

# Separability in Hamilton–Jacobi Sense in Two Degrees of Freedom and the Appell Hypergeometric Functions

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## Abstract

A huge family of separable potential perturbations of integrable billiard systems and the Jacobi problem for geodesics on an ellipsoid is given through the Appell hypergeometric functions  $F_4$  of two variables, leading to an interesting connection between two classical theories: separability in Hamilton–Jacobi sense on the one hand and the theory of hypergeometric functions on the other.

## 1 Introduction

We start with a famous thought of Jacobi [13]:

“The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied” (see [3]).

Elliptic coordinates, as the most notable substitution, were, of course, found by Jacobi himself and presented in Lecture 26 of [13]. Then they were immediately applied to separation of variables in the so-called Jacobi problem of geodesics on an ellipsoid and to the problem of attraction by two fixed centers in Lectures 28 and 29 [13]. Therein the problem of study of separable potential perturbations of a given separable system was initiated.

The basic goal of the present paper is to present our recent observation that there exists a huge family of separable potential perturbations of separable systems with two degrees of freedom, which can be simply expressed through the Appell hypergeometric functions.

Appell introduced four families of hypergeometric functions of two variables in THE 1880s. He applied them solving the Tisserand problem in Celestial Mechanics. The Appell functions have also several other applications, for example in the theory of algebraic equations, algebraic surfaces etc. Nevertheless the relationship between the Appell functions  $F_4$  and separability of variables in the Hamilton-Jacobi equations, to the best of our knowledge, remained unknown until very recently, see [9].

The equation:

$$\lambda V_{xy} + 3(yV_x - xV_y) + (y^2 - x^2)V_{xy} + xy(V_{xx} - V_{yy}) = 0 \quad (1.1)$$

is the Bertrand-Darboux equation, which represents the necessary and sufficient condition for a natural mechanical system with two degrees of freedom

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$$

to be separable in elliptical coordinates or some of their degenerations ([4, 5]).

Solutions of the equation (1.1) in the form of Laurent polynomials in  $x$  and  $y$  were described in [8]. A much more general family of solutions of equation (1.1), which are simply related to the well-known hypergeometric functions of Appell type, is presented in Section 2, see [9], but what is more important this family shows the existence of a connection between separability of classical systems on the one hand and the theory of hypergeometric functions on the other one. Furthermore in Section 2 similar formulæ for potential perturbations for the Jacobi problem for geodesics on an ellipsoid from [7] and for billiard systems on surfaces with constant curvature from [14] are given.

Some deeper explanation of the connection between the separability in elliptic coordinates and the Appell hypergeometric functions is still unknown.

We start with some basic notations concerning the Appell hypergeometric functions. The function  $F_4$  is one of the four hypergeometric functions in two variables introduced by Appell [1, 2] and is defined as a series:

$$F_4(a, b, c, d; x, y) = \sum \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m y^n}{m! n!},$$

where  $(a)_n$  is the standard Pochhammer symbol:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1),$$

$$(a)_0 = 1.$$

(For example  $m! = (1)_m$ .)

The series  $F_4$  is convergent for  $\sqrt{x} + \sqrt{y} \leq 1$ . The functions  $F_4$  can be analytically continued to the solutions of the equations:

$$\begin{aligned} x(1-x)\frac{\partial^2 F}{\partial x^2} - y^2\frac{\partial^2 F}{\partial y^2} - 2xy\frac{\partial^2 F}{\partial x\partial y} \\ + [c - (a+b+1)x]\frac{\partial F}{\partial x} - (a+b+1)y\frac{\partial F}{\partial y} - abF = 0 \\ y(1-y)\frac{\partial^2 F}{\partial y^2} - x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x\partial y} \\ + [c' - (a+b+1)y]\frac{\partial F}{\partial y} - (a+b+1)x\frac{\partial F}{\partial x} - abF = 0. \end{aligned}$$

Basic references for the Appell functions are [1, 2, 18].

## 2 The separable mechanical systems with two degrees of freedom

### 2.1 Billiards inside an ellipse

A billiard system usually describes a particle moving freely within an ellipse:

$$\frac{x^2}{A} + \frac{y^2}{B} = 1.$$

At the boundary elastic reflections with equal impact and reflection angles are assumed. It is well known that the system is completely integrable and it has an additional integral:

$$K_1 = \frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} - \frac{(\dot{x}y - \dot{y}x)^2}{AB}.$$

The search for potential perturbations,  $V = V(x, y)$ , such that the perturbed system has an integral,  $\tilde{K}_1$ , of the form

$$\tilde{K}_1 = K_1 + k_1(x, y),$$

where  $k_1 = k_1(x, y)$  depends only on coordinates, leads to the equation (1.1) on  $V$  with  $\lambda = A - B$  (see [15]).

We denote

$$V_\gamma = \tilde{y}^{-\gamma} ((1 - \gamma)\tilde{x}F_4(1, 2 - \gamma, 2, 1 - \gamma, \tilde{x}, \tilde{y}) + 1). \quad (2.1)$$

Then we have

**Theorem 1.** *Every function  $V_\gamma$  given with (2.1) and  $\gamma \in \mathbf{C}$  is a solution of the equation (1.1).*

The billiard systems perturbed with the potentials obtained have the following mechanical interpretation. With  $\gamma \in \mathbf{R}^-$  and the coefficient multiplying  $V_\gamma$  positive a potential barrier along the  $x$ -axis appears. We can consider billiard motion in the upper half-plane. Then we can assume that the cut is done along the negative part of  $y$ -axis in order to get a single-valued real function as a potential.

**Example.** The Laurent polynomial solutions of the equation (1.1) from [8] correspond to the integer values of the parameter  $\gamma$  in Theorem 1. The basic set of Laurent solutions consists of the functions

$$V_k = \sum_{i=0}^{k-2} (-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x, y, \lambda) + y^{-2k}, \quad k \in \mathbf{N},$$

$$W_k = \sum_{i=0}^{k-2} \sum_{s=1}^{k-i-1} (-1)^s U_{kis}(y, x, \lambda) + x^{-2k}, \quad k \in \mathbf{N},$$

where

$$U_{kis} = \binom{s+i-1}{i} \frac{[1 - (k-i)][2 - (k-i)] \dots [s - (k-i)]}{\lambda^{s+i} s!} x^{2s} y^{-2k+2i}.$$

### 2.2 The Jacobi problem for geodesics on an ellipsoid

The Jacobi problem for the geodesics on an ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1$$

has an additional integral

$$K_1 = \left( \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \right) \left( \frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} + \frac{\dot{z}^2}{C} \right).$$

Potential perturbations,  $V = V(x, y, z)$ , such that the perturbed systems have integrals of the form:

$$\tilde{K}_1 = K_1 + k(x, y, z),$$

satisfy the following system (see [7]):

$$\begin{aligned} & \left( \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \right) V_{xy} \frac{A-B}{AB} - 3 \frac{y}{B^2} \frac{V_x}{A} + 3 \frac{x}{A^2} \frac{V_y}{B} \\ & \quad + \left( \frac{x^2}{A^3} - \frac{y^2}{B^3} \right) V_{xy} + \frac{xy}{AB} \left( \frac{V_{yy}}{A} - \frac{V_{xx}}{B} \right) + \frac{zx}{CA^2} V_{zy} - \frac{zy}{CB^2} V_{zx} = 0 \\ & \left( \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \right) V_{yz} \frac{B-C}{BC} - 3 \frac{z}{C^2} \frac{V_y}{B} + 3 \frac{y}{B^2} \frac{V_z}{C} \\ & \quad + \left( \frac{y^2}{B^3} - \frac{z^2}{C^3} \right) V_{yz} + \frac{yz}{BC} \left( \frac{V_{zz}}{B} - \frac{V_{yy}}{C} \right) + \frac{xy}{AB^2} V_{xz} - \frac{xz}{AC^2} V_{xy} = 0 \quad (2.2) \\ & \left( \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \right) V_{zx} \frac{C-A}{AC} - 3 \frac{x}{A^2} \frac{V_z}{C} + 3 \frac{z}{C^2} \frac{V_x}{A} \\ & \quad + \left( \frac{z^2}{C^3} - \frac{x^2}{A^3} \right) V_{zx} + \frac{xz}{AC} \left( \frac{V_{xx}}{C} - \frac{V_{zz}}{A} \right) + \frac{zy}{BC^2} V_{xy} - \frac{yx}{BA^2} V_{yz} = 0. \end{aligned}$$

The system (2.2) plays the role of equation (1.1) in this problem. Solutions of the system in the Laurent polynomial form were found in [7].

Denote:

$$\frac{x^2 C(A-C)}{z^2(B-A)A} = \hat{x}, \quad \frac{y^2 C(C-B)}{z^2(B-A)B} = \hat{y}.$$

Then we have

**Theorem 2.** For every  $\gamma \in \mathbf{C}$  the function

$$V_\gamma = (-\gamma + 1) \left( \frac{z^2}{x^2} \right)^\gamma F_4(1; -\gamma + 2; 2, -\gamma + 1, \hat{x}, \hat{y})$$

is a solution of the system (2.2).

If in the formulæ above  $l_0$  is an integer, then from [7] the corresponding potential is a Laurent polynomial.

### 2.3 Billiard systems on surfaces of constant curvature

Following the notation of [14] let the billiard  $D_S$  be a subset of the surface  $\Sigma_S$ , of curvature  $S = +1$  or  $S = -1$ , bounded by the quadric  $Q_S$ , where

$$\begin{aligned}\Sigma_+ &= \{r = (x, y, z) \in \mathbf{R}^3 \mid \langle r, r \rangle_+ = 1\} \\ \Sigma_- &= \{r = (x, y, z) \in \mathbf{R}^3 \mid \langle r, r \rangle_- = -1, z > 0, \} \\ Q_S &= \Sigma_S \cap \{r \in \mathbf{R}^3 \mid \langle Qr, r \rangle_s = 0\} \neq \emptyset\end{aligned}$$

with

$$\langle r_1, r_2 \rangle_+ = x_1y_1 + x_2y_2 + x_3y_3, \quad \langle r_1, r_2 \rangle_- = x_1y_1 + x_2y_2 - x_3y_3, \quad Q = \text{diag} \left( \frac{1}{A}, \frac{1}{B}, \frac{1}{C} \right).$$

This billiard system has the integral

$$K = \frac{(\dot{x}y - \dot{y}x)^2}{AB} + S \frac{(\dot{x}z - \dot{z}x)^2}{AC} + S \frac{(\dot{z}y - \dot{y}z)^2}{BC}.$$

As before we are looking for potentials  $V = V(x, y, z)$  such that the perturbed system has an integral of the form:

$$\tilde{K} = K + k(x, y, z).$$

In this case the condition is given by the system [14]:

$$\begin{aligned}3CyV_x - 3CxV_y + V_{xy}(C(y^2 - x^2) + Kz^2(B - A)) \\ + CxyV_{xx} - CxyV_{yy} + Az yV_{zx} - BzxV_{zy} = 0, \\ 3BzV_x - K3BxV_z + V_{xz}(B(z^2 - Kx^2) + Ky^2(C - A)) \\ + BzxV_{xx} - KBzxV_{zz} + Az yV_{xy} - KCyxV_{yz} = 0, \\ 3AzV_y - K3AyV_z + V_{yz}(A(z^2 - Ky^2) + Kx^2(C - B)) \\ + Az yV_{yy} - KAz yV_{zz} + BzxV_{xy} - KCxyV_{xz} = 0.\end{aligned} \tag{2.3}$$

Starting from the solutions of [14]:

$$V_{l_0} = \frac{1}{z^{2l_0}} \sum_{\substack{0 \leq k \leq l_0 - 1 \\ 0 \leq m \leq l_0 - k - 1}} a_{m,k} x^{2m} y^{2l_0 - 2 - 2k - 2m} z^{2k},$$

where

$$a_{m,k} = K^{l_0 - k - 1} \left( \frac{C - B}{C - A} \right)^m \binom{l_0 - k - 1}{m} \binom{k + m - 1}{k} \left( \frac{A - B}{C - A} \right)^k,$$

we come to

**Theorem 3.** *The functions*

$$V_\gamma = \hat{y}^{-\gamma} ((1 - \gamma)x^2 F_4(1, 2 - \gamma, 2, 1 - \gamma, \hat{x}, \hat{y}) + 1),$$

where

$$\frac{x^2(B - C)}{y^2(C - A)} = \hat{x}, \quad K \frac{z^2(A - B)}{y^2(C - A)} = \hat{y},$$

are solutions of the system (2.3) for  $\gamma \in \mathbf{C}$ .

### 3 More than two degrees of freedom

In the previous section we have seen that integrable perturbations of separable systems with two degrees of freedom led to hypergeometric functions of two variables. Now one can expect that in the case of more than two degrees of freedom the integrable potentials are connected with hypergeometric functions again but with more than two variables. We consider the billiard system inside an ellipsoid in  $\mathbf{R}^3$  and we see that the corresponding potential perturbations are still related to the Appell function  $F_4$  of two variables but only if the ellipsoid is symmetric.

#### 3.1 Billiards inside a symmetric ellipsoid in $\mathbf{R}^3$

Consider the billiard system within an ellipsoid in  $\mathbf{R}^3$

$$\frac{x_1^2}{A} + \frac{x_2^2}{B} + \frac{x_3^2}{C} = 1.$$

Potential perturbations,  $W = W(x_1, x_2, x_3)$ , of such systems in a form of Laurent polynomials were calculated in [10]. They satisfy the following system:

$$\begin{aligned} (a_i - a_r)^{-1} (x_i^2 V_{rs} - x_i x_r V_{is}) &= (a_i - a_s)^{-1} (x_i^2 V_{rs} - x_i x_s V_{ir}), \quad \text{for } i \neq r \neq s \neq i, \\ (a_i - a_r)^{-1} x_i x_r (V_{ii} - V_{rr}) - \sum_{j \neq i, r} (a_i - a_j)^{-1} x_i x_j V_{jr} \\ &+ V_{ir} \left[ \sum_{j \neq i, r} (a_i - a_j)^{-1} x_j^2 + (a_r - a_i)^{-1} (x_i^2 - x_r^2) \right] + V_{ir} \\ &+ 3(a_i - a_r)^{-1} (x_r V_i - x_i V_r) = 0, \quad i \neq r, \end{aligned} \tag{3.1}$$

of  $(n - 1) \binom{n}{2}$  equations for  $n = 3$ . Here we denote  $V_i = \partial V / \partial x_i$  and  $V_{ij} = \partial^2 V / \partial x_i \partial x_j$ .

This system was formulated in [17] for an arbitrary number of degrees of freedom,  $n$ , and a generalization of the Bertrand-Darboux theorem was proved. According to that theorem the solutions of the last system are potentials separable in generalized elliptic coordinates. The Laurent polynomial separable potential perturbations, obtained in [10], are given by the formulæ:

$$W_{l_0} = \frac{1}{z^{2l_0}} \sum_{0 \leq m+n+k < l_0} \frac{(l_0 - k - 1)! (-1)^n}{m! n! (l_0 - 1 - k - m - n)!} \frac{P_{m,n}^k(\beta, \gamma)}{\gamma^{m+k} \beta^{n+k}} x^{2m} y^{2n} z^{2k},$$

where

$$P_{m,n}^k(\beta, \gamma) = \sum_{i=0}^k \binom{m+k-1-i}{k-i} \binom{n+i-1}{i} (-1)^i \beta^{k-i} \gamma^i$$

and  $\beta = B - C$ ,  $\gamma = C - A$ .

In the symmetric case,  $A = B$ , which corresponds to the condition  $\gamma + \beta = 0$  one has

**Lemma 1.** *If  $\gamma + \beta = 0$ , we have:*

$$P_{m,n}^k(\beta, -\beta) = \binom{k+m+n-1}{k} \beta^k.$$

**Lemma 2.** *Generalizations of the integrable potential perturbations from [10] in the symmetric case are given by:*

$$W_{l_0} = (\hat{z})^{-l_0} [(-\hat{x} + \hat{y})F_4(1, 2 - l_0; 2, 1 - l_0; -\hat{x} + \hat{y}, \hat{z}) + 1],$$

where  $l_0 \in \mathbf{C}$ .

## 4 Conclusion

It seems that general separable potential perturbations in more than two degrees of freedom are not directly related to the Lauricella functions, which are the natural generalizations of the Appell functions in the case of more than two variables. Thus the connection of higher-dimensional separable potentials and multivariable generalizations of the hypergeometric functions ([18]) and/or their deformations remains as an open problem.

From the geometric point of view it is well known that billiard systems within an ellipse are closely related to the theorems of Poncelet and Cayley [16, 12, 11]. So the Appell hypergeometric functions define natural deformations of these classical settings of projective geometry. Since all separable systems in two degrees of freedom are of the Liouville type (see [19]), this story is closely related to the study of the Liouville surfaces performed by Darboux in [6].

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