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# On Properties of Elliptic Jacobi Functions and Applications

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**Abstract**

In this paper we are interested in developments of the elliptic functions of Jacobi. In particular a trigonometric expansion of the classical theta functions introduced by the author (Algebraic methods and q-special functions, C.R.M. Proceedings and Lectures Notes, A.M.S., vol 22, Providence, 1999, 53-57) permits one to establish a differential system. This system is derived from the heat equation and is satisfied by their coefficients. Several applications may be deduced. Other types of expansions for the Jacobi elliptic functions as well as for the Zeta function are examined.

## 1 Introduction

We review briefly some known facts on Jacobi elliptic functions and theta and zeta functions for later use. (For details see, e.g. [1], [2].)
Let $\theta$ be the temperature at time $t$ at any point in a solid the conducting properties of which are uniform and isotropic. If $\rho$ be its density, $s$ its specific heat and $k$ its thermal conductivity, $\theta$ satisfies the heat equation :

$$\kappa \nabla^2 \theta = \frac{\partial \theta}{\partial t},$$

where $\kappa = k/(s\rho)$ is the diffusivity.
Let $Ouvw$ be a rectangular Cartesian frame. In the special case in which there is no variation of temperature in the $uw-$plane, the heat flow is everywhere parallel to the $v-$axis and the heat equation reduces to the form

$$\kappa \frac{\partial^2 y}{\partial v^2} = \frac{\partial y}{\partial t}, \tag{1.1}$$

where $y = \theta\,(v, t)$.

Consider the following boundary conditions

$$\theta\,(0, t) = \theta\,(1, t)\,, \quad \theta\,(v, 0) = \pi\delta\,(v - 1/2)\,, \quad 0 < v < 1,$$

where $\delta(v)$ is the Dirac delta function. Then the solution of the boundary value problem is given by

$$\theta(v,t) = 2\sum_{n\geq 0} (-1)^n e^{-(2n+1)^2\pi^2\kappa t} \sin((2n+1)\pi v). \tag{1.2}$$

Consider the change

$$\tau = 4i\pi\ \kappa t.$$

It follows that $\frac{1}{\kappa}\frac{\partial y}{\partial t} = 4i\pi\frac{\partial y}{\partial \tau}$ and Equation (1.1) becomes the partial differential equation

$$\frac{\partial^2 y}{\partial v^2} = 4i\pi\frac{\partial y}{\partial \tau}. \tag{1.3}$$

When we write $q = e^{i\pi\tau} = e^{-4\pi^2\kappa t}$, the solution (1.2) takes the form

$$\theta_1(v,\tau) = 2\sum_{n\geq 0} (-1)^n q^{\left(\frac{n+1}{2}\right)^2} \sin((2n+1)\pi v) \tag{1.4}$$

which is the first of the four theta functions of Jacobi.
When the precise value of $q$ is not important, we suppress the dependence upon $q$. If one changes the boundary conditions to

$$\frac{\partial\theta}{\partial v} = 0 \text{ on } v = 0, v = 1, \theta(v,0) = \pi\delta(v-1/2), \quad 0 < v < 1,$$

then the corresponding solution of the boundary value problem of the heat equation, (1.1), is given by

$$\theta_4(v) = \theta_4(v,\tau) = 1 + 2\sum_{n\geq 1} (-1)^n q^{n^2} \cos(2n\pi v). \tag{1.5}$$

The function, $\theta_1(v,\tau)$, is periodic with period 2. If we increment $v$ by 1/2, we obtain the second theta function

$$\theta_2(v) = \theta_2(v,\tau) = 2\sum_{n\geq 0} q^{\left(\frac{n+1}{2}\right)^2} \cos((2n+1)\pi v). \tag{1.6}$$

Similarly the increment of $v$ by 1/2 for $\theta_4(v,\tau)$ yields the third theta function

$$\theta_3(v) = \theta_3(v,\tau) = 1 + 2\sum_{n\geq 1} q^{n^2} \cos(2n\pi v). \tag{1.7}$$

It is known that the four theta functions, $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$, can be extended to complex values for $v$ and $q$ such that $|\ q\ | < 1$.
Note that Jacobi's fundamental work on the theory of elliptic functions was based on these four theta functions. His paper "Fundamenta nova theoria functionum ellipticarum" published in 1829, together with its later supplements, made fundamental contributions to the theory of elliptic functions.

We turn now to the Jacobi elliptic functions, $\mathrm{sn}u$, $\mathrm{cn}u$ and $\mathrm{dn}u$. They are defined as ratios of theta functions as

$$\mathrm{sn}u = \frac{\theta_3(0)\,\theta_1(v)}{\theta_2(0)\,\theta_4(v)}, \quad \mathrm{cn}u = \frac{\theta_4(0)\,\theta_2(v)}{\theta_2(0)\,\theta_4(v)}, \quad \mathrm{dn}u = \frac{\theta_4(0)\,\theta_3(v)}{\theta_3(0)\,\theta_4(v)}, \tag{1.8}$$

where $u = \theta_3^2(0)\,v$.
Define parameters $k$ and $k'$ by

$$k = \frac{\theta_2^2(0)}{\theta_3^2(0)}, \qquad k' = \frac{\theta_4^2(0)}{\theta_3^2(0)}.$$

They are called the modulus and the complementary modulus of the elliptic functions. When it is required to state the modulus explicitly, the elliptic functions of Jacobi are written as $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$.
Moreover as for the theta functions the three Jacobi elliptic functions are related. In particular they satisfy the following relations

$$\mathrm{sn}^2\, u + \mathrm{cn}^2\, u = 1, \quad \mathrm{dn}^2\, u + k^2\mathrm{sn}^2\, u = 1, \quad k^2\mathrm{cn}^2\, u + k'^2 = \mathrm{dn}^2\, u. \tag{1.9}$$

$$\mathrm{sn}'\, u = (\mathrm{cn}u)(\mathrm{dn}u)\,, \ \mathrm{cn}'\, u = -(\mathrm{sn}u)(\mathrm{dn}u)\,, \ \mathrm{dn}'\, u = -k^2(\mathrm{sn}u)(\mathrm{cn}u)\,. \tag{1.10}$$

The functions $\mathrm{sn}(u,k)$, $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ are doubly periodic with periods

$$\left(4K(k)\,, i2K'(k)\right), \left(4K(k)\,, 2K(k) + i2K'(k)\right), \left(2K(k)\,, i4K'(k)\right)$$

respectively. Here $K(k)$ denotes the complete elliptic integral of the first kind

$$K = 2\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2\sin^2 x}}$$

and $K'(k) = K(1-k)$. The modulus is such that $0 < k < 1$.
The limiting case $k = 0$ yields $K(0) = \frac{\pi}{2}$ and trigonometric functions:

$$\mathrm{sn}(u,0) = \sin u, \ \mathrm{cn}(u,0) = \cos u, \ \mathrm{dn}(u,0) = 1.$$

The limiting case $k = 1$ yields $K(1) = \infty$ and hyperbolic functions:

$$\mathrm{sn}(u,1) = \tanh u, \ \mathrm{cn}(u,1) = \mathrm{sech}\, u, \ \mathrm{dn}(u,1) = \mathrm{sech}\, u.$$

The Zeta function of Jacobi is defined by

$$Z(u) = \frac{d}{du}\left[\mathrm{Ln}\left(\theta_4(v)\right)\right], \qquad u = \theta_3^2(0)\,v,$$

and satisfies the following identity

$$Z(u+w) = Z(u) + Z(w) - k^2(\mathrm{sn}u)(\mathrm{sn}w)(\mathrm{sn}(u+w))\,.$$

One demonstrated in [1] a new type of trigonometric development for theta functions. This one is of course connected to the developments of classic type.
Thanks to the heat equation we deduced modular and arithmetical properties of its coefficients that seem to be of interest.
Firstly we briefly recall significant results of [1] and [3]. The proofs are omitted. In light of these results one examines thereafter properties of the elliptic and Zeta functions of Jacobi .

## 2 Theta functions

We proved the next result

**Theorem 1.** *The theta function, $\theta_4(v,\tau)$, may be expressed under the form*

$$\theta_4(v,\tau)=\theta_4(0,\tau)\ \exp\big[\sum_{p\geq 1}c_{2p}(\tau)(\sin\pi v)^{2p}\big],$$

*where the coefficients, $c_{2p}$, satisfy the recurrence relation for $p\geq 1$*

$$(A)\qquad \begin{cases} 4!\,\binom{2p+4}{4}\,c_{2p+4}=(2p+1)(2p+2)\left[(2p+2)(2p+3)+4p^2-c_0\right]c_{2p+2}\\ +(2p)^2\left[c_0-(2p)^2\right]c_{2p}-6\left[(2p+1)(2p+2)\,c_{2p+2}-2c_2-\sum_{k=1}^{p}2kc_{2k}\right]^2\end{cases}$$

*and*

$$c_0=-4\left[\theta_2^4(0,\tau)+\theta_3^4(0,\tau)\right],\quad c_2=\frac{1}{2\pi^2}\frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)}\quad \textit{and}\quad c_4=\frac{1}{3}\theta_2^4(0,\tau)\,\theta_3^4(0,\tau)+\frac{1}{3}c_2.$$

*Moreover the expression above for $\theta_4$ is valid in the strip $\mid \mathrm{Im}, v\mid<\frac{1}{2}\mathrm{Im},\tau$.*

For the other theta functions we obtain the following

**Theorem 2.** *Under the hypotheses of Theorem 1 we get the following expressions*

$$\theta_1(v,\tau)=\theta_4(0,\tau)\ \exp\left[i\pi\left(v+\tfrac{1}{4}\tau\right)+\sum_{p\geq 1}c_{2p}(\tau)\sin^{2p}\pi\left(v+\tfrac{1}{2}\tau\right)\right]$$

$$\theta_2(v,\tau)=\theta_4(0,\tau)\ \exp\left[i\pi\left(v+\tfrac{1}{4}\tau\right)+\sum_{p\geq 1}c_{2p}(\tau)\cos^{2p}\pi\left(v+\tfrac{1}{2}\tau\right)\right]$$

$$\theta_3(v,\tau)=\theta_4(0,\tau)\ \exp\left[\sum_{p\geq 1}c_{2p}(\tau)(\cos\pi v)^{2p}\right],$$

*where the coefficients $c_{2p}$ satisfy relation (A).*
*Moreover the expressions above for $\theta_1$ and $\theta_3$ are valid in the strip $\mid \mathrm{Im}\ v\mid<\mathrm{Im}\ \tau$ and $\theta_2$ is valid in the strip $\mid \mathrm{Im\ v}\ \mid<\frac{1}{2}\mathrm{Im}\ \tau$.*

Under the same hypotheses the product formula of theta functions holds, namely

$$\frac{\theta_2(v,\tau)\,\theta_3(v,\tau)\,\theta_4(v,\tau)}{\theta_4^3(0,\tau)}=e^{v+\frac{\tau}{4}}\ \exp\left[\sum_{p\geq 1}c_{2p}(\tau)\left[\sin^{2p}\pi v+\cos^{2p}\pi v+\cos^{2p}\pi\left(v+\tfrac{1}{2}\tau\right)\right]\right].$$

In particular we get

$$\theta_1'(0,\tau)=\pi\theta_4^3(0,\tau)\,q^{\frac{1}{4}}\ \exp\left[\sum_{p\geq 1}c_{2p}(\tau)\left[1+\cos^{2p}\pi\frac{\tau}{2}\right]\right].$$

The heat equation permits one to state a differential system satisfied by the coefficients $c_{2p}(\tau)$

$$(S)\quad \begin{cases} \dfrac{4}{\pi} c'_{2p} = (2p+2)(2p+1)\ \ c_{2p+2} - 4p^2\ c_{2p} \\ -4\displaystyle\sum_{m=0}^{p-1} m\ c_{2m}\left[(p-m)\ \ c_{2p-2m} - (p-m+1)\ \ c_{2p-2m+2}\right], \end{cases} \tag{2.1}$$

where $c'_{2p} = \dfrac{dc_{2p}}{d\tau}$. More precisely this system is obtained by identification after replacing the expression for $\theta_4(v,\tau) = \theta_4(0,\tau)\ \exp\left[\sum_{p\ge 1} c_{2p}(\tau)(\sin \pi v)^{2p}\right]$ in Equation (1.3).

The next theorem solves System (S) and thus an expansion of the theta function is derived.

**Theorem 3.** *The coefficients $c_{2p}(\tau)$ may be expressed as*

$$c_{2p}(\tau) = -\frac{1}{p}\sum_{k\ge 0} \frac{1}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{2p}} = -\frac{1}{p}\sum_{k\ge 0}\left[\frac{(-4)\,q^{2k+1}}{\left(1-q^{2k+1}\right)^2}\right]^p.$$

*The function $\theta_4$ has the following expansion*

$$\theta_4(v,\tau) = \theta_4(0,\tau)\ \exp\left[-\sum_{p\ge 1}\sum_{k\ge 0}\frac{1}{p}\left(\frac{\sin \pi v}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)}\right)^{2p}\right].$$

*Moreover the expression above of $\theta_4$ is valid in the strip* $\mid \mathrm{Im}, v \mid < \frac{1}{2}\mathrm{Im}, \tau$.

Of course the other theta functions $\theta_1(v,\tau)$, $\theta_2(v,\tau)$ and $\theta_3(v,\tau)$ have similar trigonometric expansions.

# 3 Elliptic functions of Jacobi

In this Section we introduce new trigonometric developments for Jacobi elliptic functions constructed from the theta functions.

**Theorem 4.** *Let $u = \theta_3^2(0)\,v$ be such that $\mid \mathrm{Im}\ v \mid < \frac{1}{2}\mathrm{Im}\ \tau$. Then the following expansions for elliptic functions hold*

$$\mathrm{sn}u = e^{i\pi v}\ \exp\left[-\sum_{p\ge 1}\frac{1}{p}\sum_{k\ge 0}\frac{1}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{2p}}\left[\sin^{2p}\pi\left(v+\frac{\tau}{2}\right) + \sin^{2p}\pi v - \cos^{2p}\pi\frac{\tau}{2} + 1\right]\right],$$

$$\mathrm{cn}u = e^{-i\pi v}\ \exp\left[-\sum_{p\ge 1}\frac{1}{p}\sum_{k\ge 0}\frac{1}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{2p}}\left[\sin^{2p}\pi v - \cos^{2p}\pi\left(v+\frac{\tau}{2}\right) - \cos^{2p}\pi\frac{\tau}{2}\right]\right],$$

$$\mathrm{dn}u = \exp\left[-\sum_{p\ge 1}\frac{1}{p}\sum_{k\ge 0}\frac{1}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{2p}}\left[\cos^{2p}\pi v - \sin^{2p}\pi v - 1\right]\right].$$

**Proof.** By Theorem 2 we get also the following expressions for ratios of theta functions

$$\frac{\theta_1(v,\tau)}{\theta_2(v,\tau)} = \exp\left[\sum_{p\geq 1} c_{2p}(\tau)\left[\sin^{2p}\pi\left(v+\frac{\tau}{2}\right) - \cos^{2p}\pi\left(v+\frac{\tau}{2}\right)\right]\right]$$

$$\frac{\theta_3(v,\tau)}{\theta_4(v,\tau)} = \exp\left[\sum_{p\geq 1} c_{2p}(\tau)\left[\cos^{2p}\pi v - \sin^{2p}\pi v\right]\right].$$

The result follows by Theorem 3 since $c_{2p}(\tau) = -\frac{1}{p}\sum_{k\geq 0}\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{-2p}$.
Moreover by definition one has $\dfrac{\mathrm{sn}u}{\mathrm{cn}u} = \dfrac{\theta_3(0)\,\theta_1(v,\tau)}{\theta_4(0)\,\theta_2(v,\tau)}$ and $\mathrm{sn}u = \dfrac{\theta_3(0)\,\theta_1(v)}{\theta_2(0)\,\theta_4(v)}$,
$\mathrm{cn}u = \dfrac{\theta_4(0)\,\theta_2(v)}{\theta_2(0)\,\theta_4(v)}$ and $\mathrm{dn}u = \dfrac{\theta_4(0)\,\theta_3(v)}{\theta_3(0)\,\theta_4(v)}$. ■

Starting from Theorem 4 we may deduce other various relations.

**Theorem 5.** *Under the hypotheses of Theorem 4 the following relations hold*

$$\frac{\mathrm{sn}u}{\mathrm{cn}u} = \exp\left[-\sum_{p\geq 1}\frac{1}{p}\sum_{k\geq 0}\frac{1}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{2p}}\left[1+\sin^{2p}\pi\left(v+\frac{1}{2}\tau\right) - \cos^{2p}\pi\left(v+\frac{1}{2}\tau\right)\right]\right].$$

$$\frac{\partial \mathrm{sn}u}{\partial u} = e^{-i\pi v}\ \exp\left[-\sum_{p\geq 1}\frac{1}{p}\sum_{k\geq 0}\frac{1}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)^{2p}}\left[\cos^{2p}\pi v - \cos^{2p}\pi\left(v+\frac{\tau}{2}\right)\right.\right.$$
$$\left.\left. - \cos^{2p}\pi\frac{\tau}{2} - 1\right]\right].$$

By the same way one obtains expansions for partial derivatives of cn $u$ and dnv$u$.

## 4 Zeta function

Consider the zeta function of Jacobi. It is defined by

$$\mathrm{Zn}(z,k) = \frac{1}{2K}\frac{d}{dz}\log\theta_4(v,\tau),$$

where $v = z/(2K)$ and $K = 2\int_0^{\frac{\pi}{2}} dx/\sqrt{1-k^2\sin^2 x}$ is the complete elliptic integral of the first kind and the modulus is such that $0 < k < 1$. Note that the zeta function has also a Fourier expansion

$$\mathrm{Zn}(z,k) = \frac{2\pi}{K}\sum_{n\geq 1}\frac{q^n}{1-q^{2n}}\sin\frac{n\pi z}{K}$$

which may be rewritten as

$$\mathrm{Zn}(z,k) = \frac{\pi}{2K}\sin(2\pi v)\sum_{k\geq 0}\frac{1}{\sin^2(\pi v) - \sin^2\left(k+\frac{1}{2}\pi\tau\right)},$$

where $v = z/(2K)$.

**Theorem 6.** *Let* $K = 2\int_0^{\frac{\pi}{2}} dx/\sqrt{1-k^2\sin^2 x}$ *be the complete elliptic integral of the first kind.*
*The zeta function of Jacobi has the following form*

$$\mathrm{Zn}\,(z,k) = \frac{\pi}{2K}\sin\left(\pi\frac{z}{K}\right)\sum_{k\geq 0}\sum_{p\geq 1}\left(\frac{\sin\dfrac{\pi z}{2K}}{\left(\sin\left(k+\frac{1}{2}\right)\pi\tau\right)}\right)^{2p}$$

*which is valid in the strip* $\mid \mathrm{Im}\ \dfrac{z}{2K} \mid < \dfrac{1}{2}\mathrm{Im}\ \tau$.

# 5 Concluding remarks

The Jacobi elliptic functions and in particular $dn\,(u,k)$ play an important role in the theory of elliptic functions as well as in many physical problems.
The previous calculations particularly indicate to us that the theory of the Jacobi elliptic functions seems not to be exhausted completely and new characterizations involving Jacobi theta functions may be found. So we may expect always to discover other properties having interesting applications, as in the works of Khare, Lakshminarayan and Sukhatme [5].

We recall some quantum mechanical facts using the function $\mathrm{dn}\,(u,k)$.
The wave functions $\psi_0^{\pm} = [\mathrm{dn}\,(u,k)]^{\mp}$ are the zero modes of the periodic supersymmetric partner potentials :

$$V_+\,(u) = \frac{2-k+2\,(k-1)}{\mathrm{dn}^2\ (u,k)} \quad \text{and} \quad V_-\,(u) = 2-k+2\mathrm{dn}^2\ (u,k).$$

This function also allows a resolution of a nonlinear Schrödinger equation. Indeed the nonlinear Schrödinger equation

$$\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + 2\psi^2\bar{\psi} = 0$$

has the following as its general periodic solutions

$$\psi\,(x,t) = r\ \exp\left[i\left(px - p^2 - \left(2-k^2\right)r^2\right)t\right]\ \mathrm{dn}\left(rx - 2prt, k^2\right),$$

where $r$, $p$ and $k$ are parameters. The cyclic identities as well as their generalized Landen formulas play an important role in showing that a kind of linear superposition of periodic solutions is valid in physically interesting nonlinear differential equations. See [5] for additional details.

# References

[1] Chouikha A R, On the expansions of elliptic functions and applications, in Algebraic methods and q-special functions, C.R.M. Proceedings and Lectures Notes, A.M.S., vol 22, Providence, 1999, 53-57.

[2] Chouikha A R, Note on trigonometric expansions of theta functions, *J. Comp. Appl. Math.* **153** (2003), 119-125.

[3] Chouikha A R, Trigonometric expansions of theta functions and applications, Math. ArXiv, http://front.math.ucdavis.edu/math.NT/0112137

[4] Erdlyi A, Magnus W, Oberhettinger F and Tricomi F (editors), Higher Transcendental Functions Voll I and III. Based on notes left by H Bateman, Robert E Krieger Publish. Co., Inc., Melbourne, Fla., 1981.

[5] Khare A, Lakshminarayan A and Sukhatme U, Cyclic identities involving Jacobi elliptic functions, *J. Math. Phys.*, **44, n 4** (2003), 1822-1841.

[6] Lawden D F, Elliptic Functions and Applications, Springer-Verlag, New-York, 1989.

[7] Mumford D, Tata lectures on Theta Functions. Vol II. Jacobian theta functions and differential equations, with C Musili, M Nori, E Previato, M Stillman and H Umemura, Progress in Mathematics, **43**, Birkhuser, Boston, 1984.

[8] Whittaker E T and Watson G N, A Course of Modern Analysis, Cambridge University Press, Cambridge UK, 1963.