into which  $k$  is transformed by the first transformation of  $n^{\text{th}}$ -order. Moreover let  $Q = K, Q' = K', L = \Lambda, L' = \Lambda'$ ; one will have

$$\frac{L'}{L} = \frac{\Lambda'}{\Lambda} = \frac{nK'}{K} = \frac{nQ'}{Q}.$$

**At § 25 Jacobi has already introduced the formula  $\frac{\Lambda'}{\Lambda} = \frac{nK'}{K}$ .**

It follows that:

$$\frac{ndk}{k(1-k^2)KK} = \frac{d\lambda}{\lambda(1-\lambda^2)\Lambda\Lambda}. \quad (2.2)$$

**The relationships  $Q = K$  and  $Q' = K'$  imply that  $a = 1, b = 0, a' = 0, b' = 1$ , while from  $L = \Lambda$  and  $L' = \Lambda'$ , it follows  $\alpha = 1, \beta = 0, \alpha' = 0, \beta' = 1$ . Thus**

$$ab' - a'b = 1 \quad \text{and} \quad \alpha\beta' - \alpha'\beta = 1.$$

**Finally:**

$$\frac{L'}{L} = \frac{nQ'}{Q} \implies d \frac{L'}{L} = n d \frac{Q'}{Q} \implies \frac{ndk}{k(1-k^2)KK} = \frac{d\lambda}{\lambda(1-\lambda^2)\Lambda\Lambda}.$$

For the first transformation we have also found that  $\Lambda = \frac{K}{nM}$  – at the end of § 24 – from which:

$$MM = \frac{1}{n} \cdot \frac{\lambda(1 - \lambda^2)dk}{k(1 - k^2)d\lambda}.$$

**This last equation is obtained from (2.2) by means of the substitution**

$$\Lambda = \frac{K}{nM}.$$

From the second transformation we have seen that  $\frac{\Lambda'_1}{\Lambda_1} = \frac{1}{n} \frac{K'}{K}$ ,  $\Lambda_1 = \frac{K}{M_1}$ , – at the end of § 25 – hence :

$$\frac{dk}{k(1 - k^2)KK} = \frac{nd\lambda_1}{\lambda_1(1 - \lambda_1^2)\Lambda_1\Lambda_1},$$

and then as before

$$M_1M_1 = \frac{1}{n} \cdot \frac{\lambda_1(1 - \lambda_1^2)dk}{k(1 - k^2)d\lambda_1}.$$

**The value  $M_1M_1$  is obtained in the same way as  $MM$ , the only difference being that in this case,  $L = \Lambda_1$  and  $L' = \Lambda'_1$ .**

In general whatever the modulus  $\lambda$  is, real or imaginary, to which the proposed modulus  $k$  can be transformed by a transformation of  $n^{th}$ -order, one will have the equation:

$$MM = \frac{1}{n} \cdot \frac{k(1 - k^2)d\lambda}{\lambda(1 - \lambda^2)dk}.$$

In order to prove it, I will write that one generally obtains equations of the form:

$$\alpha\Lambda + i\beta\Lambda' = \frac{aK + ibK'}{nM}$$

$$\alpha'\Lambda' + i\beta'\Lambda = \frac{a'K' + ib'K}{nM},$$

where  $a, a', \alpha, \alpha'$ , are odd numbers,  $b, b', \beta, \beta'$ , even numbers, and they are all either positive or negative such that  $aa' + bb' = 1$  and  $\alpha\alpha' + \beta\beta' = 1$  \*).

**Jacobi does not explain these two equations. In [6] one reads that the existence of these equations can be deduced without difficulty from the general formulas of the transformations, but the actual proof would depend on investigations that go beyond the scope of Fundamenta Nova.**

\*) A more accurate determination of the numbers  $a, a', b, b'$ , for all the transformations of the same order seems very difficult to obtain and, if we are not mistaken, this determination depends on the limits which the modulus  $k$  moves in between such that for different limits we have other solutions. The intricacy of this is well-known to an expert in the area. It is clear that the first step is to research thoroughly the nature of the imaginary modulus, which is still an open question.

Hence, assuming

$$\begin{aligned} aK + ibK' &= Q, & a'K' + ib'K &= Q' \\ \alpha\Lambda + i\beta\Lambda' &= L, & \alpha'\Lambda' + i\beta'\Lambda &= L', \end{aligned} \quad (2.3)$$

and, since  $aa' + bb' = 1$ ,  $\alpha\alpha' + \beta\beta' = 1$ , we obtain that

$$d\frac{Q'}{Q} = \frac{-n\pi dk}{2k(1-k^2)QQ}, \quad d\frac{L'}{L} = \frac{-\pi d\lambda}{2\lambda(1-\lambda^2)LL}.$$

We have seen that

$$\text{if } Q = \tilde{a}K + \tilde{b}K', \quad Q' = \tilde{a}'K + \tilde{b}'K' \implies \tilde{a}\tilde{b}' - \tilde{a}'\tilde{b} = 1$$

and

$$\text{if } L = \tilde{\alpha}\Lambda + \tilde{\beta}\Lambda', \quad L' = \tilde{\alpha}'\Lambda + \tilde{\beta}'\Lambda' \implies \tilde{\alpha}\tilde{\beta}' - \tilde{\alpha}'\tilde{\beta} = 1.$$

From (2.3) it follows that

$$\tilde{a} = a, \quad \tilde{b} = ib, \quad \tilde{a}' = ib', \quad \tilde{b}' = a'$$

and

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = i\beta, \quad \tilde{\alpha}' = i\beta', \quad \tilde{\beta}' = \alpha'$$

hence

$$1 = \tilde{a}\tilde{b}' - \tilde{a}'\tilde{b} = aa' - ib(ib') = aa' + bb'$$

and

$$1 = \tilde{\alpha}\tilde{\beta}' - \tilde{\alpha}'\tilde{\beta} = \alpha\alpha' - i\beta(i\beta') = \alpha\alpha' + \beta\beta'.$$

Thus, being  $\frac{Q'}{Q} = \frac{L'}{L}$  and  $L = \frac{Q}{nM}$ , one generally obtains:

$$MM = \frac{1}{n} \cdot \frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

I will again notice that the equation can be also put in the following form:

$$MM = \frac{1}{n} \cdot \frac{\lambda^2(1-\lambda^2)d(k^2)}{k^2(1-k^2)d(\lambda^2)} = \frac{1}{n} \cdot \frac{\lambda'^2(1-\lambda'^2)d(k'^2)}{k'^2(1-k'^2)d(\lambda'^2)}. \quad (2.4)$$

In fact

$$\frac{1}{n} \cdot \frac{\lambda^2(1-\lambda^2)d(k^2)}{k^2(1-k^2)d(\lambda^2)} = \frac{1}{n} \cdot \frac{\lambda^2(1-\lambda^2)2k dk}{k^2(1-k^2)2\lambda d\lambda}$$

so by simplifying

$$\frac{1}{n} \cdot \frac{\lambda^2(1-\lambda^2)d(k^2)}{k^2(1-k^2)d(\lambda^2)} = \frac{1}{n} \cdot \frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

The second equality of (2.4) is obtained by the relationship existing between the modulus and its complement that is:  $k^2 + k'^2 = 1$ .

Therefore the expression  $MM$  does not change when the moduli  $k$  and  $\lambda$  are replaced with their complements  $k'$  and  $\lambda'$ , which is what we have demonstrated above by the complementary transformations: the multiplier  $M$  is the same apart the sign. Moreover changing  $k$  with  $\lambda$  and  $\lambda$  with  $k$ , namely applying the supplementary transformation,  $MM$  becomes

$$\frac{1}{n} \cdot \frac{k(1-k^2)d\lambda}{\lambda(1-\lambda^2)dk} = \frac{1}{nnMM}, \quad \text{and hence } M \text{ becomes } \frac{1}{nM},$$

this is what we have proved above.

### 33.

**In the first part of this paragraph Jacobi shows that the multiplier  $M$ , when considered as a function of  $k$ , satisfies the second-order differential equation 6), while in the second part Jacobi obtains the third-order differential equation 8) which is satisfied by the moduli  $\lambda$  and  $k$ .**

Assuming  $Q = aK + bK'$ ,  $L = \alpha\Lambda + \beta\Lambda'$ , the constants  $a$ ,  $b$ ,  $\alpha$ ,  $\beta$ , can be always determined in such a way that  $L = \frac{Q}{M}$ ,  $Q = ML$ . From it one obtains the equations:

$$\begin{aligned} 1) \quad & (k - k^3) \frac{d^2 Q}{dk^2} + (1 - 3k^2) \frac{dQ}{dk} - kQ = 0 \\ 2) \quad & (\lambda - \lambda^3) \frac{d^2 L}{d\lambda^2} + (1 - 3\lambda^2) \frac{dL}{d\lambda} - \lambda L = 0, \end{aligned}$$

which can be also represented as follows:

$$\begin{aligned} 3) \quad & d \frac{\frac{(k-k^3)dQ}{dk}}{dk} - kQ = 0 \\ 4) \quad & d \frac{\frac{(\lambda-\lambda^3)dL}{d\lambda}}{d\lambda} - \lambda L = 0. \end{aligned}$$

By replacing  $Q = ML$  in the equation:

$$(k - k^3) \frac{d^2 Q}{dk^2} + (1 - 3k^2) \frac{dQ}{dk} - kQ = 0,$$

one obtains:

$$L \left\{ (k - k^3) \frac{d^2 M}{dk^2} + (1 - 3k^2) \frac{dM}{dk} - kM \right\} + \frac{dL}{dk} \left\{ 2(k - k^3) \frac{dM}{dk} + (1 - 3k^2)M \right\} + (k - k^3)M \frac{d^2 L}{dk^2} = 0,$$

which when multiplied by  $M$  can be written:

$$5) \quad LM \left\{ (k - k^3) \frac{d^2 M}{dk^2} + (1 - 3k^2) \frac{dM}{dk} - kM \right\} + d \frac{\frac{(k-k^3)M^2 dL}{dk}}{dk} = 0.$$

**One obtains equation 5) by the following procedure. Replacing  $Q = ML$  into equation 1):**

$$(k - k^3) \frac{d^2 (ML)}{dk^2} + (1 - 3k^2) \frac{d(ML)}{dk} - kML = 0$$

and applying the derivation formula of a product yields

$$(k - k^3) \frac{d}{dk} \left( \frac{dM}{dk} L + \frac{dL}{dk} M \right) + (1 - 3k^2) \left( \frac{dM}{dk} L + \frac{dL}{dk} M \right) - kML = 0$$

$$(k - k^3) \left( \frac{d^2 M}{dk^2} L + \frac{dM}{dk} \frac{dL}{dk} + \frac{d^2 L}{dk^2} M + \frac{dL}{dk} \frac{dM}{dk} \right) + (1 - 3k^2) \left( \frac{dM}{dk} L + \frac{dL}{dk} M \right) - kML = 0;$$

then collecting the terms with  $L$  and  $\frac{dL}{dk}$  gives

$$L \left\{ (k - k^3) \frac{d^2 M}{dk^2} + (1 - 3k^2) \frac{dM}{dk} - kM \right\} + \frac{dL}{dk} \left\{ 2(k - k^3) \frac{dM}{dk} + (1 - 3k^2) M \right\} \\ + (k - k^3) M \frac{d^2 L}{dk^2} = 0.$$

If one multiplies by  $M$

$$LM \left\{ (k - k^3) \frac{d^2 M}{dk^2} + (1 - 3k^2) \frac{dM}{dk} - kM \right\} + M \frac{dL}{dk} \left\{ 2(k - k^3) \frac{dM}{dk} + (1 - 3k^2) M \right\} \\ + (k - k^3) M^2 \frac{d^2 L}{dk^2} = 0,$$

and notices that

$$M \frac{dL}{dk} \left\{ 2(k - k^3) \frac{dM}{dk} + (1 - 3k^2) M \right\} + (k - k^3) M^2 \frac{d^2 L}{dk^2} \\ = (k - k^3) M^2 \frac{d^2 L}{dk^2} + 2M(k - k^3) \frac{dM}{dk} \frac{dL}{dk} + (1 - 3k^2) M^2 \frac{dL}{dk} \\ = \frac{d}{dk} \left( (k - k^3) M^2 \frac{dL}{dk} \right),$$

then equation 5) results.

However, from the previous paragraph one has:

$$M^2 = \frac{(\lambda - \lambda^3) dk}{n(k - k^3) d\lambda}, \quad \text{hence} \quad \frac{(k - k^3) M^2 dL}{dk} = \frac{(\lambda - \lambda^3) dL}{nd\lambda}.$$

Moreover from equation 4) one has:

$$d \left\{ \frac{(\lambda - \lambda^3) dL}{d\lambda} \right\} = \lambda L d\lambda, \quad \text{hence} \\ d \frac{\frac{(k - k^3) M^2 dL}{dk}}{dk} = d \frac{\frac{(\lambda - \lambda^3) dL}{d\lambda}}{ndk} = \frac{\lambda L d\lambda}{ndk}.$$

Hence equation 5) after a division by  $L$  transforms into:

$$6) \quad M \left\{ (k - k^3) \frac{d^2 M}{dk^2} + (1 - 3k^2) \frac{dM}{dk} - kM \right\} + \frac{\lambda d\lambda}{ndk} = 0.$$

We observe with Cayley [6] that this equation depends on the order of the transformation since it contains  $n$ .



When we replace  $M$  in this equation with the value obtained from the equation

$M^2 = \frac{(\lambda - \lambda^3)dk}{n(k - k^3)d\lambda}$ , then we obtain a differential equation between the moduli  $k, \lambda$  themselves, an equation which is clearly of third order. A little complicated calculation yields:

$$7) \quad \frac{3d^2\lambda^2}{dk^4} - \frac{2d\lambda}{dk} \cdot \frac{d^3\lambda}{dk^3} + \frac{d\lambda^2}{dk^2} \left\{ \left( \frac{1+k^2}{k-k^3} \right)^2 - \left( \frac{1+\lambda^2}{\lambda-\lambda^3} \right)^2 \cdot \frac{d\lambda^2}{dk^2} \right\} = 0.$$

**We have verified equation 7) in the following way. From  $M^2 = \frac{(\lambda - \lambda^3)dk}{n(k - k^3)d\lambda}$**

**we have obtained  $\frac{1}{M^2} = \frac{n(k - k^3)d\lambda}{(\lambda - \lambda^3)dk} = e_1$  and consequently**

$$\frac{dM}{dk} = -\frac{M^3}{2} \frac{de_1}{dk} \quad \frac{d^2M}{dk^2} = -\frac{3}{2} M^2 \frac{dM}{dk} \frac{de_1}{dk} - \frac{M^3}{2} \frac{d^2e_1}{dk^2}.$$

**Substitution of those values into 6) and use of REDUCE 3.7 give rise to the differential equation 7).**

In this equation  $dk$  is a constant differential.

**This means that  $k$  is the independent variable [6]. We rewrite equation 7) in modern notation as**

$$3 \left( \frac{d^2\lambda}{dk^2} \right)^2 - 2 \frac{d\lambda}{dk} \frac{d^3\lambda}{dk^3} + \left( \frac{d\lambda}{dk} \right)^2 \left\{ \left( \frac{1+k^2}{k-k^3} \right)^2 - \left( \frac{1+\lambda^2}{\lambda-\lambda^3} \right)^2 \left( \frac{d\lambda}{dk} \right)^2 \right\} = 0. \quad (2.5)$$

If one wishes to transform 7) into another equation with no constant differentials, then one has to assume:

$$\frac{d^2\lambda}{dk^2} = \frac{d^2\lambda}{dk^2} - \frac{d\lambda d^2k}{dk^3}$$

$$\frac{d^3\lambda}{dk^3} = \frac{d^3\lambda}{dk^3} - 3 \frac{d^2\lambda d^2k}{dk^4} - \frac{d\lambda d^3k}{dk^4} + 3 \frac{d\lambda d^2k^2}{dk^5}$$

hence:

$$\frac{3d^2\lambda^2}{dk^4} - \frac{2d\lambda}{dk} \cdot \frac{d^3\lambda}{dk^3} = \frac{3d^2\lambda^2}{dk^4} - \frac{3d\lambda^2 d^2k^2}{dk^6} + \frac{2d\lambda^2 d^3k}{dk^5} - \frac{2d\lambda d^3\lambda}{dk^4}.$$

Thus equation 7) when multiplied by  $dk^6$  can be transformed into the following equation, where no differential is assumed constant, and such that any differential can be considered constant:

$$8) \quad 3\{dk^2 d^2\lambda^2 - d\lambda^2 d^2k^2\} - 2dkd\lambda\{dkd^3\lambda - d\lambda d^3k\} +$$

$$+ dk^2 d\lambda^2 \left\{ \left( \frac{1+k^2}{k-k^3} \right)^2 dk^2 - \left( \frac{1+\lambda^2}{\lambda-\lambda^3} \right)^2 d\lambda^2 \right\} = 0.$$

It seems clear that this equation remains the same if we mutually replaced the elements  $k$  and  $\lambda$ : this we have proved above for the modular equations.

Now it is necessary to study this third-order differential equation with another method. For this purpose we introduce the equation we started with, namely

$$(k - k^3) \frac{d^2 Q}{dk^2} + (1 - 3k^2) \frac{dQ}{dk} - kQ = 0,$$

and the quantity  $(k - k^3)QQ = s$ . We have:

$$\begin{aligned} \frac{ds}{dk} &= (1 - 3k^2)QQ + 2(k - k^3)Q \frac{dQ}{dk} \\ \frac{d^2 s}{dk^2} &= -6kQQ + 4(1 - 3k^2)Q \frac{dQ}{dk} + 2(k - k^3) \left( \frac{dQ}{dk} \right)^2 + 2(k - k^3)Q \frac{d^2 Q}{dk^2}. \end{aligned}$$

**In the Latin text  $4(1 - 3k^2)Q \frac{dQ}{dk}$  is written as  $4(1 - 3k^2)QdQ$ , a misprint which is not listed in the Corrigenda.**

If in this equation one considers that

$$\begin{aligned} (k - k^3) \frac{d^2 Q}{dk^2} &= kQ - (1 - 3k^2) \frac{dQ}{dk}, \quad \text{then one gets :} \\ \frac{d^2 s}{dk^2} &= -4kQQ + 2(1 - 3k^2)Q \frac{dQ}{dk} + 2(k - k^3) \left( \frac{dQ}{dk} \right)^2 \\ &= 2 \frac{dQ}{dk} \left\{ (1 - 3k^2)Q + (k - k^3) \frac{dQ}{dk} \right\} - 4kQQ. \end{aligned}$$

If it is multiplied by  $2s = 2(k - k^3)QQ$ , then it becomes:

$$\frac{2sd^2 s}{dk^2} = 2(k - k^3)Q \frac{dQ}{dk} \left\{ 2(1 - 3k^2)QQ + 2(k - k^3)Q \frac{dQ}{dk} \right\} - 8k^2(1 - k^2)Q^4,$$

or also being

$$\begin{aligned} 2(k - k^3)Q \frac{dQ}{dk} &= \frac{ds}{dk} - (1 - 3k^2)QQ, \\ 2(1 - 3k^2)QQ + 2(k - k^3)Q \frac{dQ}{dk} &= \frac{ds}{dk} + (1 - 3k^2)QQ, \end{aligned}$$

we will have:

$$\frac{2sd^2 s}{dk^2} = \left( \frac{ds}{dk} \right)^2 - (1 - 3k^2)^2 Q^4 - 8k^2(1 - k^2)Q^4 = \left( \frac{ds}{dk} \right)^2 - (1 + k^2)^2 Q^4, \quad \text{or :}$$

$$9) \quad \frac{2sd^2 s}{dk^2} - \left( \frac{ds}{dk} \right)^2 + \left( \frac{1 + k^2}{k - k^3} \right)^2 ss = 0.$$

**In fact, being  $s = Q^2(k - k^3)$ , one obtains  $Q^2 = \frac{s}{(k - k^3)}$ . Then**

$$(1 + k^2)^2 Q^4 = (1 + k^2)^2 \frac{s^2}{(k - k^3)^2} = \left( \frac{1 + k^2}{k - k^3} \right)^2 s^2.$$

Assuming  $a'K + b'K' = Q'$ ,  $\frac{Q'}{Q} = t$ , we have seen that  $\frac{dt}{dk} = \frac{m}{(k - k^3)QQ} = \frac{m}{s}$ , where  $m$  is a constant: hence  $s = \frac{mdk}{dt}$ . Let us transform equation 9) into another one in which  $dt$  is assumed constant. It will be  $\frac{ds}{dk} = \frac{md^2k}{dtdk}$ ,  $\frac{d^2s}{dk^2} = \frac{md^3k}{dtdk^2} - \frac{md^2k^2}{dtdk^3}$ ; by means of these substitutions from equation 9) derives:

$$\frac{2d^3k}{dt^2dk} - \frac{3d^2k^2}{dt^2dk^2} + \left(\frac{1+k^2}{k-k^3}\right)^2 \frac{dk^2}{dt^2} = 0, \quad \text{or also :} \quad (2.6)$$

$$10) \quad 2d^3kdk - 3d^2k^2 + \left(\frac{1+k^2}{k-k^3}\right)^2 dk^4 = 0;$$

where we have to derive with respect to  $t$  as one deduces from the equation.

**In order to understand better equation (2.6) it is enough to multiply it by  $\left(\frac{dk}{dt}\right)^2$ . Thus we obtain**

$$2\frac{d^3k}{dt^3} \frac{dk}{dt} - 3\left(\frac{d^2k}{dt^2}\right)^2 + \left(\frac{1+k^2}{k(1-k^2)}\right)^2 \left(\frac{dk}{dt}\right)^4 = 0. \quad (2.7)$$

Assuming  $\frac{\alpha'\Lambda + \beta'\Lambda'}{\alpha\Lambda + \beta\Lambda'} = \omega$ , - **In the same way it was assumed above that  $t = \frac{Q'}{Q}$ , now we have  $\omega = \frac{L'}{L}$ , in accordance with the notation used by Jacobi at the beginning of §32.** - each time that  $\lambda$  becomes a transformed modulus then the constants  $\alpha$ ,  $\beta$ ,  $\alpha'$ ,  $\beta'$ , could be determined in such a way that  $t = \omega$ ; and we similarly obtain:

$$11) \quad 2d^3\lambda d\lambda - 3d^2\lambda^2 + \left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2 d\lambda^4 = 0,$$

and in this equation one will have to derive with respect to  $\omega = t$ . Let us multiply equation 10) by  $d\lambda^2$ , equation 11) by  $dk^2$ , and subtracting 11) from 10) yields:

$$12) \quad 2dkd\lambda\{d\lambda d^3k - dk d^3\lambda\} - 3\{d\lambda^2 d^2k^2 - dk^2 d^2\lambda^2\} \\ + dk^2 d\lambda^2 \left\{ \left(\frac{1+k^2}{k-k^3}\right)^2 dk^2 - \left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2 d\lambda^2 \right\} = 0.$$

But this equation coincides with 8) where we know that any differential can be considered constant; thus having proven all of this with the assumption that  $dt$  is a constant differential, then it follows that it is true for any other differential considered as such.

Here it is then a third-order equation with infinite, although particular, algebraic solutions, namely the equations that we have called modular. However, the general integral depends on the elliptic functions; since  $t = \omega$ , or  $\frac{a'K + b'K'}{aK + bK'} = \frac{\alpha'\Lambda + \beta'\Lambda'}{\alpha\Lambda + \beta\Lambda'}$ , which can be represented with the following equations:

$$mK\Lambda + m'K'\Lambda' + m''K\Lambda' + m'''K'\Lambda = 0,$$

where  $m, m', m'', m'''$ , are arbitrary constants. We consider this integration would be a fertile field for further research.

We can enquire if the modular equations for the transformations of third and fifth order satisfy our third-order differential equation, as they should. In order to avoid too lengthy calculation, it is enough to demonstrate it in the case of the transformation of second order where  $\lambda = \frac{1 - k'}{1 + k'}$ .

Assuming  $dk'$  as constant one has:

$$\begin{aligned} \lambda &= \frac{1 - k'}{1 + k'} = -1 + \frac{2}{1 + k'} & kk + k'k' &= 1 \\ \frac{d\lambda}{dk'} &= \frac{-2}{(1 + k')^2} & \frac{dk}{dk'} &= \frac{-k'}{k} \\ \frac{d^2\lambda}{dk'^2} &= \frac{4}{(1 + k')^3} & \frac{d^2k}{dk'^2} &= \frac{-1}{k} - \frac{k'k'}{k^3} = \frac{-1}{k^3} \\ \frac{d^3\lambda}{dk'^3} &= \frac{-12}{(1 + k')^4} & \frac{d^3k}{dk'^3} &= \frac{-3k'}{k^5}. \end{aligned}$$

Hence:

$$\begin{aligned} \frac{dk^2d^2\lambda^2 - d\lambda^2d^2k^2}{dk'^6} &= \frac{16k'k'}{k^2(1 + k')^6} - \frac{4}{k^6(1 + k')^4} = \\ &= \frac{4\{4k^4k'^2 - (1 + k')^2\}}{k^6(1 + k')^6} = \frac{4\{4k'^2(1 - k')^2 - 1\}}{k^6(1 + k')^4}. \end{aligned}$$

The expression

$$\frac{dk^2d^2\lambda^2 - d\lambda^2d^2k^2}{dk'^6}$$

can also be written as:

$$\left(\frac{dk}{dk'}\right)^2 \left(\frac{d^2\lambda}{dk'^2}\right)^2 - \left(\frac{d\lambda}{dk'}\right)^2 \left(\frac{d^2k}{dk'^2}\right)^2.$$

The values of these four derivatives have been found above. Then, substituting these values and performing the appropriate calculations, one obtains the previous equalities.

Moreover one gets:

$$\begin{aligned} \frac{dkd^3\lambda - d\lambda d^3k}{dk'^4} &= \frac{12k'}{k(1 + k')^4} - \frac{6k'}{k^5(1 + k')^2} = \frac{6k'\{2(1 - k')^2 - 1\}}{k^5(1 + k')^2} \tag{2.8} \\ \frac{dkd\lambda\{dkd^3\lambda - d\lambda d^3k\}}{dk'^6} &= \frac{12k'^2\{2(1 - k')^2 - 1\}}{k^6(1 + k')^4} \end{aligned}$$

from which

$$\frac{3\{dk^2d^2\lambda^2 - d\lambda^2d^2k^2\} - 2dkd\lambda\{dkd^3\lambda - d\lambda d^3k\}}{dk'^6} = \frac{12(2k'^2 - 1)}{k^6(1 + k')^4}.$$

In the Latin text formula (2.8) is written as:

$$\frac{dkd^3\lambda - d\lambda d^3k}{dk'^4} = \frac{12k'}{k(1+k')^4} + \frac{6k'}{k^5(1+k')^2} = \frac{6k'\{2(1-k')^2 + 1\}}{k^5(1-k')^2}.$$

The last denominator on the right contains a misprinted sign which is listed in the Corrigenda. After suitable checking, we found two other misprinted signs.

Moreover one has

$$\left(\frac{1+k^2}{k-k^3}\right)^2 \frac{dk^2}{dk'^2} = \frac{(1+k^2)^2}{k^4k'^2}$$

$$\left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2 \frac{d\lambda^2}{dk'^2} = \frac{4}{(1+k')^4} \left\{\frac{1+k'}{1-k'}\right\}^2 \left\{\frac{1+k'^2}{2k'}\right\}^2 = \frac{(1+k'^2)^2}{k'^2k^4}$$

from which:

$$\left\{\frac{1+k^2}{k-k^3}\right\}^2 \frac{dk^2}{dk'^2} - \left\{\frac{1+\lambda^2}{\lambda-\lambda^3}\right\}^2 \frac{d\lambda^2}{dk'^2} = \frac{3(1-2k'^2)}{k^4k'^2} \quad (2.9)$$

$$\frac{dk^2d\lambda^2}{dk'^4} \left\{\left(\frac{1+k^2}{k-k^3}\right)^2 \frac{dk^2}{dk'^2} - \left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2 \frac{d\lambda^2}{dk'^2}\right\} = \frac{12(1-2k'^2)}{k^6(1+k')^4}. \quad (2.10)$$

In the Latin text (2.9) contains  $\frac{d\lambda^2}{d\lambda'^2}$  instead of  $\frac{d\lambda^2}{dk'^2}$ . The same misprint is in formula (2.10). Both of them are not listed in the Corrigenda.

Finally one obtains:

$$\frac{3\{dk^2d^2\lambda^2 - d\lambda^2d^2k^2\} - 2dkd\lambda\{dkd^3\lambda - d\lambda d^3k\}}{dk'^6}$$

$$+ \frac{dk^2d\lambda^2}{dk'^4} \left\{\left(\frac{1+k^2}{k-k^3}\right)^2 \frac{dk^2}{dk'^2} - \left(\frac{1+\lambda^2}{\lambda-\lambda^3}\right)^2 \frac{d\lambda^2}{dk'^2}\right\} = \frac{12(2k'^2-1)}{k^6(1+k')^4} + \frac{12(1-2k'^2)}{k^6(1+k'^4)} = 0.$$

If there was a method for finding algebraic solutions of a differential equation, then it would be possible to obtain the modular equation for the transformation of any order  $n$  by using only the differential equation given above.

In [6] Cayley adds “but, the mere verification being so difficult, it does not appear that anything can be done in this manner in regard to the modular equations.”

Apart from *Condorcet* I do not know any analyst who had paid adequate attention to such an hard matter.

### 3 Application of Lie group analysis

Using ad hoc interactive REDUCE programs ([22, 23]) we have applied Lie group analysis to the differential equation (2.5), which we call Jacobi's modular differential equation, and found that it admits a six-dimensional Lie symmetry algebra isomorphic to  $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$  generated by the following operators

$$\Gamma_1 = kk'^2 K^2(k) \partial_k \quad \Gamma_2 = kk'^2 K'^2(k) \partial_k \quad \Gamma_3 = kk'^2 K(k) K'(k) \partial_k \quad (3.1)$$

$$\Gamma_4 = \lambda \lambda'^2 \Lambda^2(\lambda) \partial_\lambda \quad \Gamma_5 = \lambda \lambda'^2 \Lambda'^2(\lambda) \partial_\lambda \quad \Gamma_6 = \lambda \lambda'^2 \Lambda(\lambda) \Lambda'(\lambda) \partial_\lambda \quad (3.2)$$

which come from solving two determining equations of the following type:

$$\frac{d^3 s}{dk^3} + \frac{1+k^2}{k^3(1-k^2)^3} \left( \frac{ds}{dk} k(1-k^4) + s(1-k^4-4k^2) \right) = 0 \quad (3.3)$$

with  $s$  a function of  $k$ . Therefore a change of variable of the following type

$$t = f(k) \quad (3.4)$$

$$y = g(\lambda) \quad (3.5)$$

transforms equation (2.5) into the third-order ordinary differential equation [11]:

$$y''' = \frac{3}{2} \frac{y''^2}{y'} \quad (3.6)$$

which admits a six-dimensional Lie symmetry algebra generated by the following operators

$$X_1 = \partial_t, \quad X_2 = t \partial_t, \quad X_3 = t^2 \partial_t, \quad (3.7)$$

$$X_4 = \partial_y, \quad X_5 = y \partial_y, \quad X_6 = y^2 \partial_y. \quad (3.8)$$

Equation (3.6) is called the Schwarzian differential equation because it can be written as

$$\{y, t\} = 0$$

where  $\{.,.\}$  stands for the Schwarzian derivative [15]. It is well-known [15] that the following change of dependent variable

$$v = \frac{y''}{y'}$$

transforms (3.6) into a Riccati equation

$$v' = \frac{v^2}{2}$$

and after few quadratures the general solution of equation (3.6) is given by

$$y = \frac{c_2}{t + c_1} + c_3 \quad (3.9)$$

with  $c_i$  ( $i = 1, 3$ ) arbitrary constants. In (2.5) one can exchange  $\lambda$  with  $k$  and still obtain the same equation, therefore function  $g$  in (3.5) coincides with function  $f$  in (3.4).

Using ad hoc interactive REDUCE programs [22, 23] we have applied Lie group analysis to the differential equation (2.7) and found that it admits a six-dimensional Lie symmetry algebra generated by the operators (3.7) and (3.1). This means that, when Jacobi transformed  $\lambda$  into  $t$ , he actually found the function  $f$ , namely the transformation (3.4) and (3.5), i.e.

$$t = \frac{K'}{K} \quad (3.10)$$

$$y = \frac{\Lambda'}{\Lambda}. \quad (3.11)$$

Finally substitution of (3.10)-(3.11) into (3.9) yields the general solution of Jacobi's modular differential equation (2.5), i.e.

$$\frac{\Lambda'}{\Lambda} = \frac{c_2 K}{K' + c_1 K} + c_3. \quad (3.12)$$

Lie proved that the dimension of the Lie algebra of contact symmetries for an equation of third order cannot exceed ten [20]. Using ad hoc interactive REDUCE programs [22, 23] it is easy to show that Jacobi's modular differential equation (2.5), i.e.

$$\{\lambda, k\} = -\frac{1}{2} \left( \frac{1 + \lambda^2}{\lambda(1 - \lambda^2)} \right)^2 \left( \frac{d\lambda}{dk} \right)^2 + \frac{1}{2} \left( \frac{1 + k^2}{k(1 - k^2)} \right)^2, \quad (3.13)$$

admits ten contact symmetries of the type

$$-\frac{\partial \Omega_j}{\partial \lambda} \partial_\lambda + \left( \Omega_j - \frac{dk}{d\lambda} \frac{\partial \Omega_j}{\partial \frac{dk}{d\lambda}} \right) \partial_k \quad (j = 1, 10) \quad (3.14)$$

with

$$\Omega_j = s_j(\lambda) \frac{dk}{d\lambda} + G_j(\lambda, k) \sqrt{\frac{dk}{d\lambda}} + \alpha_j(k), \quad (3.15)$$

and

$$\begin{aligned} s_n &= 0 \quad (n = 1, 2, 3, 4, 5, 6, 7), \quad s_8 = -\lambda(1 - \lambda^2)\Lambda^2(\lambda), \\ s_9 &= -\lambda(1 - \lambda^2)\Lambda'^2(\lambda), \quad s_{10} = -\lambda(1 - \lambda^2)\Lambda(\lambda)\Lambda'(\lambda), \\ G_1 &= \sqrt{\lambda k(1 - \lambda^2)(1 - k^2)}\Lambda(y)K(k), \quad G_2 = \sqrt{\lambda k(1 - \lambda^2)(1 - k^2)}\Lambda'(y)K(k), \\ G_3 &= \sqrt{\lambda k(1 - \lambda^2)(1 - k^2)}\Lambda(y)K'(k), \quad G_4 = \sqrt{\lambda k(1 - \lambda^2)(1 - k^2)}\Lambda'(y)K'(k), \\ G_m &= 0 \quad (m = 5, 6, 7, 8, 9, 10), \quad \alpha_p = 0 \quad (p = 1, 2, 3, 4, 8, 9, 10), \\ \alpha_5 &= -k(1 - k^2)K^2(k), \quad \alpha_6 = -k(1 - k^2)K'^2(k), \quad \alpha_7 = -k(1 - k^2)K(k)K'(k). \end{aligned}$$

## 4 Final remarks

To our knowledge this is the first report of special functions in the generators of Lie symmetries admitted by a nonlinear differential equation. Another example involving Bessel functions can be found in [24].

Equation (2.7) belongs to a class of equations which were studied by Chazy in his thesis [7] in order to find if they possess the Painlevé property. Equation (2.7), i.e.

$$\{k, t\} = -\frac{1}{2} \left( \frac{1+k^2}{k(1-k^2)} \right)^2 \left( \frac{dk}{dt} \right)^2, \quad (4.1)$$

is not to be confused with the modular elliptic equation of Hermite [5, 25], i.e.:

$$\{y, x\} = \frac{1}{2} \left( \frac{1-\lambda^2}{y^2} + \frac{1-\mu^2}{(1-y)^2} + \frac{1-\lambda^2-\mu^2+\nu^2}{y(1-y)} \right) \left( \frac{dy}{dx} \right)^2. \quad (4.2)$$

It seems that Cayley's remark about Jacobi's modular differential equation (2.5) was echoed in the mathematical literature because we were unable to find any reference to such a differential equation. Everyone who is involved with elliptic integrals deals with modular equations which are the algebraic solutions of Jacobi's modular differential equation. In [3] it was stated that Srinivasa Ramanujan (1887-1920) in his famous paper [26] used modular equations to calculate a couple of simple invariants but "it seems unlikely that Ramanujan used only modular equations". However, we notice that on page 302 in [26] Ramanujan writes

"If

$$n \frac{K'}{K} = \frac{L}{L'},$$

we have

$$\frac{ndk}{kk'^2 K^2} = \frac{dl}{l'l'^2 L^2}.$$

But, by means of the modular equation connecting  $k$  and  $l$ , we can express  $dk/dl$  as an algebraic function of  $k$ , a function moreover in which all coefficients which occur are algebraic numbers."

Did Ramanujan know how to find algebraic solutions of Jacobi's modular differential equation? In [4] it was established that he owned a copy of Cayley's book [6]. We hope to address this issue in a future paper.

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<sup>§</sup>The Latin original of the paper and S G Langton's English translation can be found at [www.eulerarchive.com](http://www.eulerarchive.com)

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